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Multi-valued fixed point theorems in complex valued b-metric spaces

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Abstract. The aim of this paper is to establish and prove some results on common fixed point for a pair of multi-valued mappings in complex valued b-metric spaces. Our results generalize and extend a few results in the literature.

Keywords: Fixed point, common fixed point; multi-valued mapping, complex valued b-metric space.

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1. Introduction

Fixed point theory is an imperative field of research in mathematics. In this area, a huge involvement has been made by Banach [8], who gave the notion of contraction mapping due to a complete metric space to locate fixed point of the specified function. In 1969, Kannan [20] gave an alternate sort of contractive condition that demonstrated fixed point theorem. The distinction in Banach theorem and that of mapping in Kannan is that continuity is necessary for contraction of Banach maps but Kannan maps are not necessarily continuous. Additionally, Chaterjea [12] gave similar kind of contraction. In the case of single-valued mappings, the aforementioned results have been generalized by many researchers in various ways (see, for example, [5, 7, 12, 16]) and the references therein. One may also consult Rhoades [25] for multitude definitions of contractive type mappings. Two obvious intersecting properties of most generalizations of the Banach

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fixed point theorem is that their proofs are similar and the contractive conditions consist of linear combinations of the distances between two distinct points and their images. The first-two most embraced extensions of Banach principle involving rational inequalities were presented by Dass-Gupta [13] and Jaggi [18]. On the other hand, the earliest known fixed point theorem whose statement and proof are significantly different from Banach fixed point theorem was presented in 1976 by Caristi [11, Theorem 2.1].

Away from single-valued mappings, in 1969, Nadler [24] initiated the study of fixed point theorems for multi-valued mappings. Nadler's contraction principle motivated many researchers and hence the idea has been refined in different directions (see, for instance, [4, 6, 10, 22]). Moreover, all the generalizations of Banach fixed point theorem is further classified in two directions-either the contractive condition is replaced with a more generalized one or the axioms characterizing the ground set is enlarged or weakened. In the second case, some of these metric-like spaces are called semimetric, quasimetric, pseuodometric, b-metric, K-metric. Along this line, by replacing the set of real numbers as the usual co-domain of a metric, Huang and Zhang [17] launched the concept of cone metric as a generalization of metric spaces, thereby, establishing some fixed point theorems for contractive mappings on cone metric spaces. Starting from the year 2007, many authors have come up with various significant fixed point results in the setting of cone metric spaces (see, for example, [19, 27]). The interested researcher may also want to go deep into a comprehensive new survey on cone metric spaces by Aleksić et al. [3].

It is well-known that fixed point results regarding rational contractive conditions cannot be extended or even meaningless in cone metric spaces. To overcome this restriction, Azam et al. [5] initiated the concept of complex valued metric spaces and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions involving rational expressions. Thereafter, the study of fixed point theorems concerning rational inequalities in complex values metric spaces have been growing vigorously (see, for example, [1, 2, 4, 14, 15, 21]). Along the line, the idea of b-metric space was presented by Bakhtin [9] in 1989. Also, Rao [26] introduced the notion of fixed point results on complex valued b-metric spaces, which is broader than complex valued metric spaces. However, every complex valued b-metric space is a cone b-metric space over Banach algebra $\mathbb{C}$ in which the cone is normal with the coefficient of normality $K = 1$, and where the cone has non-empty interior (that is, solid cone). Following [26], various authors have demonstrated fixed point results for different mappings fulfilling rational inequalities with regards to complex valued b-metric spaces (see, for instance, [6, 23]).

In this paper, we adopt the methods in [1, 2, 5] to extend some of the famous fixed point results to multi-valued mappings in complex valued b-metric spaces.

2. Preliminaries

To begin with, we give some basic definitions and results which will be useful in the sequel. Let $\mathbb{C}$ be the set of complex numbers and $u_1, u_2 \in \mathbb{C}$. Also, we define a partial order $\prec$ and $\preceq$ on $\mathbb{C}$ as follows:

(i) $u_1 \prec u_2$ if and only if $\mathbb{R}e(u_1) < \mathbb{R}e(u_2)$ and $\mathbb{I}m(u_1) < \mathbb{I}m(u_2)$.
(ii) $u_1 \preceq u_2$ if and only if $\mathbb{R}e(u_1) \leqslant \mathbb{R}e(u_2)$ and $\mathbb{I}m(u_1) \leqslant \mathbb{I}m(u_2)$.

Definition 2.1 Let X^o be a non empty set and $\tau \geqslant 1$ be a real number. A function $d_c : X^o \times X^o \to \mathbb{C}$ is called complex valued b-metric, if for all $\xi, \eta, \zeta \in X^o$, the following conditions hold.

(i) $0 \preceq d_c(\xi,\eta)$ and $d_c(\xi,\eta)=0$ if and only if $\xi=\eta$;
(ii) $d_c(\xi,\eta)=d_c(\eta,\xi)$;
(iii) $d_c(\xi,\eta)\preceq \tau[d_c(\xi,\zeta)+d_c(\zeta,\eta)]$.

Then the pair (X^o,d_c) is called a complex valued b-metric space.

***Example* 2.2** Let $X^o=[0,1]$. Define a mapping $d_c: X^o\times X^o\to\mathbb{C}$ by

$$d_c(\xi,\eta)=|\xi-\eta|^2+i|\xi-\eta|^2$$

for all $\xi,\eta\in X^o$. Then (X^o,d_c) is a complex valued b-metric space with $\tau=2$.

