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Common fixed point results for graph preserving mappings in parametric N_b-metric spaces

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Abstract. In this paper, we discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of graph preserving mappings in parametric N_b-metric spaces. As some consequences of this study, we obtain several important results in parametric b-metric spaces, parametric S-metric spaces and parametric A-metric spaces. Finally, we provide some illustrative examples to justify the validity of our main result.

Keywords: Parametric N_b-metric, digraph, coincidence point, fixed point.

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1. Introduction

Fixed point theory is an important branch of nonlinear analysis which can be applied to many areas of mathematics and applied sciences such as variational and linear inequalities, control theory, convex optimization, linear algebra, differential equations and mathematical economics. The most celebrated result in this field is the Banach contraction principle [6]. It becomes very famous due to its wide applications. In particular, it is an important tool for solving existence and uniqueness problems in nonlinear functional analysis. Several authors successfully generalized this result in many directions. In last three decades, different types of generalized metric spaces have been developed by different mathematicians. One such generalized metric space is a parametric metric space

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introduced and studied by Hussain et al. [16]. Some other generalized metric spaces are b-metric space [5], parametric b-metric space [16], parametric S-metric space [35] etc.

In 2012, Sedghi et al. [34] introduced the notion of S-metric space. Afterwards, the definition of S-metric is generated by extending to n-tuple by Abbas et al. [1] and called it A-metric. Recently, Priyobarta et al. [31] introduced the concept of parametric A-metric space as a generalization of A-metric space. Very recently, Nihal et al. [36] extended the concept of parametric A-metric space to parametric N_b-metric space and studied some fixed point results. After examining the proofs of the results in [36], we noticed that there is something wrong with the proof of the Cauchy sequence in Theorem 3.1 [36]. This leads to subsequent errors in Theorems 4.1 and 5.1 [36]. The detailed reasons are as follows: On page number 950 in [36], the authors used

$$(n-1)ba^k\,[1+b^2a+b^4a^2+\cdots]\,N_{u_0,u_1,t} \leqslant (n-1)\frac{ba^k}{1-b^2a}\,N_{u_0,u_1,t}.$$

This is incorrect unless $b^2a < 1$. In this paper, we would like to modify the contractive type condition to achieve their claim (see Corollary 3.4).

In recent investigations, the study of fixed point theory combining a graph is a new development in the domain of contractive type single valued and multi valued theory. In 2005, Echenique [13] studied fixed point theory by using graphs. Later on, Espinola and Kirk [14] applied fixed point results in graph theory. Afterwards, combining fixed point theory and graph theory, a series of articles (see [3, 4, 8, 9, 18, 23–27] and references therein) have been dedicated to the improvement of fixed point theory. Many important results of [1, 11, 21, 22, 28–33, 36] have become the source of motivation for many researchers that do research in fixed point theory. The main purpose of this article is to investigate the existence and uniqueness of points of coincidence and common fixed points for a pair of mappings under various contractive conditions in parametric N_b-metric spaces. Further, we prove some fixed point theorems for expansive mappings in parametric b-metric space and parametric S-metric space.

2. Some Basic Concepts

We begin with some basic notations, definitions and results which will be used in the sequel.

Definition 2.1 [34] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each x, y, z, $a \in X$,

(i) $S(x,y,z) \geqslant 0$,
(ii) $S(x,y,z) = 0$ if and only if $x = y = z$,
(iii) $S(x,y,z) \leqslant S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

It is to be noted that an S-metric is not symmetric, in general. The following examples illustrate the above fact.

***Example* 2.2** [34] Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X. Then

$$S(x,y,z) = \| y+z-2x \| + \| y-z \|$$

is an S-metric on X.

***Example* 2.3** [34] Let X be a nonempty set and d be an ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Definition 2.4 [16] Let X be a nonempty set and $P : X \times X \times (0,\infty) \to [0,\infty)$ be a function. Then P is called a parametric metric on X if

(i) $P(x,y,t) = 0$ if and only if $x = y$,
(ii) $P(x,y,t) = P(y,x,t)$,
(iii) $P(x,y,t) \leqslant P(x,z,t) + P(z,y,t)$

for each x, y, $z \in X$ and all $t > 0$. The pair (X,P) is called a parametric metric space.

***Example* 2.5** [16] Let X denote the set of all functions $f : (0,\infty) \to \mathbb{R}$. Define $P : X \times X \times (0,\infty) \to [0,\infty)$ by $P(f,g,t) =\mid f(t) - g(t) \mid$ for all f, $g \in X$ and all $t > 0$. Then (X,P) is a parametric metric space.

Definition 2.6 [17] Let X be a nonempty set, $b \geqslant 1$ be a real number, and $P : X \times X \times (0,\infty) \to [0,\infty)$ be a map satisfying the following conditions:

(i) $P(x,y,t) = 0$ if and only if $x = y$,
(ii) $P(x,y,t) = P(y,x,t)$,
(iii) $P(x,y,t) \leqslant b[P(x,z,t) + P(z,y,t)]$

for each x, y, $z \in X$ and all $t > 0$. Then P is called a parametric b-metric on X and the pair (X,P) is called a parametric b-metric space.

Definition 2.7 [35] Let X be a nonempty set and $P_S : X \times X \times X \times (0,\infty) \to [0,\infty)$ be a function. P_S is called a parametric S-metric on X if

(PS1) $P_S(x,y,z,t) = 0$ if and only if $x = y = z$,
(PS2) $P_S(x,y,z,t) \leqslant P_S(x,x,a,t) + P_S(y,y,a,t) + P_S(z,z,a,t)$

for each x, y, z, $a \in X$ and all $t > 0$. The pair (X,P_S) is called a parametric S-metric space.

***Example* 2.8** [35] Let $X = \{f \mid f : (0,\infty) \to \mathbb{R} \textit{ be a function}\}$ and the function $P_S : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by

$$P_S(f,g,h,t) =\mid f(t) - h(t) \mid + \mid g(t) - h(t) \mid$$

for each f, g, $h \in X$ and all $t > 0$. Then P_S is a parametric S-metric and the pair (X,P_S) is a parametric S-metric space.

Lemma 2.9 [35] Let (X,P_S) be a parametric S-metric space. Then we have

$$P_S(x,x,y,t) = P_S(y,y,x,t)$$

for each x, $y \in X$ and all $t > 0$.

