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Approximate $n-$ideal amenability of module extension Banach algebras

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Abstract. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}-$bimodule. We study the notion of approximate $n-$ideal amenability for module extension Banach algebras $\mathcal{A} \oplus X$. First, we describe the structure of ideals of this kind of algebras and we present the necessary and sufficient conditions for a module extension Banach algebra to be approximately n-ideally amenable.

Keywords: Amenability, ideal amenability, module extension Banach algebras.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}-$bimodule. Then X^* (the topological dual of X) is a Banach $\mathcal{A}-$bimodule with the following module actions:

$$\langle a \cdot x^*, x\rangle = \langle x^*, x \cdot a\rangle \qquad ; \qquad \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle,$$

where $a \in \mathcal{A}, x \in X$ and $x^* \in X^*$. If I is a two-sided closed ideal in $\mathcal{A}$, then I^* also is a Banach $\mathcal{A}-$bimodule with the corresponding actions. Also, $I^{(n)}$ the n-th dual space of I is a Banach $\mathcal{A}-$bimodule for all $n \in \mathbb{N}$.

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A derivation from $\mathcal{A}$ into X is a linear mapping $D : \mathcal{A} \to X$ satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b, \qquad (a, b \in \mathcal{A}).$$

For $x \in X$, the map $\delta_x : \mathcal{A} \to X$ defined by $\delta_x(a) = a \cdot x - x \cdot a$ is a derivation for each $a \in \mathcal{A}$. This kind of derivations are called inner derivations. We denote by $\mathcal{Z}^1(\mathcal{A}, X)$, the space of all continuous derivations from $\mathcal{A}$ into X and we denote by $\mathcal{N}^1(\mathcal{A}, X)$, the space of all inner derivations from $\mathcal{A}$ into X. The quotient space $\mathcal{H}^1(\mathcal{A}, X) = \mathcal{Z}^1(\mathcal{A}, X)/\mathcal{N}^1(\mathcal{A}, X)$ is called the first cohomology group of $\mathcal{A}$ with coefficients in X (see [3, 10]).

The Banach algebra $\mathcal{A}$ is called amenable if every continuous derivation from $\mathcal{A}$ into Banach $\mathcal{A}-$bimodule X^* is inner, i.e. $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}-$bimodule X. This notion was introduced by B. E. Johnson in ([10]). Bade, Curtis and Dales in [1, 3] defined the concept of weak amenability for commutative Banach algebras. Later, Dales, Ghahramani and Gronbaek [4] introduced the concept of $n-$weak amenability of Banach algebras. The Banach algebra $\mathcal{A}$ is $n-$weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, $(n \in \mathbb{N})$.

More recently, Eshaghi-Gordji and Yazdanpanah [8] introduced a notion of amenability as follows: $\mathcal{A}$ is ideally amenable [$n-$ideally amenable $(n \in \mathbb{N})$] if $\mathcal{H}^1(\mathcal{A}, I^*) = \{0\}$ [$\mathcal{H}^1(\mathcal{A}, I^{(n)}) = \{0\}$] for every closed two-sided ideal I in $\mathcal{A}$.

In 2008, Monfared [11] discussed another version of amenability named the right character amenability and after that in 2013, Bodaghi et al. [2] turned to the generalized notion of character amenability and the relevant properties. Recently, in [12], Rahimi and Amini studied the concept of amenability modulo an ideal. They proved some results about this issue that inducing the amenability of $l^1(S)$ modulo ideals by certain categories of group congruences on S is equivalent to the amenability of S. Along with this work, one can find other newly-published papers on the amenability modulo an ideals of a Banach algebra such as [6, 9, 13, 14].

Ghahramani and Loy introduced a generalized notion of amenability [5]. This new notion was approximate amenability of a Banach algebra. The continuous Derivation $D : \mathcal{A} \to X$ is called approximately inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha(ax_\alpha - x_\alpha a)$. Then a Banach algebra $\mathcal{A}$ is approximately amenable if every continuous derivation from $\mathcal{A}$ into X^* is approximately inner for each Banach $\mathcal{A}-$bimodule X. Also, $\mathcal{A}$ is approximately $n-$weakly amenable if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is approximately inner $(n \in \mathbb{N})$.

Similarly, we have the notions approximate ideal [$n-$ideal] amenability for $n \in \mathbb{N}$ in [7, 15].

***Example* 1.1** [15] (i) Let G be a locally compact group. Then $M(G)$ is approximately $n - L^1(G)$ amenable.
(ii) Let G be a compact group. Then $L^1(G)^{**}$ is approximately $n - L^1(G)$ amenable.

The direct l_1-sum of $\mathcal{A}$ and X is the Banach space $\mathcal{A} \oplus X$ with the following norm

$$\|(a, x)\| = \|a\| + \|x\|, \qquad (a \in \mathcal{A}, x \in X).$$

Also, $\mathcal{A} \oplus X$ is a Banach algebra with the following product

$$(a_1, x_1) \cdot (a_2, x_2) = (a_1a_2, x_1 \cdot a_2 + a_1 \cdot x_2).$$

$\mathcal{A} \oplus X$ is called module extension Banach algebra corresponding to $\mathcal{A}$ and X ([16]). On the other hand, we know that $(0 \oplus X)^\perp$ and $(\mathcal{A} \oplus 0)^\perp$ are isometrically isomorph with

X^* and $\mathcal{A}^*$ as $\mathcal{A}-$bimodules, respectively. So, we have

$$(\mathcal{A} \oplus X)^* = (0 \oplus X)^\perp \dot{+} (\mathcal{A} \oplus 0)^\perp$$

where $\dot{+}$ denotes direct $\mathcal{A}-$bimodule $l_\infty-$sum. But, for simplicity, we can write

$$(\mathcal{A} \oplus X)^* = \mathcal{A}^* \dot{+} X^*.$$

Now, Consider $\mathcal{A}^{(n)} \dot{+} X^{(n)}$ as the underlying space $(\mathcal{A} \oplus X)^{(n)}$. Then

$$(\mathcal{A} \oplus X)^{(2n)} = \mathcal{A}^{(2n)} \oplus_1 X^{(2n)};$$
$$(\mathcal{A} \oplus X)^{(2n+1)} = \mathcal{A}^{(2n+1)} \oplus_\infty X^{(2n+1)}.$$