Definition 2.3 [4] Let (X^o,d_c) be a complex valued b-metric space.
(i) We say that a point $\xi\in X^o$ is an interior point of a set $\mathbb{P}\subseteq X^o$, whenever there exists $0\prec r\in\mathbb{C}$ such that $B(\xi,r)=\{\eta\in X^o: d_c(\xi,\eta)\prec r\}\subseteq\mathbb{P}$.
(ii) We say that a point $\xi\in X^o$ is the limit point of a set $\mathbb{P}\subseteq X^o$ whenever for every $0\prec r\in\mathbb{C}$, $B(\xi,r)\cap(\mathbb{P}\setminus\xi)\neq\emptyset$.
(iii) $\mathbb{P}\subseteq X^o$ is called an open set if each element of $\mathbb{P}$ is an interior point of $\mathbb{P}$.

Definition 2.4 [26] Let $\{\xi_p\}$ be a sequence in a complex valued b-metric space (X^o,d_c) and $\xi\in X$, then

(i) ξ is the limit point of a sequence $\{\xi_p\}$ if for every $c\in\mathbb{C}$ with $0\prec c$ there is $p_o\in\mathbb{Q}$ such that $d_c(\xi_p,\xi)\prec c$ for all $p>p_o$ and we write $\lim\limits_{p\to\infty}\xi_p=\xi$.
(ii) If for every $c\in\mathbb{C}$ with $0\prec c$ there is $p_o\in\mathbb{Q}$ such that $d_c(\xi_p,\xi_{p+q})\prec c$ for all $p>p_o$ and $p,q\in\mathbb{Q}$. Then $\{\xi_p\}$ is a Cauchy sequence in (X^o,d_c).
(iii) A metric space (X^o,d_c) is complete if every Cauchy sequence is convergent in (X^o,d_c).

Lemma 2.5 [14] Let (X^o,d_c) be a complex valued b-metric space and $\{\xi_p\}$ be a sequence in (X^o,d_c). Then $\{\xi_p\}$ converges to ξ iff $|d_c(\xi_p,\xi)|\to 0$ as $p\to\infty$.

Lemma 2.6 [14] Let (X^o,d_c) be a complex valued b-metric space and $\{\xi_p\}$ be a sequence in (X^o,d_c). Then $\{\xi_p\}$ is a Cauchy sequence iff $|d_c(\xi_p,\xi_{p+q})|\to 0$ as $p\to\infty$.

Definition 2.7 Let (X^o,d_c) be a complex valued b-metric space. We denote $s(u)=\{z\in\mathbb{C}: u\preceq z\}$ and

$$s(\xi,\mathbb{Q})=\bigcup_{n\in\mathbb{Q}}s(d_c(\xi,n))=\bigcup_{n\in\mathbb{Q}}\{z\in\mathbb{C}: d_c(\xi,n)\preceq z\}$$

for $\xi\in X^o$ and $\mathbb{Q}\in CB(X^o)$. Also, we have

$$s(\mathbb{P},\mathbb{Q})=\Big(\bigcap_{m\in\mathbb{P}}s(m,\mathbb{Q})\Big)\cap\Big(\bigcap_{n\in\mathbb{Q}}s(n,\mathbb{P})\Big).$$

for $\mathbb{P},\mathbb{Q}\in CB(X^o)$.

Definition 2.8 [26] Let (X^o,d_c) be a complex-valued b-metric space.

(i) Let $\mathbb{T}: X^o\to CB(X^o)$ be a multi-valued mapping. For $\xi\in X^o$ and $\mathbb{P}\in CB(X^o)$, define $W_\xi(\mathbb{P})=\{d_c(\xi,a): a\in\mathbb{P}\}$, and for $\xi,\eta\in X^o$ and $\mathbb{T}\eta\in CB(X^o)$, we have $W_\xi(\mathbb{T}_\eta)=\{d_c(\xi,u): u\in\mathbb{T}_\eta\}$.

(ii) A mapping $F : X^o \to 2^{\mathbb{C}}$ is said to be bounded below if for each $\xi \in X^o$ there exists $z_\xi \in \mathbb{C}$ such that $z_\xi \preceq w$ for all $w \in F_\xi$.
(iii) For a multi-valued mapping $J : X^o \to CB(X^o)$, we say that it has lower bound property on (X^o, d_c) if for any $\xi \in X^o$ the mapping $F_\xi : X^o \to 2^c$ defined by $F_\xi(Jv) = W_\xi(Fv)$ is bounded below. This means that for $\xi, v \in X^o$ there is an element $l_\xi(Jv) \in \mathbb{C}$ such that $l_\xi(Jv) \preceq a$ for all $a \in W_\xi(Jv)$, where $l_\xi(Jv)$ is said to be a lower bound of J corresponding to (ξ, v).
(iv) For a multi-valued mapping $J : X^o \to CB(X^o)$, we say that it has greatest lower bound property (g.l.b property) on (X^o, d_c) if the g.l.b of $W_\xi(Jv)$ exists in $\mathbb{C}$ for all $\xi, v \in X^o$. We denote the g.l.b of $W_\xi(Jv)$ by $d_c(\xi, Jv)$ and define it as:

$$d_c(\xi, Jv) = inf\{d_c(\xi, a) : a \in Jv\}.$$

Definition 2.9 Let (X^o, d_c) be a complex valued b- metric space and $\mathbb{S}, \mathbb{T} : X^o \to CB(X^o)$ be multi- valued mappings.
(i) A point $\xi \in X^o$ is called a fixed point of $\mathbb{T}$ if $\xi \in \mathbb{T}\xi$.
(ii) A point $\xi \in X^o$ is called a common fixed point of $\mathbb{S}$ and $\mathbb{T}$ if $\xi \in \mathbb{S}\xi$ and $\xi \in \mathbb{T}\xi$.