Definition 2.10 [1] Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if for any x_i, $a \in X$, $i = 1, 2 \cdots, n$, the following conditions hold:

(A1) $A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) \geqslant 0$,
(A2) $A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_n$,

(A3) $$A(x_1, x_2, x_3, \cdots, x_{n-1}, x_n) \leqslant \left\{ \begin{array}{l} A(x_1, x_1, x_1, \cdots, (x_1)_{n-1}, a) \\ \\ +A(x_2, x_2, x_2, \cdots, (x_2)_{n-1}, a) \\ \\ + \cdots + A(x_n, x_n, x_n, \cdots, (x_n)_{n-1}, a) \end{array} \right\}.$$

The pair (X, A) is called an A-metric space.

Definition 2.11 [31] Let X be a nonempty set and $P_A : X^n \times (0, \infty) \to [0, \infty)$ be a function. P_A is called a parametric A-metric on X if,

(PA1) $P_A(x_1, x_2, \cdots, x_n, t) = 0$ if and only if $x_1 = x_2 = \cdots = x_n$,

(PA2) $$P_A(x_1, x_2, \cdots, x_n, t) \leqslant \left\{ \begin{array}{l} P_A(x_1, x_1, \cdots, (x_1)_{n-1}, a, t) \\ \\ +P_A(x_2, x_2, \cdots, (x_2)_{n-1}, a, t) \\ \\ + \cdots + P_A(x_n, x_n, \cdots, (x_n)_{n-1}, a, t) \end{array} \right\}$$

for each $x_i,\ a \in X,\ i = 1, 2, 3, \cdots, n$ and all $t > 0$. The pair (X, P_A) is called a parametric A-metric space.

***Example* 2.12** [31] Let $X = \mathbb{R}$ and let the function $P_A : X^n \times (0, \infty) \to [0, \infty)$ be defined by

$$P_A(x_1, x_2, \cdots, x_n, t) = g(t) \left(| x_1 - x_2 | + | x_2 - x_3 | + \cdots + | x_n - x_1 | \right),$$

for each $x_1,\ x_2, \cdots, x_n \in X$ and all $t > 0$, where $g : (0, \infty) \to (0, \infty)$ is a continuous function. Then P_A is a parametric A-metric and the pair (X, P_A) is a parametric A-metric space.

Lemma 2.13 [31] Let (X, P_A) be a parametric A-metric space. Then we have

$$P_A(x, x, \cdots, x, y, t) = P_A(y, y, \cdots, y, x, t),$$

for each $x,\ y \in X$ and all $t > 0$.

Definition 2.14 [36] Let X be a nonempty set, $b \geqslant 1$ be a given real number, $n(\geqslant 2) \in \mathbb{N}$ and $N : X^n \times (0, \infty) \to [0, \infty)$ be a function. N is called a parametric N_b-metric on X if,

(N1) $N(x_1, x_2, \cdots, x_n, t) = 0$ if and only if $x_1 = x_2 = \cdots = x_n$,

(N2) $$N(x_1, x_2, \cdots, x_n, t) \leqslant b \left\{ \begin{array}{l} N(x_1, x_1, \cdots, (x_1)_{n-1}, a, t) \\ \\ +N(x_2, x_2, \cdots, (x_2)_{n-1}, a, t) \\ \\ + \cdots + N(x_n, x_n, \cdots, (x_n)_{n-1}, a, t) \end{array} \right\}$$

for each $x_i,\ a \in X,\ i = 1, 2, 3, \cdots, n$ and all $t > 0$. The pair (X, N) is called a parametric N_b-metric space. If $n = 3$, then N is called a parametric S_b-metric on X and the pair (X, N) is called a parametric S_b-metric space.

Throughout the paper, we will denote $N(x, x, \cdots, (x)_{n-1}, y, t)$ by $N_{x,y,t}$.

***Example* 2.15** Let $X = \{f \mid f : (0, \infty) \to \mathbb{R}\ \textit{be a function}\}$ and let the function

$N : X^3 \times (0, \infty) \to [0, \infty)$ be defined by

$$N(f, g, h, t) = (| f(t) - g(t) | + | f(t) - h(t) | + | g(t) - h(t) |)^2,$$

for each f, g, $h \in X$ and all $t > 0$. Then (X, N) is a parametric N_b-metric space with $b = 3$ and $n = 3$. Because,

$$\begin{aligned} N(f, g, h, t) &= (| f(t) - g(t) | + | f(t) - h(t) | + | g(t) - h(t) |)^2 \\ &\leqslant 4\,(| f(t) - \alpha(t) | + | g(t) - \alpha(t) | + | h(t) - \alpha(t) |)^2 \\ &\leqslant 12\,(| f(t) - \alpha(t) |^2 + | g(t) - \alpha(t) |^2 + | h(t) - \alpha(t) |^2) \\ &= 3\,(N_{f,\alpha,t} + N_{g,\alpha,t} + N_{h,\alpha,t}) \end{aligned}$$

for each f, g, h, $\alpha \in X$ and all $t > 0$. But it is not a parametric S-metric space. In fact, $(PS2)$ does not hold for $f(t) = 4$, $g(t) = 6$, $h(t) = 8$, $\alpha(t) = 5$.

Lemma 2.16 [36] Let (X, N) be a parametric N_b-metric space. Then we have $N_{x,y,t} \leqslant b\,N_{y,x,t}$ and $N_{y,x,t} \leqslant b\,N_{x,y,t}$ for each x, $y \in X$ and all $t > 0$.

Lemma 2.17 [36] Let (X, N) be a parametric N_b-metric space. Then we have

$$N_{x,y,t} \leqslant b\,[(n-1)N_{x,z,t} + N_{y,z,t}] \text{ and } N_{x,y,t} \leqslant b\,[(n-1)N_{x,z,t} + bN_{z,y,t}]$$

for each x, y, $z \in X$ and all $t > 0$.

Definition 2.18 [36] Let (X, N) be a parametric N_b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $N_{x_n,x,t} < \epsilon$, for all $n \geqslant n_0$ and all $t > 0$, that is, $\lim\limits_{n\to\infty} N_{x_n,x,t} = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.

(ii) (x_n) is called a Cauchy sequence if and only if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $N_{x_n,x_m,t} < \epsilon$ for all $n, m \geqslant n_0$ and all $t > 0$, that is, $\lim\limits_{n,m\to\infty} N_{x_n,x_m,t} = 0$.

(iii) (X, N) is called complete if and only if every Cauchy sequence in X is convergent.