One can easily prove that $(\mathcal{A} \oplus X)^{(n)}$ is a Banach $(\mathcal{A} \oplus X)-$bimodule with the following module actions:
(i) If n is odd:

$$(a, x) \cdot (a^{(n)}, x^{(n)}) = (aa^{(n)} + xx^{(n)}, ax^{(n)}),$$
$$(a^{(n)}, x^{(n)}) \cdot (a, x) = (a^{(n)}a + x^{(n)}x, x^{(n)}a);$$

(ii) If n is even:

$$(a, x) \cdot (a^{(n)}, x^{(n)}) = (aa^{(n)}, ax^{(n)} + xa^{(n)}),$$
$$(a^{(n)}, x^{(n)}) \cdot (a, x) = (a^{(n)}a, a^{(n)}x + x^{(n)}a);$$

where $(a, x) \in \mathcal{A} \oplus X$ and $(a^{(n)}, x^{(n)}) \in \mathcal{A}^{(n)} \dot{+} X^{(n)} = (\mathcal{A} \oplus X)^{(n)}$.

Remark 1 *Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}-$bimodule. Then J is a closed ideal in $\mathcal{A} \oplus X$ if and only if there are a closed ideal I in $\mathcal{A}$ and a closed submodule Y of X such that $J = I \oplus Y$ and that $IX \cup XI \subseteq Y$.*

In this paper, we study the approximate $n-$ideal amenability of module extension Banach algebras. Throughout this paper, we consider $I \oplus Y$ as an ideal of Banach algebra $\mathcal{A} \oplus X$. Since $(\mathcal{A} \oplus X)-$bimodule actions on $(\mathcal{A} \oplus X)^{(n)}$ is different whenever n is odd or even, thus approximate $n-$ideal amenability of $\mathcal{A} \oplus X$ is investigated in two separate sections 2 and 3.

2. approximate $(2n + 1)-$ideal amenability of $\mathcal{A} \oplus X$

Throughout this section, n is a non-negative integer. To prove the main theorem of this section, we need the following lemmas.

Lemma 2.1 Let $T : X \to I^{(2n+1)}$ be a continuous $\mathcal{A}-$bimodule homomorphism. Then $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{T}((a, x)) = (T(x), 0)$ is a continuous derivation. Moreover, $\bar{T}$ is approximately inner if and only if there exists a net $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that for every $a \in \mathcal{A}$, $\lim_\alpha(aF_\alpha - F_\alpha a) = 0$ and $T(x) = \lim_\alpha(xF_\alpha - F_\alpha x)$ for each $x \in X$.

Proof. Let (a_1, x_1) and (a_2, x_2) be two arbitrary elements of $\mathcal{A} \oplus X$. We have

$$\begin{aligned}\bar{T}((a_1, x_1)(a_2, x_2)) &= \bar{T}((a_1a_2, a_1x_2 + x_1a_2)) \\ &= (T(a_1x_2 + x_1a_2), 0) \\ &= (a_1T(x_2) + T(x_1)a_2, 0).\end{aligned}$$

On the other hand,

$$\begin{aligned}(a_1, x_1)\bar{T}((a_2, x_2)) + \bar{T}((a_1, x_1))(a_2, x_2) &= (a_1, x_1)(T(x_2), 0) + (T(x_1), 0)(a_2, x_2) \\ &= (a_1T(x_2), 0) + (T(x_1)a_2, 0) \\ &= (a_1T(x_2) + T(x_1)a_2, 0).\end{aligned}$$

Therefore $\bar{T}$ is a derivation. Now, Let $\bar{T}$ be approximately inner. Then there exist nets $(G_\alpha)_\alpha \subseteq I^{(2n+1)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that

$$\begin{aligned}\bar{T}((a, x)) &= \lim_\alpha((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)) \\ &= \lim_\alpha((aG_\alpha + xF_\alpha, aF_\alpha) - (G_\alpha a + F_\alpha x, F_\alpha a)) \\ &= \lim_\alpha(aG_\alpha + xF_\alpha - G_\alpha a - F_\alpha x, aF_\alpha - F_\alpha a).\end{aligned}$$

Now, for every $x \in X$, we have

$$\begin{aligned}(T(x), 0) = \bar{T}((0, x)) &= \lim_\alpha[(0, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (0, x)] \\ &= \lim_\alpha[(xF_\alpha, 0) - (F_\alpha x, 0)] \\ &= [\lim_\alpha(xF_\alpha - F_\alpha x), 0] \\ &= \lim_\alpha(xF_\alpha - F_\alpha x, 0).\end{aligned}$$

Also

$$\begin{aligned}(0, 0) = \bar{T}((a, 0)) &= \lim_\alpha((a, 0) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, 0)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).\end{aligned}$$

So, it is clear that for every $a \in \mathcal{A}$, $\lim_\alpha(aF_\alpha - F_\alpha a) = 0$ and for every $x \in X$, $T(x) = \lim_\alpha(xF_\alpha - F_\alpha x)$.

Conversely, let there exists such a net $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$. Then

$$\begin{aligned}\bar{T}((a, x)) = (T(x), 0) &= \lim_\alpha(xF_\alpha - F_\alpha x, aF_\alpha - F_\alpha a) \\ &= \lim_\alpha(a, x) \cdot (0, F_\alpha) - (0, F_\alpha) \cdot (a, x).\end{aligned}$$

This shows that $\bar{T}$ is approximately inner. ■

Lemma 2.2 Let $D : \mathcal{A} \to Y^{(2n+1)}$ be a continuous derivation such that for every $a_1, a_2 \in \mathcal{A}$ and $x_1, x_2 \in X$, $x_1D(a_2) = D(a_1)x_2$. Then mapping $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$

defined by $\bar{D}((a,x)) = (0, D(a))$ is a continuous derivation. Moreover

(i) If $\bar{D}$ is approximately inner then D is so.