3. Main results

In this section, we prove our main results and provide some examples to justify their hypotheses.

3.1 *Banach type contractive mapping*

Theorem 3.1 Let (X^o, d_c) be a complete complex valued b-metric space and $\mathbb{S}, \mathbb{T} : X^o \to CB(X^o)$ be a pair of multi-valued mappings satisfying the g.l.b property such that

$$\alpha d_c(\xi, \eta) + \frac{\mu d_c(\xi, \mathbb{S}\xi) d_c(\eta, \mathbb{T}\eta) + \lambda d_c(\eta, \mathbb{S}\xi) d_c(\xi, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \in s(\mathbb{S}\xi, \mathbb{T}\eta) \tag{1}$$

for all $\xi, \eta \in X^o$ and α, μ, λ are non negative real numbers with $\tau\alpha + \mu + \lambda < 1$, where $\tau \geqslant 1$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point in X^o.

Proof. Let $\xi_o \in X^o$ be arbitrary but fixed element. Then $T\xi_o$ is not empty so we take $\xi_1 \in T\xi_o$. Thus, from (1), setting $\xi = \xi_1$ and $\eta = \xi_1$, we have

$$\alpha d_c(\xi_o, \xi_1) + \frac{\mu d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1) + \lambda d_c(\xi_1, \mathbb{S}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)} \in s(\mathbb{S}\xi_o, \mathbb{T}\xi_1).$$

This implies

$$\alpha d_c(\xi_o, \xi_1) + \frac{\mu d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1) + \lambda d_c(\xi_1, \mathbb{S}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)} \in \bigcap_{a' \in \mathbb{S}\xi_o} s(a', \mathbb{T}\xi_1)$$

and

$$\alpha d_c(\xi_o,\xi_1)+\frac{\mu d_c(\xi_o,\mathbb{S}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)+\lambda d_c(\xi_1,\mathbb{S}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_0,\xi_1)}\in s(a',\mathbb{T}\xi_1)$$

for all $a'\in\mathbb{S}\xi_o$. Since $\xi_1\in\mathbb{S}\xi_0$, we get

$$\alpha d_c(\xi_o,\xi_1)+\frac{\mu d_c(\xi_o,\mathbb{S}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)+\lambda d_c(\xi_1,\mathbb{S}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_0,\xi_1)}\in s(\xi_1,\mathbb{T}\xi_1)$$

or

$$\alpha d_c(\xi_o,\xi_1)+\frac{\mu d_c(\xi_o,\mathbb{S}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)+\lambda d_c(\xi_1,\mathbb{S}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_0,\xi_1)}\in\bigcup_{b'\in\mathbb{T}\xi_1}s(d_c(\xi_1,b')).$$

Therefore, there exists $\xi_2\in\mathbb{T}\xi_2$ such that

$$\alpha d_c(\xi_o,\xi_1)+\frac{\mu d_c(\xi_o,\mathbb{S}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)+\lambda d_c(\xi_1,\mathbb{S}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_0,\xi_1)}\in s(d_c(\xi_1,\xi_2)).$$

Using Definition 2.7 and g.l.b property, we get

$$d_c(\xi_1,\xi_2)\preceq\alpha d_c(\xi_o,\xi_1)+\frac{\mu d_c(\xi_o,\mathbb{S}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)+\lambda d_c(\xi_1,\mathbb{S}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_0,\xi_1)}.$$

From which we have

$$d_c(\xi_1,\xi_2)\preceq\alpha d_c(\xi_o,\xi_1)+\frac{\mu d_c(\xi_o,\xi_1)d_c(\xi_1,\xi_2)+\lambda d_c(\xi_1,\xi_1)d_c(\xi_o,\xi_2)}{1+d_c(\xi_0,\xi_1)}.$$

This implies

$$\begin{aligned}|d_c(\xi_1,\xi_2)|&\leqslant\alpha|d_c(\xi_o,\xi_1)|+\frac{\mu|d_c(\xi_o,\xi_1)||d_c(\xi_1,\xi_2)|}{1+|d_c(\xi_0,\xi_1)|}\\&=\alpha|d_c(\xi_o,\xi_1)|+\mu|d_c(\xi_1,\xi_2)|\frac{|d_c(\xi_o,\xi_1)|}{1+|d_c(\xi_0,\xi_1)|};\end{aligned}$$

that is,

$$|d_c(\xi_1,\xi_2)|\leqslant\alpha|d_c(\xi_o,\xi_1)|+\mu|d_c(\xi_1,\xi_2)|.$$

This gives

$$(1-\mu)|d_c(\xi_1,\xi_2)|\leqslant\alpha|d_c(\xi_o,\xi_1)|$$

or

$$|d_c(\xi_1,\xi_2)|\leqslant\frac{\alpha}{(1-\mu)}|d_c(\xi_o,\xi_1)|.$$

Inductively, we can develop a sequence $\{\xi_p\}$ in X^o such that $|d_c(\xi_p, \xi_{p+q})| \leqslant \kappa^p |d_c(\xi_0, \xi_1)|$, where $\kappa = \frac{\alpha}{1-\mu}$. Now, for $q \in \mathbb{N}$ and as (X^o, d_c) is a complex valued b-metric space, we have

$$d_c(\xi_p, \xi_{p+q}) \preceq \tau[d_c(\xi_p, \xi_{p+1}) + d_c(\xi_{p+1}, \xi_{p+q})].$$