Remark 1 *[36] In a parametric N_b-metric space (X, N), the following assertions hold:*

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

Lemma 2.19 [36] Let (X, N) be a parametric N_b-metric space and (u_k), (v_k) be two sequences converge to u and v, respectively. Then we have

$$\frac{1}{b^2} N_{u,v,t} \leqslant \liminf_{k\to\infty} N_{u_k,v_k,t} \leqslant \limsup_{k\to\infty} N_{u_k,v_k,t} \leqslant b^2 N_{u,v,t}$$

for all $t > 0$. In particular, if (v_k) is a constant sequence such that $v_k = v$ for all k, then we get

$$\frac{1}{b^2} N_{u,v,t} \leqslant \liminf_{k\to\infty} N_{u_k,v,t} \leqslant \limsup_{k\to\infty} N_{u_k,v,t} \leqslant b^2 N_{u,v,t}$$

for all $t > 0$. Also if $u = v$, then we have $\lim_{k\to\infty} N_{u_k,v,t} = 0$ for all $t > 0$.

Definition 2.20 [2] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.21 [19] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, $T(Sx) = S(Tx)$ whenever $Sx = Tx$.

Proposition 2.22 [2] Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.

Definition 2.23 Let (X, N) be a parametric N_b-metric space. A mapping $f : X \to X$ is called expansive if there exists a positive number $k > b^3$ such that $N_{fx,fy,t} \geqslant k\, N_{x,y,t}$ for all $x, y \in X$ and all $t > 0$.

Let (X, N) be a parametric N_b-metric space and ρ be a binary relation over X. Denote $R = \rho \cup \rho^{-1}$. Then xRy if and only if $x\rho y$ or $y\rho x$ for all $x, y \in X$.

Definition 2.24 We say that (X, N, R) is regular if the following condition holds:

If the sequence (x_k) in X and the point $x \in X$ are such that $x_k R x_{k+1}$ for all $k \geqslant 1$ and $\lim_{k\to\infty} N_{x_k,x,t} = 0$, then there exists a subsequence (x_{k_i}) of (x_k) such that $x_{k_i} R x$ for all $i \geqslant 1$.

Definition 2.25 Let (X, N) be a parametric N_b-metric space and ρ be a binary relation over X. Then the mapping $T : X \to X$ is called comparative if T maps comparable elements into comparable elements, that is,

$$x, y \in X,\ xRy \Rightarrow Tx\, R\, Ty.$$

Similarly, for $f,\, g : X \to X$, we call f is comparative w.r.t. g if

$$x, y \in X,\ gxRgy \Rightarrow fx\, R\, fy.$$

We next review some basic notions in graph theory.

Let (X, N) be a parametric N_b-metric space. We consider a directed graph G such that the set of its vertices $V(G) = X$ and the set of its edges $E(G)$ contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the directions of the edges of G. Therefore, we consider G as a directed graph which is symmetric. Thus, $E(\tilde{G}) = E(G) \cup E(G^{-1})$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 12, 15].

Definition 2.26 Let (X, N) be a parametric N_b-metric space and $G = (V(G), E(G))$ be a graph. Then the mapping $T : X \to X$ is called edge preserving if

$$x, y \in X,\ (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G).$$

Definition 2.27 Let (X, N) be a parametric N_b-metric space endowed with a digraph $G = (V(G), E(G))$ and $f,\, g : X \to X$ be two mappings. Then f is called edge preserving w.r.t. g if , $(gx, gy) \in E(G)$ implies that $(fx, fy) \in E(G)$ for all $x, y \in X$.

Definition 2.28 Let (X, N) be a parametric N_b-metric space and $f : X \to X$ be a mapping. Then f is called continuous if given $x \in X$ and a sequence $(x_k)_{k\in\mathbb{N}}$, $x_k \to x$ implies $fx_k \to fx$.

Definition 2.29 Let (X, N) be a parametric N_b-metric space and f, $g : X \to X$ be two mappings. Then f is called continuous w.r.t. g if given $x \in X$ and a sequence $(gx_k)_{k\in\mathbb{N}}$, $gx_k \to gx$ implies $fx_k \to fx$.

Definition 2.30 Let (X, N) be a parametric N_b-metric space endowed with a graph $G = (V(G), E(G))$. A mapping $f : X \to X$ is called G-continuous if given $x \in X$ and a sequence $(x_k)_{k\in\mathbb{N}}$, $x_k \to x$ and $(x_k, x_{k+1}) \in E(G)$ for $k \in \mathbb{N}$ imply $fx_k \to fx$.

Definition 2.31 Let (X, N) be a parametric N_b-metric space endowed with a graph $G = (V(G), E(G))$ and let f, $g : X \to X$ be two mappings. Then f is called G-continuous w.r.t. g if given $x \in X$ and a sequence $(gx_k)_{k\in\mathbb{N}}$, $gx_k \to gx$ and $(gx_k, gx_{k+1}) \in E(G)$ for $k \in \mathbb{N}$ imply $fx_k \to fx$.

3. Fixed Points in Parametric N_b-Metric Space

In this section, we assume that (X, N) is a parametric N_b-metric space and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges. Let the mappings f, $g : X \to X$ be such that $f(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $gx_1 = fx_0$. Continuing in this way, we can construct a sequence (gx_k) such that $gx_k = fx_{k-1},\ k = 1, 2, 3, \cdots$.

Before presenting our main result, we state a property of the graph G, call it property $(*)$.

Property $(*)$**:** If (gx_k) is a sequence in X such that $gx_k \to x$ and $(gx_k, gx_{k+1}) \in E(G)$ for all $k \geqslant 1$, then there exists a subsequence (gx_{k_i}) of (gx_k) such that $(gx_{k_i}, x) \in E(G)$ for all $i \geqslant 1$.

Taking $g = I$, the above property reduces to property $(*')$:

Property $(*')$**:** If (x_k) is a sequence in X such that $x_k \to x$ and $(x_k, x_{k+1}) \in E(G)$ for all $k \geqslant 1$, then there exists a subsequence (x_{k_i}) of (x_k) such that $(x_{k_i}, x) \in E(G)$ for all $i \geqslant 1$.

Theorem 3.1 Let (X, N) be a parametric N_b-metric space endowed with a digraph G and let the mappings f, $g : X \to X$ be such that

$$N_{fx,fy,t} \leqslant h\, max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{2(n-1)b}\} \tag{1}$$

for all x, $y \in X$ with $(gx, gy) \in E(\tilde{G})$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Suppose that f is edge preserving w.r.t. g, $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X. Assume that at least one of the following conditions holds:

(i) f is G-continuous w.r.t. g.
(ii) The graph G has the property $(*)$.

If there exists $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$, then f and g have a point of coincidence in $g(X)$. Moreover, f and g have a unique point of coincidence in $g(X)$ if the graph G has the following property:

(**) If x, y are points of coincidence of f and g in $g(X)$, then $(x, y) \in E(\tilde{G})$.
Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. Suppose there exists $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$. Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 = fx_0$. So, $(gx_0, gx_1) \in E(G)$. As f is edge preserving w.r.t. g, it follows that $(fx_0, fx_1) \in E(G)$, that is, $(gx_1, fx_1) \in E(G)$. Continuing this process, we can construct a sequence (x_k) in X such that $gx_k = fx_{k-1}$, for all $k \geqslant 1$ and $(gx_k, gx_{k+1}) \in E(G)$ for all $k \in \mathbb{N} \cup \{0\}$.