(ii) If D is approximately inner then there is a net of continuous derivations $\tilde{D}_\alpha : \mathcal{A}\oplus X \to (I\oplus Y)^{(2n+1)}$ such that for all α and for each $a \in \mathcal{A}$, we have $\tilde{D}_\alpha((a,0)) = 0$ and $\bar{D} - \tilde{D}_\alpha$ is inner.

Proof. For every $(a_1, x_1), (a_2, x_2) \in \mathcal{A} \oplus X$, we have

$$\begin{aligned}\bar{D}((a_1,x_1)(a_2,x_2)) &= \bar{D}((a_1a_2, a_1x_2 + x_1a_2)) \\ &= (0, D(a_1a_2)) \\ &= (0, a_1D(a_2) + D(a_1)a_2).\end{aligned}$$

On the other hand,

$$(a_1,x_1)\bar{D}((a_2,x_2)) = (a_1,x_1)(0, D(a_2)) = (x_1D(a_2), a_1D(a_2))$$

and

$$\bar{D}((a_1,x_1))(a_2,x_2) = (0, D(a_1))(a_2,x_2) = (D(a_1)x_2, D(a_1)a_2).$$

It is seen that $\bar{D}$ is a derivation.

Now, let $\bar{D}$ be approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n+1)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ provided that

$$\bar{D}((a,x)) = \lim_\alpha((a,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,x)).$$

But we have

$$\begin{aligned}(0, D(a)) = \bar{D}((a,0)) &= \lim_\alpha((a,0)\cdot(G_\alpha,F_\alpha) - (G_\alpha,F_\alpha)\cdot(a,0)) \\ &= \lim_\alpha((aG_\alpha, aF_\alpha) - (G_\alpha a, F_\alpha a)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).\end{aligned}$$

Hence it follows that $D(a) = \lim_\alpha(aF_\alpha - F_\alpha a)$ for all $a \in \mathcal{A}$; so D is approximately inner. This completes the proof of (i).

(ii) Let D be approximately inner. Then there is a net $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that for all $a \in \mathcal{A}$, $D(a) = \lim_\alpha(aF_\alpha - F_\alpha a)$. Suppose that $T_\alpha : X \to I^{(2n+1)}$ is defined by

$$T_\alpha(x) = F_\alpha x - xF_\alpha, \qquad (x \in X).$$

Also, let $\bar{T}_\alpha : \mathcal{A}\oplus X \to (I\oplus Y)^{(2n+1)}$ be defined by

$$\bar{T}_\alpha((a,x)) = (T_\alpha(x), 0), \qquad (a \in \mathcal{A}, x \in X).$$

Take $\tilde{D}_\alpha = \bar{T}_\alpha$. Then for all α and for each $a \in \mathcal{A}$ we can write

$$\tilde{D}_\alpha((a,0)) = \bar{T}_\alpha((a,0)) = (T_\alpha(0), 0) = 0.$$

Thus $\tilde{D}_\alpha((a,0)) = 0$. On the other hand, we have

$$\begin{aligned}(\bar{D} - \tilde{D}_\alpha)((a,x)) &= (\bar{D} - \bar{T}_\alpha)((a,x)) \\ &= \bar{D}((a,x)) - \bar{T}_\alpha((a,x)) \\ &= (0, D(a)) - (T_\alpha(x), 0) \\ &= (-T_\alpha(x), D(a)) \\ &= (xF_\alpha - F_\alpha x, aF_\alpha - F_\alpha a) \\ &= (a,x)\cdot(0,F_\alpha) - (0,F_\alpha)\cdot(a,x).\end{aligned}$$

Therefore $(\bar{D} - \tilde{D}_\alpha)$ is inner. ■

Lemma 2.3 Suppose that $D : \mathcal{A} \to I^{(2n+1)}$ is a continuous derivation. Then the mapping $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{D}((a,x)) = (D(a), 0)$ is a continuous derivation. Moreover, $\bar{D}$ is approximately inner if and only if D is approximately inner.

Proof. Let $(a_1, x_1), (a_2, x_2) \in \mathcal{A} \oplus X$ be two arbitrary elements. We have

$$\begin{aligned}\bar{D}((a_1,x_1)\cdot(a_2,x_2)) &= \bar{D}((a_1a_2, x_1a_2 + a_1x_2)) = (D(a_1a_2), 0) \\ &= (D(a_1)a_2 + a_1D)a_2), 0) \\ &= (D(a_1), 0)(a_2, x_2) + (a_1, x_1)(D(a_2), 0) \\ &= \bar{D}((a_1,x_1))(a_2,x_2) + (a_1,x_1)\bar{D}((a_2,x_2)).\end{aligned}$$

So, $\bar{D}$ is a derivation. Now, let $\bar{D}$ be approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n+1)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that

$$\bar{D}((a,x)) = \lim_\alpha((a,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,x)).$$

But $\bar{D}((a,0)) = (D(a), 0)$. Then it follows that

$$\begin{aligned}(D(a),0) &= \bar{D}((a,0)) \\ &= \lim_\alpha((a,0)\cdot(G_\alpha,F_\alpha) - (G_\alpha,F_\alpha)\cdot(a,0)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).\end{aligned}$$

Consequently, $D(a) = \lim_\alpha(aG_\alpha - G_\alpha a)$ for $(G_\alpha)_\alpha \subseteq I^{(2n+1)}$; i.e. D is approximately inner.

Conversely, we assume that D is approximately inner. Then there is a net $(G_\alpha)_\alpha \subseteq I^{(2n+1)}$ such that for all $a \in \mathcal{A}$, $D(a) = \lim_\alpha(aG_\alpha - G_\alpha a)$. We can write

$$\begin{aligned}\bar{D}((a,x)) &= (D(a),0) = (\lim_\alpha(aG_\alpha - G_\alpha a), 0) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, 0) \\ &= \lim_\alpha((a,x)\cdot(G_\alpha,0) - (G_\alpha,0)\cdot(a,x)).\end{aligned}$$

By letting $u_\alpha = (G_\alpha, 0) \subseteq (T \oplus Y)^{(2n+1)}$, we have $\bar{D}((a,x)) = \lim_\alpha((a,x)\cdot u_\alpha - u_\alpha \cdot (a,x))$ where $(u_\alpha)_\alpha \subseteq (I \oplus Y)^{(2n+1)}$. Thus $\bar{D}$ is approximately inner. ■

Lemma 2.4 Let $T : X \to Y^{(2n+1)}$ be a continuous $\mathcal{A}-$bimodule homomorphism satisfying $x_1T(x_2) + T(x_1)x_2 = 0$ for each $x_1, x_2 \in X$. Then the mapping $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{T}((a,x)) = (0, T(x))$ is a continuous derivation. Moreover, $\bar{T}$ is approximately inner if and only if $T = 0$.