From which we get

$$d_c(\xi_p, \xi_{p+q}) \preceq \tau d_c(\xi_p, \xi_{p+1}) + \tau^2 d_c(\xi_{p+1}, \xi_{p+2}) + ... + \tau^q d_c(\xi_{p+q-1}, \xi_{p+q});$$

that is,

$$d_c(\xi_p, \xi_{p+q}) \preceq \tau\kappa^p d_c(\xi_o, \xi_1) + \tau^2\kappa^{p+1} d_c(\xi_o, \xi_1) + ... + \tau^q\kappa^{p+q-1} d_c(\xi_o, \xi_1).$$

This yields

$$d_c(\xi_p, \xi_{p+q}) \preceq \tau\kappa^p d_c(\xi_o, \xi_1)[1 + \tau\kappa + (\tau\kappa)^2 + ... + (\tau\kappa)^{q-1}]$$

or

$$|d_c(\xi_p, \xi_{p+q})| \leqslant |\tau\kappa^p d_c(\xi_o, \xi_1)[1 + \tau\kappa + (\tau\kappa)^2 + ... + (\tau\kappa)^{q-1}]|.$$

From which we have

$$|d_c(\xi_p, \xi_{p+q})| \leqslant \frac{\tau\kappa^p}{1 - \tau\kappa} |d_c(\xi_o, \xi_1)|.$$

Since $\tau\alpha + \mu + \lambda < 1$, then $\frac{\tau\alpha}{1-\mu} < 1$. Letting $p, q \to \infty$, gives $|d_c(\xi_p, \xi_{p+q})| \to 0$. Hence, by Lemma 2.6, $\{\xi_p\}$ is a Cauchy sequence in X^o. The completeness of X^o implies that there exists $\zeta \in X^o$ such that $\lim_{p\to\infty} \xi_p = \zeta$.

Now, we show that $\zeta \in \mathbb{S}\zeta$ and $\zeta \in \mathbb{T}\zeta$. From (1), we have

$$\alpha d_c(\xi_{2p}, \zeta) + \frac{\mu d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta) + \lambda d_c(\zeta, \mathbb{S}\xi_{2p}) d_c(\xi_{2p}, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in s(\mathbb{S}\xi_{2p}, \mathbb{T}\zeta).$$

This implies

$$\alpha d_c(\xi_{2p}, \zeta) + \frac{\mu d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta) + \lambda d_c(\zeta, \mathbb{S}\xi_{2p}) d_c(\xi_{2p}, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in \bigcap_{a' \in \mathbb{S}\xi_{2p}} s(a', \mathbb{T}\zeta).$$

Since $\xi_{2p+1} \in \mathbb{S}\xi_{2p}$, we have

$$\alpha d_c(\xi_{2p}, \zeta) + \frac{\mu d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta) + \lambda d_c(\zeta, \mathbb{S}\xi_{2p}) d_c(\xi_{2p}, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in s(\xi_{2p+1}, \mathbb{T}\zeta).$$

This gives

$$\alpha d_c(\xi_{2p}, \zeta) + \frac{\mu d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta) + \lambda d_c(\zeta, \mathbb{S}\xi_{2p}) d_c(\xi_{2p}, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in \bigcup_{b' \in \mathbb{T}\zeta} s(d_c(\xi_{2p+1}, b')).$$

This implies there exists $\zeta_p \in \mathbb{T}\zeta$ such that

$$\alpha d_c(\xi_{2p}, \zeta) + \frac{\mu d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta) + \lambda d_c(\zeta, \mathbb{S}\xi_{2p}) d_c(\xi_{2p}, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in s(d_c(\xi_{2p+1}, \zeta_p)).$$

This means

$$d_c(\xi_{2p+1}, \zeta_p) \preceq \alpha d_c(\xi_{2p}, \zeta) + \frac{\mu d_c(\xi_{2p}, \xi_{2p+1}) d_c(\zeta, \zeta_p) + \lambda d_c(\zeta, \mathbb{S}\xi_{2p}) d_c(\xi_{2p}, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)}.$$

Now, $d_c(\zeta, \zeta_p) \preceq \tau[d_c(\zeta, \xi_{2p+1}) + d_c(\xi_{2p+1}, \zeta_p)]$. Therefore,

$$\begin{aligned} d_c(\zeta, \zeta_p) \preceq{}& \tau d_c(\zeta, \xi_{2p+1}) + \tau\alpha d_c(\xi_{2p}, \zeta) \\ &+ \frac{\tau\mu d_c(\xi_{2p}, \xi_{2p+1}) d_c(\zeta, \zeta_p) + \tau\lambda d_c(\zeta, \xi_{2p+1}) d_c(\xi_{2p}, \zeta_p)}{1 + d_c(\xi_{2p}, \zeta)}. \end{aligned}$$