We assume that $fx_k \neq fx_{k-1}$ for every $k \in \mathbb{N}$. If $fx_k = fx_{k-1}$ for some $k \in \mathbb{N}$, then $gx_k = fx_{k-1} = fx_k$ which shows that fx_{k-1} is a point of coincidence of f and g in $g(X)$. By using condition (1) and Lemmas 2.16 and 2.17, we have

$$
\begin{aligned}
N_{fx_k,fx_{k+1},t} &\leqslant h\,max\begin{Bmatrix} N_{gx_k,gx_{k+1},t}, N_{fx_k,gx_k,t}, N_{fx_{k+1},gx_{k+1},t}, \\ \frac{N_{gx_k,fx_{k+1},t}+N_{gx_{k+1},fx_k,t}}{2(n-1)b} \end{Bmatrix} \\
&\leqslant h\,max\begin{Bmatrix} N_{fx_{k-1},fx_k,t}, N_{fx_k,fx_{k-1},t}, N_{fx_{k+1},fx_k,t}, \\ \frac{b\,N_{fx_{k+1},fx_{k-1},t}+N_{fx_k,fx_k,t}}{2(n-1)b} \end{Bmatrix} \\
&\leqslant h\,max\begin{Bmatrix} N_{fx_{k-1},fx_k,t}, b\,N_{fx_{k-1},fx_k,t}, \\ b\,N_{fx_k,fx_{k+1},t}, \frac{N_{fx_{k+1},fx_{k-1},t}}{2(n-1)} \end{Bmatrix} \\
&\leqslant h\,max\begin{Bmatrix} b\,N_{fx_{k-1},fx_k,t}, b\,N_{fx_k,fx_{k+1},t}, \\ \frac{(n-1)b\,N_{fx_{k+1},fx_k,t}+b\,N_{fx_{k-1},fx_k,t}}{2(n-1)} \end{Bmatrix} \\
&\leqslant h\,max\begin{Bmatrix} b\,N_{fx_{k-1},fx_k,t}, b^2\,N_{fx_k,fx_{k+1},t}, \\ \frac{(n-1)b^2\,N_{fx_k,fx_{k+1},t}+b(n-1)\,N_{fx_{k-1},fx_k,t}}{2(n-1)} \end{Bmatrix} \\
&= h\,max\begin{Bmatrix} b\,N_{fx_{k-1},fx_k,t}, b^2\,N_{fx_k,fx_{k+1},t}, \\ \frac{b^2\,N_{fx_k,fx_{k+1},t}+b\,N_{fx_{k-1},fx_k,t}}{2} \end{Bmatrix} \\
&= h\,max\{b\,N_{fx_{k-1},fx_k,t}, b^2\,N_{fx_k,fx_{k+1},t}\}. \qquad (2)
\end{aligned}
$$

If $max\{b\,N_{fx_{k-1},fx_k,t}, b^2\,N_{fx_k,fx_{k+1},t}\} = b^2\,N_{fx_k,fx_{k+1},t}$, then condition (2) and $0 \leqslant h < \frac{1}{b^3}$ assure that $N_{fx_k,fx_{k+1},t} \leqslant hb^2\,N_{fx_k,fx_{k+1},t} < N_{fx_k,fx_{k+1},t}$, which is a contradiction. Thus, $max\{b\,N_{fx_{k-1},fx_k,t}, b^2\,N_{fx_k,fx_{k+1},t}\} = b\,N_{fx_{k-1},fx_k,t}$. So, it follows from condition (2) that $N_{fx_k,fx_{k+1},t} \leqslant hb\,N_{fx_{k-1},fx_k,t}$ for all $k \in \mathbb{N}$. Put $\alpha = hb$. Then, $0 \leqslant \alpha < \frac{1}{b^2}$. Therefore,

$$N_{fx_k,fx_{k+1},t} \leqslant \alpha\,N_{fx_{k-1},fx_k,t}, \ \textit{for all } k \in \mathbb{N}. \qquad (3)$$

By repeated use of condition (3), we get

$$N_{fx_k,fx_{k+1},t} \leqslant \alpha^k\,N_{fx_0,fx_1,t}, \ \textit{for all } k \geqslant 0. \qquad (4)$$

Therefore, $\lim_{k\to\infty} N_{fx_k,fx_{k+1},t} = 0$. Now, we show that (fx_k) is a Cauchy sequence in $f(X)$. For $m,k \in \mathbb{N}$ with $m > k$ and using conditions (4), $(N2)$, Lemmas 2.16, 2.17, we have

$$\begin{aligned}
N_{fx_k,fx_m,t} &\leqslant (n-1)bN_{fx_k,fx_{k+1},t} + bN_{fx_m,fx_{k+1},t} \\
&\leqslant (n-1)bN_{fx_k,fx_{k+1},t} + b^2N_{fx_{k+1},fx_m,t} \\
&\leqslant (n-1)bN_{fx_k,fx_{k+1},t} + (n-1)b^3N_{fx_{k+1},fx_{k+2},t} + b^3N_{fx_m,fx_{k+2},t} \\
&\leqslant (n-1)bN_{fx_k,fx_{k+1},t} + (n-1)b^3N_{fx_{k+1},fx_{k+2},t} + (n-1)b^5N_{fx_{k+2},fx_{k+3},t} \\
&\quad + \cdots + (n-1)b^{2m-2k-3}N_{fx_{m-2},fx_{m-1},t} + b^{2m-2k-2}N_{fx_{m-1},fx_m,t} \\
&\leqslant (n-1)b[\alpha^k + b^2\alpha^{k+1} + b^4\alpha^{k+2} + \cdots + b^{2(m-k-2)}\alpha^{m-2}]N_{fx_0,fx_1,t} \\
&\quad +(n-1)b\, b^{2(m-k-1)}\alpha^{m-1}N_{fx_0,fx_1,t} \\
&\leqslant (n-1)b\alpha^k[1 + (b^2\alpha) + (b^2\alpha)^2 + \cdots]N_{fx_0,fx_1,t} \\
&= (n-1)\frac{b\alpha^k}{1-b^2\alpha}N_{fx_0,fx_1,t} \to 0 \text{ as } k \to \infty, \ \textit{since } b^2\alpha < 1
\end{aligned}$$

Thus, $\lim_{k,m\to\infty} N_{fx_k,fx_m,t} = 0$. This proves that (fx_k) is a Cauchy sequence in $f(X)$. Let $f(X)$ be a complete subspace of X. Then there exists $u \in f(X) \subseteq g(X)$ so that $fx_k \to u$ and also $gx_k \to u$. In case, $g(X)$ is complete, this holds also with $u \in g(X)$. Let $u = gs$ for some $s \in X$. Now, we prove that u is a point of coincidence of f and g in $g(X)$.