Proof. First, we show that $\bar{T}$ is a derivation. Let $(a_1, x_1), (a_2, x_2) \in \mathcal{A} \oplus X$ be two arbitrary elements. We have

$$\begin{aligned}\bar{T}((a_1,x_1)\cdot(a_2,x_2)) &= \bar{T}((a_1a_2, a_1x_2 + x_1a_2))\\ &= (0, T(a_1x_2 + x_1a_2))\\ &= (0, a_1T(x_2) + T(x_1)a_2).\end{aligned}$$

On the other hand,

$$\bar{T}((a_1,x_1))\cdot(a_2,x_2) = (0, T(x_1))(a_2,x_2) = (T(x_1)x_2, T(x_1)a_2)$$

and

$$(a_1,x_1)\cdot\bar{T}((a_2,x_2)) = (a_1,x_1)(0,T(x_2)) = (x_1T(x_2), a_1T(x_2)).$$

It follows that $\tilde{T}$ is a derivation.

Now, let $\bar{T}$ is approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n+1)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that

$$\bar{T}((a,x)) = \lim_\alpha((a,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,x)).$$

Since $\bar{T}((a,x)) = \bar{T}((0,x))$, thus

$$\begin{aligned}(0,T(x)) = \bar{T}((0,x)) &= \lim_\alpha((0,x)\cdot(G_\alpha,F_\alpha) - (G_\alpha,F_\alpha)\cdot(0,x))\\ &= \lim_\alpha((xF_\alpha, 0) - (F_\alpha x, 0))\\ &= \lim_\alpha(xF_\alpha - F_\alpha x, 0).\end{aligned}$$

Hence T is trivial; i.e. $T = 0$. Converse is clear. ■

Now, we present the necessary and sufficient conditions for module extension Banach algebra $\mathcal{A} \oplus X$ to be approximately $(2n+1)-$ideally amenable.

Theorem 2.5 Let $\mathcal{A} \oplus X$ be a module extension Banach algebra and $I \oplus Y$ be an arbitrary closed ideal in $\mathcal{A} \oplus X$. Then $\mathcal{A} \oplus X$ is approximately $(2n+1)-$ideally amenable if and only if the following conditions hold:

(i) $\mathcal{A}$ is approximately $(2n+1) - I-$weakly amenable;
(ii) Every derivation from $\mathcal{A}$ into $Y^{(2n+1)}$ is approximately inner;
(iii) For every continuous $\mathcal{A}-$bimodule homomorphism $T : X \to I^{(2n+1)}$, there is net $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that for each $a \in \mathcal{A}$, $\lim_\alpha(aF_\alpha - F_\alpha a) = 0$ and for every $x \in X$, $T(x) = \lim_\alpha(xF_\alpha - F_\alpha x)$;

(iv) The only continuous $\mathcal{A}-$bimodule homomorphism $T : X \to Y^{(2n+1)}$ for which $x_1T(x_2) + T(x_1)x_2 = 0$ $(x_1, x_2 \in X)$ in $I^{(2n+1)}$ is $T = 0$.

Proof. First, we prove the necessity. Let $\mathcal{A} \oplus X$ be approximately $(2n + 1)-$ideally amenable and $I \oplus Y$ be a closed ideal in $\mathcal{A} \oplus X$. Then every continuous derivation from $\mathcal{A} \oplus X$ into $(I \oplus Y)^{(2n+1)}$ is approximately inner. Let $D : \mathcal{A} \to I^{(2n+1)}$ be a continuous derivation. By Lemma 2.3, the derivation $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{D}((a, x)) = (D(a), 0)$ is approximately inner, thus D is so. That is, $\mathcal{A}$ is approximately $(2n + 1) - I-$weakly amenable. Therefore condition (i) holds.

Now, suppose that $D : \mathcal{A} \to Y^{(2n+1)}$ is a continuous derivation. Since derivation $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{D}((a, x)) = (0, D(a))$ is approximately inner. So by Lemma 2.2, D is approximately inner and consequently the condition (ii) is complete.

If $T : X \to I^{(2n+1)}$ is an arbitrary continuous $\mathcal{A}-$bimodule homomorphism then since $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{T}((a, x)) = (T(x), 0)$ is approximately inner, by Lemma 2.1, it follows that there exists a net $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that for each $a \in \mathcal{A}$, $\lim_\alpha(aF_\alpha - F_\alpha a) = 0$ and for every $x \in X$, we have $T(x) = \lim_\alpha(xF_\alpha - F_\alpha x)$. Thus, condition (iii) follows.

Finally, let $T : X \to Y^{(2n+1)}$ be a continuous $\mathcal{A}-$bimodule homomorphism satisfying $x_1T(x_2) + T(x_1)x_2 = 0$ in $I^{(2n+1)}$ for each $x_1, x_2 \in X$. Since derivation $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\bar{T}((a, x)) = (0, T(x))$ is approximately inner, thus by Lemma 2.4, we have $T = 0$ and this completes the proof of (iv).