Thus,

$$\begin{aligned} |d_c(\zeta, \zeta_p)| \leqslant{}& \tau |d_c(\zeta, \xi_{2p+1})| + \tau\alpha |d_c(\xi_{2p}, \zeta)| \\ &+ \frac{\tau\mu |d_c(\xi_{2p}, \xi_{2p+1})| |d_c(\zeta, \zeta_p)| + \tau\lambda |d_c(\zeta, \xi_{2p+1})| |d_c(\xi_{2p}, \zeta_p)|}{1 + |d_c(\xi_{2p}, \zeta)|}. \end{aligned}$$

As $p \to \infty$, we get $|d_c(\zeta, \zeta_p)| \to 0$. By lemma 2.5, it follows that $\zeta_p \to \zeta$. Since $\mathbb{T}\zeta$ is closed, so $\zeta \in \mathbb{T}\zeta$. Similarly, one can show that $\zeta \in \mathbb{S}\zeta$. Thus, $\mathbb{T}$ and $\mathbb{S}$ have a common fixed point ζ in X^o. ■

By setting $\lambda = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.2 Let (X^o, d) ba a complete complex valued b-metric space and $\mathbb{S}, \mathbb{T} : X^o \to CB(X^o)$ be a pair of multi-valued mappings satisfying the g.l.b property such that

$$\alpha d_c(\xi, \eta) + \frac{\mu d_c(\xi, \mathbb{S}\xi) d_c(\eta, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \in s(\mathbb{S}\xi, \mathbb{T}\eta) \tag{2}$$

for all $\xi, \eta \in X^o$ and α, μ are non negative reals with $\tau\alpha + \mu < 1$, where $\tau \geqslant 1$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point in X^o.

By putting $\mathbb{S} = \mathbb{T}$ in Theorem 3.1, we have the following corollary.

Corollary 3.3 Let (X^o, d) be a complete complex valued b metric space and $\mathbb{T} : X^o \to CB(X^o)$ be a multi-valued mappings satisfying the g.l.b property such that

$$\alpha d_c(\xi, \eta) + \frac{\mu d_c(\xi, \mathbb{T}\xi) d_c(\eta, \mathbb{T}\eta) + \lambda d_c(\eta, \mathbb{T}\xi) d_c(\xi, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \in s(\mathbb{T}\xi, \mathbb{T}\eta) \tag{3}$$

for all $\xi, \eta \in X^o$ and α, μ, λ are non negative reals with $\tau\alpha + \mu + \lambda < 1$, where $\tau \geqslant 1$. Then $\mathbb{T}$ has a fixed point in X^o.

***Example* 3.4** Let $X^o = [0, 1]$. Define a mapping $d_c : X^o \times X^o \to \mathbb{C}$ by $d_c(\xi, \eta) = |\xi - \eta|^2 e^{i\psi}$, where $\psi = tan^{-1}\left|\frac{\eta}{\xi}\right|$. Then (X^o, d_c) is a complete complex valued b-metric

space. Consider the mapping $\mathbb{S}, \mathbb{T} : X^o \to CB(X^o)$, defined by

$$\mathbb{S}\xi = \left[0, \frac{\xi}{5}\right], \ \mathbb{T}\xi = \left[0, \frac{\xi}{7}\right]$$

The contractive condition of Theorem 3.1 becomes trivial when $\xi = \eta = 0$. Now, for non zero ξ, η, define d_c by

$$d_c(\xi, \eta) = |\xi - \eta|^2 e^{\iota\psi} \ , \ d_c(\xi, \mathbb{S}\xi) = \left|\xi - \frac{\xi}{5}\right|^2 e^{i\psi},$$

$$d_c(\eta, \mathbb{T}\eta) = \left|\eta - \frac{\eta}{7}\right|^2 e^{i\psi} \ , \ d_c(\eta, \mathbb{S}\xi) = \left|\eta - \frac{\xi}{5}\right|^2 e^{\iota\psi},$$

$$d_c(\xi, \mathbb{T}\eta) = \left|\xi - \frac{\eta}{7}\right|^2 e^{i\psi} \ , \ (d_c(\mathbb{S}\xi, \mathbb{T}\eta)) = \left(\left|\frac{\xi}{5} - \frac{\eta}{7}\right|^2 e^{i\psi}\right).$$

Consider,

$$\alpha|d_c(\xi, \eta)| + \frac{\mu|d_c(\xi, \mathbb{S}\xi)||d_c(\eta, \mathbb{T}\eta)| + \lambda|d_c(\eta, \mathbb{S}\xi)||d_c(\xi, \mathbb{T}\eta)|}{|1 + d_c(\xi, \eta)|}.$$

This implies

$$\alpha|\xi - \eta|^2 + \frac{\mu|\xi - \frac{\xi}{5}|^2|\eta - \frac{\eta}{7}|^2 + \lambda|\eta - \frac{\xi}{5}|^2|\xi - \frac{\eta}{7}|^2}{|1 + d_c(\xi, \eta)|}.$$