Suppose that condition (i) holds, that is, f is assumed to be G-continuous w.r.t. g. Then, $gx_k \to u = gs$ and $(gx_k, gx_{k+1}) \in E(G)$ imply that $fx_k \to fs$. As the limit of a convergent sequence is unique, it follows that $u = fs = gs$. Therefore, u is a point of coincidence of f and g in $g(X)$.

Now, suppose that condition (ii) holds, that is, we assume that the graph G has the property $(*)$. As $gx_k \to u$ and $(gx_k, gx_{k+1}) \in E(G)$ for all $k \geqslant 1$, by property $(*)$, there exists a subsequence (gx_{k_i}) of (gx_k) such that $(gx_{k_i}, u) \in E(G)$ for all $i \geqslant 1$.

By using condition (1), we obtain

$$\begin{aligned}
N_{gs,fs,t} &\leqslant b[(n-1)N_{gs,fx_{k_i},t} + N_{fs,fx_{k_i},t}] \\
&\leqslant (n-1)b^2N_{fx_{k_i},gs,t} + bN_{fs,fx_{k_i},t} \\
&\leqslant (n-1)b^2N_{fx_{k_i},gs,t} + bh\,max\left\{\begin{array}{l} N_{gs,gx_{k_i},t}, N_{fx_{k_i},gx_{k_i},t}, N_{fs,gs,t}, \\ \frac{N_{gs,fx_{k_i},t}+N_{gx_{k_i},fs,t}}{2(n-1)b} \end{array}\right\} \\
&\leqslant (n-1)b^2N_{fx_{k_i},gs,t} + bh\,max\left\{\begin{array}{l} b\,N_{gx_{k_i},gs,t}, (n-1)b\,N_{fx_{k_i},gs,t} + b\,N_{gx_{k_i},gs,t}, \\ N_{fs,gs,t}, \frac{b\,N_{fx_{k_i},gs,t}+(n-1)b\,N_{gx_{k_i},gs,t}+b\,N_{fs,gs,t}}{2b} \end{array}\right\}.
\end{aligned}$$

Taking the limit as $i \to \infty$, we get

$$N_{gs,fs,t} \leqslant bh\,max\left\{N_{fs,gs,t}, \frac{1}{2}N_{fs,gs,t}\right\} \leqslant b^2h\,N_{gs,fs,t},$$

which gives that $N_{gs,fs,t} = 0$, since $0 \leqslant hb^2 < \frac{1}{b}$. Thus, $gs = fs = u$ and so u is a point of coincidence of f and g in $g(X)$. For uniqueness, let there exist another $v \in g(X)$ such that $v = gx = fx$ for some $x \in X$. By property $(**)$, it follows that $(u,v) \in E(\tilde{G})$. Then,

$$\begin{aligned}
N_{u,v,t} &= N_{fs,fx,t} \\
&\leqslant h\,max \left\{ \begin{array}{l} N_{gs,gx,t}, N_{fs,gs,t}, N_{fx,gx,t}, \\ \frac{N_{gs,fx,t}+N_{gx,fs,t}}{2(n-1)b} \end{array} \right\} \\
&\leqslant h\,max \left\{ \begin{array}{l} N_{u,v,t}, N_{u,u,t}, N_{v,v,t}, \\ \frac{N_{u,v,t}+N_{v,u,t}}{2b} \end{array} \right\} \\
&\leqslant h\,max \left\{ N_{u,v,t}, \frac{b\,N_{u,v,t} + b\,N_{u,v,t}}{2b} \right\} \\
&= h\,max\,\{N_{u,v,t}, N_{u,v,t}\} \\
&= h\,N_{u,v,t}.
\end{aligned}$$

This gives that $N_{u,v,t} = 0$, since $0 \leqslant h < \frac{1}{b^3}$ and hence $u = v$. Thus, f and g have a unique point of coincidence in $g(X)$. If f and g are weakly compatible, then by Proposition 2.22, f and g have a unique common fixed point in $g(X)$. ■

Corollary 3.2 Let (X,N) be a parametric N_b-metric space with the coefficient $b \geqslant 1$ and let the mappings $f,\, g : X \to X$ be such that $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X and satisfy the following condition:

$$N_{fx,fy,t} \leqslant h\,max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{2(n-1)b}\}$$

for all $x,\, y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. ■

Corollary 3.3 Let (X,N) be a complete parametric N_b-metric space endowed with a digraph G. Suppose that the mapping $f : X \to X$ is edge preserving and satisfies the following condition

$$N_{fx,fy,t} \leqslant h\,max\{N_{x,y,t}, N_{fx,x,t}, N_{fy,y,t}, \frac{N_{x,fy,t} + N_{y,fx,t}}{2(n-1)b}\}$$

for all $x,\, y \in X$ with $(x,y) \in E(\tilde{G})$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Assume that at least one of the following conditions holds:

(i) f is G-continuous.
(ii) The graph G has the property $(*')$.

If there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the graph G has the following property:

$(**')$ If $x,\, y$ are fixed points of f in X, then $(x,y) \in E(\tilde{G})$.

Proof. The proof follows from Theorem 3.1 by taking $g = I$, the identity map on X. ■

Corollary 3.4 Let (X, N) be a complete parametric N_b-metric space with the coefficient $b \geqslant 1$ and let $f : X \to X$ be such that

$$N_{fx,fy,t} \leqslant h\, max\{N_{x,y,t}, N_{fx,x,t}, N_{fy,y,t}, \frac{N_{x,fy,t} + N_{y,fx,t}}{2(n-1)b}\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Then f has a unique fixed point in X.

Proof. The proof can be obtained from Theorem 3.1 by taking $G = G_0$ and $g = I$. ■

Corollary 3.5 Let (X, N) be a complete parametric N_b-metric space with the coefficient $b \geqslant 1$ and let $g : X \to X$ be an onto mapping satisfying

$$N_{x,y,t} \leqslant h\, max\{N_{gx,gy,t}, N_{x,gx,t}, N_{y,gy,t}, \frac{N_{gx,y,t} + N_{gy,x,t}}{2(n-1)b}\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Then g has a unique fixed point in X.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$ and $f = I$. ■

The following result generalizes Theorem 3.1[31] in parametric A-metric spaces to parametric N_b-metric spaces.