Now, we prove the sufficiency. Let conditions (i)-(iv) hold and $I \oplus Y$ be a closed ideal in $\mathcal{A} \oplus X$. Also, let $D : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ be a continuous derivation. We show that D is approximately inner. For this, consider the following projection maps:

$$p_1 : (I \oplus Y)^{(2n+1)} \to I^{(2n+1)} \quad ; \quad p_2 : (I \oplus Y)^{(2n+1)} \to Y^{(2n+1)}.$$

Also, consider the inclusion maps $k_1 : \mathcal{A} \to \mathcal{A} \oplus X$ and $k_2 : X \to \mathcal{A} \oplus X$ by $k_1(a) = (a, 0)$ and $k_2(x) = (0, x)$, respectively. It is clear that p_1 and p_2 are $\mathcal{A}-$bimodule homomorphisms and k_1 is algebraic homomorphism. Since D is a continuous derivation, then $D \circ k_1 : \mathcal{A} \to (I \oplus Y)^{(2n+1)}$ is so. This implies that

$$p_1 \circ D \circ k_1 : \mathcal{A} \to I^{(2n+1)} \quad , \quad p_2 \circ D \circ k_1 : \mathcal{A} \to Y^{(2n+1)}$$

are continuous derivations. In this case, by conditions (i) and (ii), $p_1 \circ D \circ k_1$ and $p_2 \circ D \circ k_1$ are approximately inner. Therefore $D \circ k_1$ is approximately inner. by Lemmas 2.2, 2.3 and 2.4

$$\overline{D \circ k_1} = \overline{p_1 \circ D \circ k_1} + \overline{p_2 \circ D \circ k_1} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$$

is a continuous derivation. Thus there exists a net of continuous derivations $\tilde{D}_\alpha : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ such that for every α and for each $a \in \mathcal{A}$, $\tilde{D}_\alpha((a, 0)) = 0$ and $\overline{D \circ k_1} - \tilde{D}_\alpha$ is inner.

On the other hand, for each $a \in \mathcal{A}$ we have

$$\begin{aligned}(D - \overline{D \circ k_1})((a, 0)) &= D((a, 0)) - \overline{D \circ k_1}((a, 0)) \\ &= D \circ k_1(a) - D \circ k_1(a) = 0.\end{aligned}$$

Take $\hat{D}_\alpha = D - \overline{D \circ k_1} + \tilde{D}_\alpha$. Then $\hat{D}_\alpha : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ is a continuous derivation satisfying $\hat{D}_\alpha((a, 0)) = 0$ for each $a \in \mathcal{A}$.

Moreover, for every $a \in \mathcal{A}$ and $x \in X$ we have

$$\hat{D}_\alpha((0, ax)) = \hat{D}_\alpha((a, 0)(0, x)) = (a, 0)\hat{D}_\alpha((0, x)) = a\hat{D}_\alpha((0, x))$$

and

$$\hat{D}_\alpha((0, xa)) = \hat{D}_\alpha((0, x)(a, 0)) = \hat{D}_\alpha((0, x)(a, 0)) = \hat{D}_\alpha((0, x))a.$$

Then $\hat{D}_\alpha \circ k_2 : X \to (I \oplus Y)^{(2n+1)}$ is a continuous $\mathcal{A}-$bimodule homomorphism. By condition (iii), for each α there is net $(F^\alpha_\beta)_\beta \subseteq Y^{(2n+1)}$ such that for each $a \in \mathcal{A}$, $\lim_\beta(aF^\alpha_\beta - F^\alpha_\beta a) = 0$ and for all $x \in X$, $p_1 \circ \hat{D}_\alpha \circ k_2(x) = \lim_\beta(xF^\alpha_\beta - F^\alpha_\beta x)$.

Also, for every $x_1, x_2 \in X$ we can write

$$\begin{aligned}
([p_2 \circ \hat{D}_\alpha \circ k_2(x_1)]x_2 + x_1[p_2 \circ \hat{D}_\alpha \circ k_2(x_2)], 0) &= ([p_2 \circ \hat{D}_\alpha(0, x_1)]x_2, 0) \\
&+ (x_1[p_2 \circ \hat{D}_\alpha(0, x_2)], 0) \\
&= \hat{D}_\alpha((0, x_1))(0, x_2) + (0, x_1)\hat{D}_\alpha((0, x_2)) \\
&= \hat{D}_\alpha((0, x_1)(0, x_2)) + \hat{D}_\alpha((0, x_1)(0, x_2)) \\
&= \hat{D}_\alpha((0, 0)) + \hat{D}_\alpha((0, 0)) \\
&= (0, 0).
\end{aligned}$$

Consequently, for every $x_1, x_2 \in X$

$$[p_2 \circ \hat{D}_\alpha \circ k_2(x_1)]x_2 + x_1[p_2 \circ \hat{D}_\alpha \circ k_2(x_2)] = 0.$$

Therefore by the condition (iv), $p_2 \circ \hat{D}_\alpha \circ k_2 = 0$. Thus, one can write

$$\begin{aligned}
\hat{D}_\alpha((a, x)) = \hat{D}_\alpha((0, x)) &= \hat{D}_\alpha \circ k_2(x) \\
&= (p_1 \circ \hat{D}_\alpha \circ k_2(x), p_2 \circ \hat{D}_\alpha \circ k_2(x)) \\
&= \lim_\beta (xF^\alpha_\beta - F^\alpha_\beta x, 0) \\
&= \lim_\beta ((a, x) \cdot (0, F^\alpha_\beta) - (0, F^\alpha_\beta) \cdot (a, x)).
\end{aligned}$$

So, $\hat{D}_\alpha$ is approximately inner. By letting $D = \hat{D}_\alpha + (\overline{D \circ k_1} - \tilde{D}_\alpha)$, we easily observe that D is approximately inner. Hence $\mathcal{A} \oplus X$ is approximately $(2n+1)-$ideally amenable and proof is complete. ■

3. approximate $(2n)-$ideal amenability of $\mathcal{A} \oplus X$

Throughout this section, suppose that $n \in \mathbb{N}$. First, we prove some lemmas.

Lemma 3.1 Let $T : X \to I^{(2n)}$ be a continuous $\mathcal{A}-$bimodule homomorphism satisfying $x_1T(x_2) + T(x_1)x_2 = 0$ for every $x_1, x_2 \in X$. Then the mapping $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\bar{T}((a, x)) = (T(x), 0)$ is a continuous derivation. Moreover, $\bar{T}$ is approximately inner if and only if $T = 0$.