Then, clearly for $\alpha = \frac{1}{25}$ and any value of μ and λ, we have

$$\left|\frac{\xi}{5} - \frac{\eta}{7}\right|^2 \leqslant \frac{1}{25}|\xi - \eta|^2 + \frac{\mu|\xi - \frac{\xi}{5}|^2|\eta - \frac{\eta}{7}|^2 + \lambda|\eta - \frac{\xi}{5}|^2|\xi - \frac{\eta}{7}|^2}{|1 + d_c(\xi, \eta)|}.$$

Thus,

$$\alpha d_c(\xi, \eta) + \frac{\mu d_c(\xi, \mathbb{S}\xi) d_c(\eta, \mathbb{T}\eta) + \lambda d_c(\eta, \mathbb{S}\xi) d_c(\xi, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \in s(\mathbb{S}\xi, \mathbb{T}\eta).$$

Hence, all the conditions of theorem 3.1 are satisfied and 0 is a common fixed point of $\mathbb{S}$ and $\mathbb{T}$.

Theorem 3.5 Let (X^o, d_c) be a complex-valued b-metric space and $S, T : X^0 \longrightarrow CB(X)$ be a pair of multi-valued mappings satisfying the g.l.b. property such that

$$\alpha d_c(\xi, \eta) + \frac{\mu d_c(\xi, S\xi) d_c(\eta, T\eta) + \lambda d_c(\eta, S\xi) d_c(\xi, T\eta)}{1 + d_c(\xi, \eta)} \in s\left(S\xi, T\eta\right) \tag{4}$$

for all $\xi, \eta \in \overline{B}(\xi_0, r)$ and

$$\frac{(1 - \kappa)r}{\tau} \in s(\xi_0, S\xi_0), \tag{5}$$

where α, μ, λ are nonnegative real numbers with $\tau\alpha + \mu + \lambda < 1$, and $\kappa = \frac{\alpha}{1-\mu} < 1$ for any $\tau \geqslant 1$. Then there exists u in $\overline{B}(\xi_0, r)$ such that $u \in Su \cap Tu$.

Proof. Let ξ_0 be an arbitrary point in X. Since $S\xi_0 \in CB(X)$, so there exists some $\xi_1 \in S\xi_0$ such that $\frac{(1-\kappa)r}{\tau} \in s(d_c(\xi_0, \xi_1))$. From (5), it is easy to see that $d_c(\xi_0, \xi_1) \preceq \frac{(1-\kappa)r}{\tau}$, which implies that

$$|d_c(\xi_0, \xi_1)| \leqslant \frac{(1-\kappa)}{\tau} |r| \leqslant (1-\kappa)|r|. \tag{6}$$

Hence, $\xi_1 \in \overline{B}(\xi_0, r)$. From (4), we have

$$\alpha d_c(\xi_0, \xi_1) + \frac{\mu d_c(\xi_0, S\xi_0) d_c(\xi_1, T\xi_1) + \lambda d_c(\xi_1, S\xi_1) d_c(\xi_0, T\xi_1)}{1 + d_c(\xi_0, \xi_1)} \in s\left(S\xi_0, T\xi_1\right). \tag{7}$$

From here, by following the remaining steps in the proof of Theorem 1 and using (6), we obtain

$$|d_c(\xi_1, \xi_2)| \leqslant \kappa |d_c(\xi_0, \xi_1)| \leqslant \frac{\kappa(1-\kappa)}{\tau} |r| \leqslant \kappa(1-\kappa)|r|.$$

Notice that

$$\begin{aligned} |d_c(\xi_0, \xi_2)| &\leqslant \tau |d_c(\xi_0, \xi_1)| + \tau |d_c(\xi_1, \xi_2)| \\ &\leqslant \tau \frac{(1-\kappa)|r|}{\tau} + \tau\kappa \frac{(1-\kappa)|r|}{\tau} \\ &\leqslant (1-\kappa^2)|r|. \end{aligned}$$

It follows that $\xi_2 \in \overline{B}(\xi_0, r)$. From (4), we get

$$\alpha d_c(\xi_1, \xi_2) + \frac{\mu d_c(\xi_1, T\xi_1) d_c(\xi_2, S\xi_2) + \lambda d_c(\xi_1, S\xi_2) d_c(\xi_2, T\xi_1)}{1 + d_c(\xi_1, \xi_2)} \in s\left(T\xi_1, S\xi_2\right). \tag{8}$$

By repeating the above steps and using the fact that (X^o, d_c) is a complex-valued b-metric space, we can generate a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in $\overline{B}(\xi_0, r)$ such that $|d_c(\xi_{2n}, \xi_{2n+1})| \leqslant \kappa^{2n} |d_c(\xi_0, \xi_1)|$, $|d_c(\xi_{2n+1}, \xi_{2n+2})| \leqslant \kappa^{2n+1} |d_c(\xi_0, \xi_1)|$, where $\xi_{2n+1} \in S\xi_{2n}$ and $\xi_{2n+2} \in T\xi_{2n+1}$. Inductively, we can construct a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in X such that

$$|d_c(\xi_n, \xi_{n+1})| \leqslant \kappa^n |d_c(\xi_0, \xi_1)|. \tag{9}$$

Now, for $m, n \in \mathbb{N}$ with $n < m$, by triangle inequality, we have

$$d_c(\xi_n, \xi_m) \preceq \tau d_c(\xi_n, \xi_{n+1}) + \tau^2 d_c(\xi_{n+1}, \xi_{n+2}) + \cdots + \tau^{m-n} d_c(\xi_{m-1}, \xi_m).$$