Corollary 3.6 Let (X, N) be a complete parametric N_b-metric space with the coefficient $b \geqslant 1$ and let $g : X \to X$ be an onto mapping satisfying

$$N_{gx,gy,t} \geqslant k\, N_{x,y,t} \tag{5}$$

for all x, $y \in X$, all $t > 0$ and some $k > b^3$. Then g has a unique fixed point in X.

Proof. Taking $h = \frac{1}{k}$ and using condition (5), it follows that

$$N_{x,y,t} \leqslant \frac{1}{k} N_{gx,gy,t} \leqslant h\, max\{N_{gx,gy,t}, N_{x,gx,t}, N_{y,gy,t}, \frac{N_{gx,y,t} + N_{gy,x,t}}{2(n-1)b}\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Now the result follows from Corollary 3.5. ■

Remark 2 *Corollary 3.6 ensures the existence of unique fixed point for expansive mappings in parametric N_b-metric spaces.*

Corollary 3.7 Let (X, N) be a parametric N_b-metric space endowed with a binary relation ρ over X and let the mappings f, $g : X \to X$ satisfy the following condition:

$$N_{fx,fy,t} \leqslant h\, max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{2(n-1)b}\}$$

for all x, $y \in X$ with $gxRgy$, where $R = \rho \cup \rho^{-1}$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Suppose that f is comparative w.r.t. g, $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X. Suppose also that the following conditions hold:

(i) (X, N, R) is regular,
(ii) there exists $x_0 \in X$ such that $gx_0 R f x_0$.

Then f and g have a point of coincidence in $g(X)$. Moreover, f and g have a unique point of coincidence in $g(X)$ if the following property holds:

If x, y are points of coincidence of f and g in $g(X)$, then xRy.

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Theorem 3.1 by taking $G = (V(G), E(G))$ where $V(G) = X,\ E(G) = \{(x, y) \in X \times X : xRy\} \cup \Delta$. ■

Corollary 3.8 Let (X, N) be a complete parametric N_b-metric space endowed with a partial ordering $\preceq$. Suppose the mapping $f : X \to X$ maps comparable elements into comparable elements and satisfies the following condition:

$$N_{fx,fy,t} \leqslant h\,max\{N_{x,y,t}, N_{fx,x,t}, N_{fy,y,t}, \frac{N_{x,fy,t} + N_{y,fx,t}}{2(n-1)b}\}$$

for all $x, y \in X$ with $x \preceq y$ or, $y \preceq x$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Suppose the triple $(X, N, \preceq)$ has the following property:

(†) If (x_k) is a sequence in X such that $x_k \to x$ and x_k, x_{k+1} are comparable for all $k \geqslant 1$, then there exists a subsequence (x_{k_i}) of (x_k) such that x_{k_i}, x are comparable for all $i \geqslant 1$.

If there exists $x_0 \in X$ such that x_0, fx_0 are comparable, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:

(††) If x, y are fixed points of f in X, then x, y are comparable.

Proof. The proof can be obtained from Theorem 3.1 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. ■

Corollary 3.9 Let (X, N) be a complete parametric N_b-metric space with the coefficient $b \geqslant 1$ and let $f : X \to X$ be such that

$$N_{fx,fy,t} \leqslant \alpha N_{x,y,t} + \beta N_{fx,x,t} + \gamma N_{fy,y,t} + \delta[N_{x,fy,t} + N_{y,fx,t}] \tag{6}$$

for all x, $y \in X$, all $t > 0$ and α, β, γ, $\delta \geqslant 0$ with $\alpha + \beta + \gamma + 2(n-1)b\delta < \frac{1}{b^3}$. Then f has a unique fixed point in X.

Proof. Condition (6) gives that

$$N_{fx,fy,t} \leqslant (\alpha + \beta + \gamma + 2(n-1)b\delta)\, max \left\{ \begin{array}{c} N_{x,y,t}, N_{fx,x,t}, N_{fy,y,t}, \\ \\ \frac{N_{x,fy,t} + N_{y,fx,t}}{2(n-1)b} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$. Taking $h = \alpha + \beta + \gamma + 2(n-1)b\delta$, it follows that $0 \leqslant h < \frac{1}{b^3}$. Now applying Corollary 3.4, we obtain the desired result. ■

Remark 3 *We note that several important fixed point results including fixed points for expansive mappings in parametric A-metric spaces can be obtained by putting $b = 1$ and choosing different digraphs in Theorem 3.1.*

Now we furnish some examples to justify the validity of our main result. The first example shows that the existence and uniqueness of the common fixed point can not follows easily by working in the setting of a usual metric space without any graph. It should be noticed that Theorem 3.3 [10] can not assure the existence of a common fixed point in the following example.

***Example* 3.10** Let $X = \{1, 2, 3\} \cup [4, \infty)$ and define $N : X^3 \times (0, \infty) \to [0, \infty)$ by

$$N(x, y, z, t) = t^3(\mid x - y \mid + \mid x - z \mid + \mid y - z \mid)^2$$

for all x, y, $z \in X$ and all $t > 0$. Then (X, N) is a complete parametric N_b-metric space with $b = 3$, $n = 3$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1, 3)\}$. Let f, $g : X \to X$ be defined by

$$fx = \begin{cases} 1, & if \;\; x = 1, 3, \\ 3, & if \;\; x = 2, \\ x^2, & if \;\; x \geqslant 4 \end{cases} \quad \text{and } gx = \begin{cases} x, & if \;\; x = 1, 2, 3, \\ x+2, & if \;\; x \geqslant 4. \end{cases}$$

Then, $f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X. Moreover, f and g are weakly compatible. Obviously, f and g does not satisfy the contractive condition in Berinde's [10] meaning. In fact, in the setting of a usual metric space, for $x = 1$, $y = 2$, we have

$$d(fx, fy) = d(1, 3) = 2 > 1 = d(gx, gy).$$

So, Theorem 3.3 [10] can not assure the existence of a common fixed point of f and g.

On the other hand, f is edge preserving w.r.t. g with $(gx_0, fx_0) \in E(G)$ for $x_0 = 1$. Furthermore, condition (1) holds trivially and it is easy to compute that properties $(*)$ and $(**)$ hold true. Thus, we have all the conditions of Theorem 3.1 which ensures the existence of a unique common fixed point 1 of f and g in $g(X)$.