Proof. By Lamma 2.1, it is clear that $\bar{T}$ is a derivation. Let $\bar{T}$ be approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that for every $(a,x) \in \mathcal{A} \oplus X$,

$$\bar{T}((a,x)) = \lim_\alpha((a,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,x)).$$

Consequently,

$$\begin{aligned}(T(x),0) &= \lim_\alpha((aG_\alpha, aF_\alpha + xG_\alpha) - (G_\alpha a, G_\alpha x + F_\alpha a)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x).\end{aligned}$$

But, since $(T(x),0) = \bar{T}((0,x))$ so

$$\begin{aligned}(T(x),0) = \bar{T}((0,x)) &= \lim_\alpha((0,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(0,x)) \\ &= \lim_\alpha((0,xG_\alpha) - (0,G_\alpha x)) \\ &= \lim_\alpha((0,xG_\alpha - G_\alpha x)).\end{aligned}$$

Therefore, $T(x) = 0$ for each $x \in X$. The converse is clear. ■

Lemma 3.2 Let $D : \mathcal{A} \to Y^{(2n)}$ is a continuous derivation. Then the mapping $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\bar{D}((a,x)) = (0, D(a))$ is a continuous derivation. Moreover, $\bar{D}$ is approximately inner if and only if D is approximately inner.

Proof. It is clear that $\bar{D}$ is a derivation. Let $\bar{D}$ be approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that for each $(a,x) \in \mathcal{A} \oplus X$, we have

$$\begin{aligned}\bar{D}((a,x)) &= \lim_\alpha((a,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,x)) \\ &= \lim_\alpha((aG_\alpha, aF_\alpha + xG_\alpha) - (G_\alpha a, G_\alpha x + F_\alpha a)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x).\end{aligned}$$

But we know that

$$\begin{aligned}(0, D(a)) = \bar{D}((a,0)) &= \lim_\alpha((a,0)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,0)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a)\end{aligned}$$

and

$$\begin{aligned}(0,0) = \bar{D}((0,x)) &= \lim_\alpha((0,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(0,x)) \\ &= \lim_\alpha(0, xG_\alpha - G_\alpha x).\end{aligned}$$

Hence for some $(F_\alpha)_\alpha \subseteq Y^{(2n)}$, we have $D(a) = \lim_\alpha(aF_\alpha - F_\alpha a)$. So D is approximately inner.

Conversely, let D be approximately inner. Then there is net $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha(aF_\alpha - F_\alpha a)$. Then

$$\bar{D}((a,x)) = (0, D(a)) = (0, \lim_\alpha(aF_\alpha - F_\alpha a)) = \lim_\alpha((a,x)\cdot(0,F_\alpha) - (0,F_\alpha)\cdot(a,x)).$$

Take $(G_\alpha)_\alpha = (0, F_\alpha) \subseteq (I \oplus Y)^{(2n)}$. Then $\bar{D}((a,x)) = \lim_\alpha((a,x)\cdot G_\alpha - G_\alpha \cdot (a,x))$; i.e. $\bar{D}$ is approximately inner. ■

Lemma 3.3 Let $D : \mathcal{A} \to I^{(2n)}$ be a continuous derivation. Then the mapping $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\bar{D}((a,x)) = (D(a), 0)$ is a continuous derivation. Moreover, $\bar{D}$ is approximately inner if and only if D is approximately inner.

Proof. The proof is similar to that of Lemma 2.3. ■

Lemma 3.4 Let $T : X \to Y^{(2n)}$ be a continuous $\mathcal{A}$−bimodule homomorphism. Then the mapping $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\bar{T}((a,x)) = (0, T(x))$ is a continuous derivation. Moreover, $\bar{T}$ is approximately inner if and only if there exists net $(G_\alpha)_\alpha \subseteq I^{(2n)}$ such that for every $a \in \mathcal{A}$, $\lim_\alpha(aG_\alpha - G_\alpha a) = 0$ and for each $x \in X$, $T(x) = \lim_\alpha(xG_\alpha - G_\alpha x)$.

Proof. First, we show that $\bar{T}$ is a derivation. Let $(a_1,x_1),(a_2,x_2) \in \mathcal{A} \oplus X$ be two arbitrary elements. We have

$$\begin{aligned}\bar{T}((a_1,x_1)\cdot(a_2,x_2)) &= \bar{T}((a_1a_2, a_1x_2 + x_1a_2)) \\ &= (0, T(a_1x_2 + x_1a_2)) \\ &= (0, a_1T(x_2) + T(x_1)a_2).\end{aligned}$$

On the other hand,

$$\bar{T}((a_1,x_1))\cdot(a_2,x_2) = (0, T(x_1))(a_2,x_2) = (0, T(x_1)a_2)$$

and

$$(a_1,x_1)\cdot\bar{T}((a_2,x_2)) = (a_1,x_1)(0,T(x_2)) = (0, a_1T(x_2)).$$

This shows that $\tilde{T}$ is a derivation.

Suppose that $\bar{T}$ is approximately inner. Then there exist nets $(G_\alpha)_\alpha \subseteq I^{(2n)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that for every $(a,x) \in \mathcal{A} \oplus X$, we have

$$\begin{aligned}\bar{T}((a,x)) &= \lim_\alpha((a,x)\cdot(G_\alpha,F_\alpha) - (G_\alpha,F_\alpha)\cdot(a,x)) \\ &= \lim_\alpha(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x).\end{aligned}$$

But

$$(0, T(x)) = \bar{T}((0,x)) = \lim_\alpha(0, xG_\alpha - G_\alpha x)$$

and

$$(0,0) = \bar{T}((a,0)) = \lim_{\alpha}(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).$$

This follows that for every $a \in \mathcal{A}$, $\lim_\alpha(aG_\alpha - G_\alpha a) = 0$ and for every $x \in X$, $T(x) = \lim_\alpha(xG_\alpha - G_\alpha x)$.