Hence, the iterative scheme (9) yields

$$\begin{aligned}|d_c(\xi_n,\xi_m)| &\leqslant \tau|d_c(\xi_n,\xi_{n+1})| + \tau^2|d_c(\xi_{n+1},\xi_{n+2}) + \cdots + \tau^{m-n}|d_c(\xi_{m-1},\xi_m)|) \\ &\leqslant \tau\kappa^n\left(|1+\tau\kappa+\cdots\tau^{m-n-1}\kappa^{m-n-1}|\right)|d_c(\xi_0,\xi_1)| \\ &\leqslant \frac{\tau\kappa^n}{1-\tau\kappa}|d_c(\xi_0,\xi_1)|.\end{aligned}$$

Consequently, $|d_c(\xi_n,\xi_m)| \leqslant \frac{\tau\kappa^n}{1-\kappa\tau}|d_c(\xi_0,\xi_1)| \longrightarrow 0$ as $n,m \longrightarrow \infty$. Hence, $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\overline{B}(\xi_0,r)$. Since $\overline{B}(\xi_0,r)$ is a closed subspace of a complete space X^o, therefore there exists $u \in \overline{B}(\xi_0,r)$ such that $\xi_n \longrightarrow u$ as $n \longrightarrow \infty$. Again, to show that $u \in Su \cap Tu$, we follow the same steps as in Theorem 3.1 to have $|d_c(u,u_n)| \longrightarrow 0$. This implies that $u_n \longrightarrow u$ as $n \longrightarrow \infty$. Since Tu is closed, therefore $u \in Tu$. Analogously, one can show that $u \in Su$. Consequently, $u \in Su \cap Tu$. ■

By setting $S = T$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.6 Let (X^o, d_c) be a complex-valued b-metric space and $T : X^0 \longrightarrow CB(X)$ be a multi-valued mapping satisfying the g.l.b. property such that

$$\alpha d_c(\xi,\eta) + \frac{\mu d_c(\xi,T\xi)d_c(\eta,T\eta) + \lambda d_c(\eta,T\xi)d_c(\xi,T\eta)}{1+d_c(\xi,\eta)} \in s\left(T\xi,T\eta\right), \tag{10}$$

for all $\xi,\eta \in \overline{B}(\xi_0,r)$ and

$$\frac{(1-\kappa)r}{\tau} \in s(\xi_0,T\xi_0), \tag{11}$$

where α,μ,λ are nonnegative real numbers with $\tau\alpha+\mu+\lambda<1$ and $\kappa = \frac{\alpha}{1-\mu} < 1$ for any $\tau \geqslant 1$. Then there exists u in $\overline{B}(\xi_0,r)$ such that $u \in Tu$.

3.2 *Kannan type contractive mapping*

Theorem 3.7 Let (X^o,d) be a complete complex valued b- metric space and $\mathbb{S},\mathbb{T} : X^o \to CB(X^o)$ be a pair of multi-valued mappings satisfying the g.l.b property such that

$$\alpha d_c(\xi,\mathbb{S}\xi) + \mu d_c(\eta,\mathbb{T}\eta) + \frac{\lambda d_c(\xi,\mathbb{S}\xi)d_c(\eta,\mathbb{T}\eta)}{1+d_c(\xi,\eta)} \in s(\mathbb{S}\xi,\mathbb{T}\eta) \tag{12}$$

for all $\xi,\eta \in X^o$ and α,μ,λ are non negative reals with $\tau\alpha+\mu+\lambda<1$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

Proof. Let $\xi_o \in X^o$. Then $T\xi_o$ is non empty, and so we can take $\xi_1 \in T\xi_o$. From (12), we have

$$\alpha d_c(\xi_o,\mathbb{S}\xi_o) + \mu d_c(\xi_1,\mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o,\mathbb{S}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)}{1+d_c(\xi_o,\xi_1)} \in s(\mathbb{S}\xi_o,\mathbb{T}\xi_1).$$

This implies

$$\alpha d_c(\xi_o, \mathbb{S}\xi_o) + \mu d_c(\xi_1, \mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in \bigcap_{a' \in \mathbb{S}\xi_o} s(a', \mathbb{T}\xi_1)$$

and

$$\alpha d_c(\xi_o, \mathbb{S}\xi_o) + \mu d_c(\xi_1, \mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(a', \mathbb{T}\xi_1)$$

for all $a' \in \mathbb{S}\xi_o$. Since $\xi_1 \in \mathbb{S}\xi_o$, then

$$\alpha d_c(\xi_o, \mathbb{S}\xi_o) + \mu d_c(\xi_1, \mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(\xi_1, \mathbb{T}\xi_1)$$

or

$$\alpha d_c(\xi_o, \mathbb{S}\xi_o) + \mu d_c(\xi_1, \mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in \bigcup_{b' \in \mathbb{T}\xi_1} s(d_c(\xi_1, b')).$$

Therefore, there exist $\xi_2 \in \mathbb{T}\xi_1$ such that

$$\alpha d_c(\xi_o, \mathbb{S}\xi_o) + \mu d_c(\xi_1, \mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(d_c(\xi_1, \xi_2)).$$