Remark 4 *It is interesting to note that in Example 3.10, the condition*

$$N_{fx,fy,t} \leqslant h \; max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\}$$

does not hold for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. In fact, for $x = 1$, $y = 4$, we have $fx = 1$, $fy = 16$, $gx = 1$, $gy = 6$. Therefore, $N_{fx,fy,t} = 4t^3 \mid fx - fy \mid^2 = 900t^3$ and

$$max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\} = 4t^3 \, max\{25, 0, 100, \frac{125}{6}\} = 400t^3.$$

Now it follows that,

$$\begin{aligned} N_{fx,fy,t} = 900t^3 &= \frac{9}{4}.400t^3 \\ &= \frac{9}{4} \, max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\} \\ &> \frac{1}{b^3} \, max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\}. \end{aligned}$$

Moreover, we find that for $x = 1$, $y = 4$, *we have* $N_{fx,fy,t} = 900t^3 = \frac{225}{9} N_{x,y,t} > N_{x,y,t}$.

***Example* 3.11** Let $X = \{0, 2, 4\} \cup [5, \infty)$ and define $N : X^3 \times (0, \infty) \to [0, \infty)$ by

$$N(x, y, z, t) = t^3(| x - y | + | x - z | + | y - z |)^2$$

for all x, y, $z \in X$ and all $t > 0$. Then (X, N) is a complete parametric N_b-metric space with $b = 3$, $n = 3$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(2, 4)\}$. Let f, $g : X \to X$ be defined by

$$fx = \begin{cases} 0, & if\ \ x = 0, \\ 4, & if\ \ x = 2,\, 4, \\ x^2, & if\ \ x \geqslant 5 \end{cases} \quad \text{and } gx = \begin{cases} x, & if\ \ x = 0,\, 2,\, 4, \\ x + 1, & if\ \ x \geqslant 5. \end{cases}$$

Then, $f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X. Moreover, f and g are weakly compatible. Obviously, f and g does not satisfy the contractive condition in Berinde's [10] meaning. In fact, in the setting of a usual metric space, for $x = 0$, $y = 2$, we have

$$d(fx, fy) = d(0, 4) = 4 > 2 = d(gx, gy).$$

So, Theorem 3.3 [10] can not assure the existence of a common fixed point of f and g.

On the other hand, f is edge preserving w.r.t. g with $(gx_0, fx_0) \in E(G)$ for $x_0 = 0$. Furthermore, condition (1) holds trivially and it is easy to verify that property $(*)$ holds true. Thus, we have all the conditions of Theorem 3.1 except property $(**)$. We find that 0 and 4 are common fixed points of f and g in $g(X)$ and hence they are also points of coincidence of f and g in $g(X)$, but $(0, 4) \notin E(\tilde{G})$. Thus, we can not find unique common fixed point of f and g without property $(**)$ although f and g are weakly compatible.

4. Fixed Points in Parametric *b*-Metric Space

In this section, we note that every parametric b-metric is a parametric N_b-metric with $n = 2$.

Theorem 4.1 Let (X, P) be a parametric b-metric space with the coefficient $b \geqslant 1$ and let the mappings f, $g : X \to X$ be such that $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X and satisfy the following condition:

$$P(fx, fy, t) \leqslant h\, max \left\{ \begin{array}{l} P(gx, gy, t), P(fx, gx, t), P(fy, gy, t), \\ \frac{P(gx, fy, t) + P(gy, fx, t)}{2b} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Corollary 3.2 by taking $n = 2$. ∎

Corollary 4.2 Let (X, P) be a complete parametric b-metric space with the coefficient $b \geqslant 1$ and let $f : X \to X$ be such that

$$P(fx, fy, t) \leqslant h\, max \left\{ \begin{array}{l} P(x, y, t), P(fx, x, t), P(fy, y, t), \\ \frac{P(x, fy, t) + P(y, fx, t)}{2b} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Then f has a unique fixed point in X.

Proof. The proof can be obtained from Theorem 4.1 by taking $g = I$. ■

Corollary 4.3 Let (X, P) be a complete parametric b-metric space with the coefficient $b \geqslant 1$ and let $g : X \to X$ be an onto mapping satisfying

$$P(x, y, t) \leqslant h\, max \left\{ \begin{array}{l} P(gx, gy, t), P(x, gx, t), P(y, gy, t), \\ \frac{P(gx, y, t) + P(gy, x, t)}{2b} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Then g has a unique fixed point in X.

Proof. The proof follows from Theorem 4.1 by taking $f = I$. ■

Corollary 4.4 Let (X, P) be a complete parametric b-metric space with the coefficient $b \geqslant 1$ and let $g : X \to X$ be an onto mapping satisfying

$$P(gx, gy, t) \geqslant k\, P(x, y, t) \tag{7}$$

for all x, $y \in X$, all $t > 0$ and some $k > b^3$. Then g has a unique fixed point in X.

Proof. Taking $h = \frac{1}{k}$ and using condition (7), it follows that

$$\begin{aligned} P(x, y, t) &\leqslant \frac{1}{k} P(gx, gy, t) \\ &\leqslant h\, max\{P(gx, gy, t), P(x, gx, t), P(y, gy, t), \frac{P(gx, y, t) + P(gy, x, t)}{2b}\} \end{aligned}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < \frac{1}{b^3}$. Now the result follows from Corollary 4.3. ■

Remark 5 *Corollary 4.4 ensures the existence of unique fixed point for expansive mappings in parametric b-metric spaces.*

The following result is the analogue of Banach contraction theorem in parametric b-metric spaces.

Theorem 4.5 Let (X, P) be a complete parametric b-metric space with the coefficient $b \geqslant 1$ and let $f : X \to X$ be such that

$$P(fx, fy, t) \leqslant \alpha\, P(x, y, t) \tag{8}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant \alpha < \frac{1}{b^3}$. Then f has a unique fixed point in X.

Proof. By using condition (8), it follows that

$$\begin{aligned} P(fx, fy, t) &\leqslant \alpha\, P(x, y, t) \\ &\leqslant \alpha\, max\{P(x, y, t), P(fx, x, t), P(fy, y, t), \frac{P(x, fy, t) + P(y, fx, t)}{2b}\} \end{aligned}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant \alpha < \frac{1}{b^3}$. Now the result follows from Theorem 4.1 by taking $g = I$. ■

By an argument similar to that used in Theorem 4.5, we can obtain the following results.

Theorem 4.6 Let (X, P) be a complete parametric b-metric space with the coefficient $b \geqslant 1$ and let $f : X \to X$ be such that

$$P(fx, fy, t) \leqslant \alpha\, P(fx, x, t) + \beta\, P(fy, y, t)$$

for all x, $y \in X$, all $t > 0$ and α, $\beta \geqslant 0$ with $\alpha + \beta < \frac{1}{b^3}$. Then f has a unique fixed point in X.