Conversely, suppose that there exists such net $(G_\alpha)_\alpha \subseteq I^{(2n)}$ satisfying $\lim_\alpha(aG_\alpha - G_\alpha a) = 0$ and $T(x) = \lim_\alpha(xG_\alpha - G_\alpha x)$. Then we have

$$\begin{aligned}\bar{T}((a,x)) = (0, T(x)) &= \lim_{\alpha}(aG_\alpha - G_\alpha a, xG_\alpha - G_\alpha x)\\ &= \lim_{\alpha}((a,x)\cdot(G_\alpha,0) - (G_\alpha,0)\cdot(a,x)).\end{aligned}$$

By letting $(u_\alpha)_\alpha = (G_\alpha, 0) \subseteq (I \oplus Y)^{(2n)}$, it follows that

$$\bar{T}((a,x)) = \lim_{\alpha}((a,x)\cdot u_\alpha - u_\alpha \cdot (a,x));$$

i.e. $\bar{T}$ is approximately inner. ∎

Now, we can find the necessary and sufficient conditions for module extension Banach algebra $\mathcal{A} \oplus X$ to be approximately $(2n)-$ideally amenable.

Theorem 3.5 Let $\mathcal{A} \oplus X$ be a module extension Banach algebra and $I \oplus Y$ be a closed ideal in $\mathcal{A} \oplus X$. Then $\mathcal{A} \oplus X$ is approximately $(2n)-$ideally amenable if and only if the following conditions hold:

(i) The only continuous derivations $D : \mathcal{A} \to I^{(2n)}$ for which there is a continuous operator $T : X \to Y^{(2n)}$ such that $T(ax) = D(a)x + aT(x)$ and $T(xa) = xD(a) + T(x)a$ $(a \in \mathcal{A}, x \in X)$ are approximately inner derivations;

(ii) Every continuous derivation from $\mathcal{A}$ into $Y^{(2n)}$ is approximately inner;

(iii) The only continuous $\mathcal{A}-$bimodule homomorphism $T : X \to I^{(2n)}$ for which $x_1T(x_2) + T(x_1)x_2 = 0$ $(x_1, x_2 \in X)$ in $Y^{(2n)}$ is zero;

(iv) For every continuous $\mathcal{A}-$bimodule homomorphism $T : X \to Y^{(2n)}$, there is net $(G_\alpha)_\alpha \subseteq I^{(2n)}$ such that for each $a \in \mathcal{A}$, $\lim_\alpha(aG_\alpha - G_\alpha a) = 0$ and for every $x \in X$, $T(x) = \lim_\alpha(xG_\alpha - G_\alpha x)$.

Proof. First, we prove the necessity. Let $\mathcal{A}\oplus X$ be approximately $(2n)-$ideally amenable and $I \oplus Y$ be a closed ideal of it. Then every continuous derivation from $\mathcal{A} \oplus X$ into $(I\oplus Y)^{(2n)}$ is approximately inner. Let $D : \mathcal{A} \to I^{(2n)}$ be a continuous derivation including the properties mentioned in the condition (i). We define $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ by

$$\bar{D}((a,x)) = (D(a), T(x))(a \in \mathcal{A}, x \in X).$$

Clearly, $\bar{D}$ is a continuous derivation. Also, $\bar{D}$ is approximately inner. Thus there are nets $(G_\alpha)_\alpha \subseteq I^{(2n)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that

$$\bar{D}((a,x)) = \lim_{\alpha}((a,x)\cdot(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha)\cdot(a,x)).$$

Consequently

$$\begin{aligned}(D(a),T(x)) &= \lim_{\alpha}((aG_\alpha, aF_\alpha + xG_\alpha) - (G_\alpha a, G_\alpha x + F_\alpha a))\\ &= \lim_{\alpha}(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x).\end{aligned}$$

Therefore $D(a) = \lim_\alpha (aG_\alpha - G_\alpha a)$ where $(G_\alpha)_\alpha \subseteq I^{(2n)}$. So D is approximately inner and the condition (i) holds.

Let $D : \mathcal{A} \to Y^{(2n)}$ be a continuous derivation. Because the continuous derivation $\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\bar{D}((a,x)) = (0, D(a))$ is approximately inner, so by Lemma 3.2, D is approximately inner and the condition (ii) is proved.

Now, let $T : X \to I^{(2n)}$ be a continuous $\mathcal{A}-$bimodule homomorphism satisfying $x_1T(x_2)+T(x_1)x_2 = 0$ $(x_1, x_2 \in X)$ in $Y^{(2n)}$. Since the mapping $\bar{T} : \mathcal{A}\oplus X \to (I\oplus Y)^{(2n)}$ defined by $\bar{T}((a,x)) = (T(x), 0)$ is approximately inner so, by Lemma 3.1 we have $T = 0$ and the condition (iii) is completed.

Finally, let $T : X \to Y^{(2n)}$ be a continuous $\mathcal{A}-$bimodule homomorphism. Since $\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\bar{T}((a,x)) = (0, T(x))$ is approximately inner, thus by Lemma 3.4, there is net $(G_\alpha)_\alpha \subseteq I^{(2n)}$ such that for every $a \in \mathcal{A}$, $\lim_\alpha(aG_\alpha - G_\alpha a) = 0$ and for each $x \in X$, $T(x) = \lim_\alpha(xG_\alpha - G_\alpha x)$. Hence condition (iv) is proved.

Now, for proving the sufficiency we assume that the conditions (i)-(iv) hold and that $I \oplus Y$ is an arbitrary closed ideal in $\mathcal{A} \oplus X$. Also, let $D : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ be a continuous derivation. We show that D is approximately inner. For this, consider the following projection maps:

$$p_1 : (I \oplus Y)^{(2n)} \to I^{(2n)} \quad ; \quad p_2 : (I \oplus Y)^{(2n)} \to Y^{(2n)}.$$

Also, consider the following inclusion maps:

$$k_1 : \mathcal{A} \to \mathcal{A} \oplus X \quad ; \quad k_2 : X \to \mathcal{A} \oplus X.$$

Clearly, p_1 and p_2 are $\mathcal{A}-$bimodule homomorphisms. Since D is a continuous derivation, thus $D \circ k_1 : \mathcal{A} \to (I \oplus Y)^{(2n)}$ is so. On the other hand,

$$p_1 \circ D \circ k_1 : \mathcal{A} \to I^{(2n)} \quad , \quad p_2 \circ D \circ k_1 : \mathcal{A} \to Y^{(2n)}$$

are continuous derivations.