Theorem 4.7 Let (X, P) be a complete parametric b-metric space with the coefficient $b \geqslant 1$ and let $f : X \to X$ be such that

$$P(fx, fy, t) \leqslant \alpha\, [P(x, fy, t) + P(y, fx, t)]$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant 2\alpha < \frac{1}{b^4}$. Then f has a unique fixed point in X.

The following theorem is a generalization of Theorem 3.3[20] which assures the existence of unique fixed point without continuity of the function.

Theorem 4.8 Let (X, P) be a complete parametric metric space and let $f : X \to X$ be a mapping satisfying the following condition:

$$P(fx, fy, t) \leqslant \beta\, [P(fx, x, t) + P(fy, y, t)] + \delta\, [P(x, fy, t) + P(y, fx, t)]$$

for all x, $y \in X$, all $t > 0$ and β, $\delta \geqslant 0$ with $\beta + \delta < \frac{1}{2}$. Then f has a unique fixed point in X.

Proof. The proof follows from Corollary 3.9 by taking $n = 2$, $b = 1, \alpha = 0$, $\gamma = \beta$. ■

Remark 6 *It is worth mentioning that several important fixed point results in parametric metric spaces can be obtained by putting $n = 2$, $b = 1$ in Theorem 3.1.*

5. Fixed Points in Parametric S-Metric Space

In this section, we note that every parametric S-metric is a parametric N_b-metric with $n = 3$ and $b = 1$.

Theorem 5.1 Let (X, P_S) be a parametric S-metric space and let the mappings f, $g : X \to X$ be such that $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X and satisfy the following condition:

$$P_S(fx, fx, fy, t) \leqslant h\,max \left\{ \begin{array}{l} P_S(gx, gx, gy, t), P_S(fx, fx, gx, t), \\ \\ P_S(fy, fy, gy, t), \frac{P_S(gx,gx,fy,t)+P_S(gy,gy,fx,t)}{4} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < 1$. Then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Corollary 3.2 by taking $n = 3$ and $b = 1$. ■

Corollary 5.2 Let (X, P_S) be a complete parametric S-metric space and let $g : X \to X$ be an onto mapping satisfying

$$P_S(x, x, y, t) \leqslant h\,max \left\{ \begin{array}{l} P_S(gx, gx, gy, t), P_S(x, x, gx, t), P_S(y, y, gy, t), \\ \frac{P_S(gx,gx,y,t)+P_S(gy,gy,x,t)}{4} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < 1$. Then g has a unique fixed point in X.

Proof. The proof follows from Theorem 5.1 by taking $f = I$. ■

The following result gives fixed point for expansive mappings in a parametric S-metric space. In fact, this is a generalization of Theorem 21[35].

Corollary 5.3 Let (X, P_S) be a complete parametric S-metric space and let $g : X \to X$ be an onto mapping satisfying

$$P_S(gx, gx, gy, t) \geqslant k\,P_S(x, x, y, t) \tag{9}$$

for all x, $y \in X$, all $t > 0$ and some $k > 1$. Then g has a unique fixed point in X.

Proof. Taking $h = \frac{1}{k}$ and using condition (9), it follows that

$$\begin{aligned} P_S(x, x, y, t) &\leqslant \frac{1}{k}\,P_S(gx, gx, gy, t) \\ &\leqslant h\,max \left\{ \begin{array}{l} P_S(gx, gx, gy, t), P_S(x, x, gx, t), P_S(y, y, gy, t), \\ \frac{P_S(gx,gx,y,t)+P_S(gy,gy,x,t)}{4} \end{array} \right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and some $0 \leqslant h < 1$. Now the result follows from Corollary 5.2. ■

Corollary 5.4 Let (X, P_S) be a complete parametric S-metric space and let $f : X \to X$ be such that

$$P_S(fx, fx, fy, t) \leqslant h\,max \left\{ \begin{array}{l} P_S(x, x, y, t), P_S(fx, fx, x, t), \\ \\ P_S(fy, fy, y, t), \frac{P_S(x,x,fy,t)+P_S(y,y,fx,t)}{4} \end{array} \right\}$$

for all x, $y \in X$, all $t > 0$ and some $0 \leqslant h < 1$. Then f has a unique fixed point in X.

Proof. The proof can be obtained from Theorem 5.1 by taking $g = I$. ■

Corollary 5.5 Let (X, P_S) be a complete parametric S-metric space and let $f : X \to X$ be such that

$$P_S(fx, fx, fy, t) \leqslant \alpha P_S(x, x, y, t) + \beta P_S(fx, fx, x, t) + \gamma P_S(fy, fy, y, t) \\ +\delta[P_S(x, x, fy, t) + P_S(y, y, fx, t)] \quad (10)$$

for all $x, y \in X$, all $t > 0$ and $\alpha, \beta, \gamma, \delta \geqslant 0$ with $\alpha + \beta + \gamma + 4\delta < 1$. Then f has a unique fixed point in X.

Proof. It follows from condition (10) that

$$P_S(fx, fx, fy, t) \leqslant (\alpha + \beta + \gamma + 4\delta)\, max \begin{Bmatrix} P_S(x, x, y, t), P_S(fx, fx, x, t), \\ P_S(fy, fy, y, t), \frac{P_S(x,x,fy,t)+P_S(y,y,fx,t)}{4} \end{Bmatrix}$$

for all $x, y \in X$, all $t > 0$. Taking $h = \alpha + \beta + \gamma + 4\delta$, it follows that $0 \leqslant h < 1$. Now applying Corollary 5.4, we obtain the desired result. ■

Theorem 5.6 Let (X, P_S) be a complete parametric S-metric space endowed with a binary relation ρ over X. Assume that $f : X \to X$ is a comparative map which satisfies the following condition:

$$P_S(fx, fx, fy, t) \leqslant h\, max \begin{Bmatrix} P_S(x, x, y, t), P_S(fx, fx, x, t), \\ P_S(fy, fy, y, t), \frac{P_S(x,x,fy,t)+P_S(y,y,fx,t)}{4} \end{Bmatrix}$$

for all $x, y \in X$ with xRy, where $R = \rho \cup \rho^{-1}$, all $t > 0$ and some $0 \leqslant h < 1$. Suppose also that the following conditions hold:

(i) (X, P_S, R) is regular,
(ii) there exists $x_0 \in X$ such that $x_0 R f x_0$.

Then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:
If x, y are fixed points of f in X, then xRy.

Proof. The proof follows from Corollary 3.7 by taking $n = 3$, $b = 1$ and $g = I$. ■

Remark 7 *It is valuable to note that several important fixed point results in parametric S_b-metric spaces can be obtained by putting $n = 3$ and choosing different digraphs G in Theorem 3.1.*

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