Claim 1: $p_1 \circ D \circ k_2 : X \to I^{(2n)}$ is trivial.

Take $\Delta := p_1 \circ D \circ k_2$. To prove claim 1, it is sufficient to show that Δ is a continuous $\mathcal{A}-$bimodule homomorphism satisfying $x_1\Delta(x_2) + \Delta(x_1)x_2 = 0$ for every $x_1, x_2 \in X$ by condition (iii). We have

$$\begin{aligned}\Delta(ax) = p_1 \circ D \circ k_2(ax) &= p_1 \circ D((0, ax))\\ &= p_1 \circ D((a,0)(0,x))\\ &= p_1(D((a,0))(0,x) + (a,0)D((0,x)))\\ &= p_1((a,0)D((0,x)))\\ &= p_1(aD \circ k_2(x))\\ &= a(p_1 \circ D \circ k_2)(x)\\ &= a\Delta(x).\end{aligned}$$

Similarly, $\Delta(xa) = \Delta(x)a$. So $\Delta = p_1 \circ D \circ k_2$ is $\mathcal{A}-$bimodule homomorphism. Also, we have

$$\begin{aligned} 0 = D((0,0)) &= D((0,x_1)(0,x_2)) \\ &= D((0,x_1)(0,x_2) + (0,x_1)D((0,x_2)) \\ &= (0,\Delta(x_1)x_2) + (0,x_1\Delta(x_2)) \\ &= (0,x_1\Delta(x_2) + \Delta(x_1)x_2). \end{aligned}$$

Therefore claim 1 holds. Now, we take $T := p_2 \circ D \circ k_2 : X \to Y^{(2n)}$ and $D_1 := p_1 \circ D \circ k_1 : \mathcal{A} \to I^{(2n)}$.

Claim 2: $T(ax) = D_1(a)x + aT(x)$ and $T(xa) = xD_1(a) + T(x)a$ for every $a \in \mathcal{A}, x \in X$.

To prove the above claim, we have

$$\begin{aligned} (0,T(ax)) &= (0, p_2 \circ D \circ k_2(ax)) \\ &= (0, p_2 \circ D((0,ax))) \\ &= D((0,ax)) \\ &= D((a,0)(0,x)) \\ &= D((a,0))(0,x) + (a,0)D((0,x)) \\ &= (0, D_1(a)x) + a(0,T(x)) \\ &= (0, D_1(a)x + aT(x)). \end{aligned}$$

Similarly, for each $a \in \mathcal{A}$ and $x \in X$ we have

$$(0,T(xa)) = (0, xD_1(a) + T(x)a).$$

Hence the claim 2 holds. Consequently, derivation $D_1 = p_1 \circ D \circ k_1$ is approximately inner by condition (i).

Now, let there exists net $(G_\alpha)_\alpha \subseteq I^{(2n)}$ such that for every $a \in \mathcal{A}$,

$$D_1(a) = \lim_\alpha (aG_\alpha - G_\alpha a).$$

Also, let $T_1 : X \to Y^{(2n)}$ be defined by $T_1(x) = \lim_\alpha (xG_\alpha - G_\alpha x)$ for each $x \in X$. Then by claim 2, for $T - T_1 : X \to Y^{(2n)}$ we have

$$\begin{aligned} (T - T_1)(ax) &= T(ax) - T_1(ax) \\ &= (D_1(a)x + aT(x)) - \lim_\alpha (axG_\alpha - G_\alpha ax) \\ &= \lim_\alpha (aG_\alpha - G_\alpha a)x + aT(x) = \lim_\alpha (axG_\alpha - G_\alpha ax) \\ &= a \lim_\alpha (G_\alpha x - xG_\alpha) + aT(x) \\ &= a(T - T_1)(x) \end{aligned}$$

where $a \in \mathcal{A}$ and $x \in X$. Similarly, $(T - T_1)(xa) = (T - T_1)(x)a$. Therefore $T - T_1$ is a continuous $\mathcal{A}-$bimodule homomorphism. Now, by condition (iv), there is net $(v_\beta)_\beta \subseteq I^{(2n)}$ such that for each $a \in \mathcal{A}$, $\lim_\beta(av_\beta - v_\beta a) = 0$ and for every $x \in X$, $(T - T_1)(x) = \lim_\beta(xv_\beta - v_\beta x)$. From Lemma 3.4, we know that $\overline{T - T_1} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by

$$\overline{T - T_1}((a, x)) = (0, (T - T_1)(x))$$

is approximately inner derivation. Since $p_2 \circ D \circ k_1 : \mathcal{A} \to Y^{(2n)}$ is a continuous derivation, so by the condition (ii), it is approximately inner. On the other hand, by Lemma 3.2, the mapping $\overline{p_2 \circ D \circ k_1} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}$ defined by

$$\overline{p_2 \circ D \circ k_1}((a, x)) = (0, p_2 \circ D \circ k_1(a))$$

is approximately inner derivation. Now, by using claim 1, we have

$$\begin{aligned} D((a, x)) &= (D_1(a), p_2 \circ D \circ k_1(a) + T(x)) \\ &= \overline{p_2 \circ D \circ k_1}((a, x)) + (\overline{T - T_1})((a, x)) + (D_1(a), T(x)). \end{aligned}$$

Since every three summands are approximately inner derivations, so D is approximately inner derivation from $\mathcal{A} \oplus X$ into $(I \oplus Y)^{(2n)}$. Consequently, $\mathcal{A} \oplus X$ is approximately $(2n)-$ideally amenable. ■

***Example* 3.6** Let $\mathcal{A}^\sharp =: \mathcal{A} \oplus \mathbb{C}$ be the unitization of a Banach algebra $\mathcal{A}$ and $n \in \mathbb{N}$. In this case, we have:

(i) if $\mathcal{A}^\sharp$ is approximately $n-$ideally amenable, then $\mathcal{A}$ is approximately $n-$ideally amenable.
(ii) if $\mathcal{A}$ is approximately $(2n-1)-$ideally amenable, then $\mathcal{A}^\sharp$ is approximately $(2n-1)-$ideally amenable.

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