By Definition 2.7, we get

$$\begin{aligned} d_c(\xi_1, \xi_2) &\preceq \alpha d_c(\xi_o, \mathbb{S}\xi_o) + \mu d_c(\xi_1, \mathbb{T}\xi_1) + \frac{\lambda d_c(\xi_o, \mathbb{S}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \\ &\preceq \alpha d_c(\xi_o, \xi_1) + \mu d_c(\xi_1, \xi_2) + \frac{\lambda d_c(\xi_o, \xi_1) d_c(\xi_1, \xi_2)}{1 + d_c(\xi_o, \xi_1)}. \end{aligned}$$

This implies that

$$\begin{aligned} |d_c(\xi_1, \xi_2)| &\leqslant \alpha |d_c(\xi_o, \xi_1)| + \mu |d_c(\xi_1, \xi_2)| + \frac{\lambda |d_c(\xi_o, \xi_1)||d_c(\xi_1, \xi_2)|}{1 + |d_c(\xi_o, \xi_1)|} \\ &\leqslant \alpha |d_c(\xi_o, \xi_1)| + \mu |d_c(\xi_1, \xi_2)| + \lambda |d_c(\xi_1, \xi_2)|. \end{aligned}$$

Thus,

$$(1 - \mu - \lambda)|d_c(\xi_1, \xi_2)| \leqslant \alpha |d_c(\xi_o, \xi_1)|$$

or

$$|d_c(\xi_1, \xi_2)| \leqslant \frac{\alpha}{(1 - \mu - \lambda)} |d_c(\xi_o, \xi_1)|.$$

Inductively, we can develop a sequence $\{\xi_p\}$ in X^o such that $|d_c(\xi_p, \xi_{p+q})| \leqslant \kappa^p |d_c(\xi_o, \xi_1)|$,

where $\kappa = \frac{\alpha}{1-\mu-\lambda}$. Now, for $q \in \mathbb{N}$ and as (X^o, d_c) is a complex valued b-metric space

$$d_c(\xi_p, \xi_{p+q}) \preceq \tau[d_c(\xi_p, \xi_{p+1}) + d_c(\xi_{p+1}, \xi_{p+q}).]$$

From which we have

$$d_c(\xi_p, \xi_{p+q}) \preceq \tau d_c(\xi_p, \xi_{p+1}) + \tau^2 d_c(\xi_{p+1}, \xi_{p+2}) + ... + \tau^q d_c(\xi_{p+q-1,\xi_p+q});$$

that is,

$$\begin{aligned} d_c(\xi_p, \xi_{p+q}) &\preceq \tau\kappa^p d_c(\xi_o, \xi_1) + \tau^2\kappa^{p+1} d_c(\xi_o, \xi_1) + ... + \tau^q\kappa^{p+q-1} d_c(\xi_o, \xi_1) \\ &\preceq \tau\kappa^p d_c(\xi_o, \xi_1)[1 + \tau\kappa + (\tau\kappa)^2 + ... + (\tau\kappa)^{q-1}]. \end{aligned}$$

Hence,

$$\begin{aligned} |d_c(\xi_p, \xi_{p+q})| &\leqslant |\tau\kappa^p d_c(\xi_o, \xi_1)[1 + \tau\kappa + (\tau\kappa)^2 + ... + (\tau\kappa)^{q-1}]| \\ &\leqslant \tau\kappa^p |d_c(\xi_o, \xi_1)|[1 + \tau\kappa + (\tau\kappa)^2 + ... + (\tau\kappa)^{q-1}] \end{aligned}$$

Thus,

$$|d_c(\xi_p, \xi_{p+q})| \leqslant \frac{\tau\kappa^p}{1 - \tau\kappa} |d_c(\xi_o, \xi_1)|.$$

Since $\tau\alpha + \mu + \lambda < 1$ then $\Rightarrow \frac{\tau\alpha}{1-\mu-\lambda} < 1$. Thus, $\tau\kappa < 1$. As $p, q \to \infty$, we get $|d_c(\xi_p, \xi_{p+q})| \longrightarrow 0$. Hence, by lemma 2.6, $\{\xi_p\}$ is a Cauchy sequence in X^o. The completeness of X^o ensures that there exists a point $\zeta \in X^o$ such that $limit_{p\to\infty}\xi_p = \zeta$. Now, we show that $\zeta \in \mathbb{S}\zeta$ and $\zeta \in \mathbb{T}\zeta$. From (12), we have

$$\alpha d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) + \mu d_c(\zeta, \mathbb{T}\zeta) + \frac{\lambda d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in s(\mathbb{S}\xi_{2p}, \mathbb{T}\zeta)$$

and

$$\alpha d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) + \mu d_c(\zeta, \mathbb{T}\zeta) + \frac{\lambda d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in \bigcap_{a' \in S\xi_{2p}} s(a', \mathbb{T}\zeta).$$

Since $\xi_{2p+1} \in \mathbb{S}\xi_{2p}$,

$$\alpha d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) + \mu d_c(\zeta, \mathbb{T}\zeta) + \frac{\lambda d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in s(\xi_{2p+1}, \mathbb{T}\zeta)$$

and

$$\alpha d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) + \mu d_c(\zeta, \mathbb{T}\zeta) + \frac{\lambda d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in \bigcup_{b' \in \mathbb{T}\zeta} s(d_c(\xi_{2p+1}, b')).$$

So, there exists an element $\zeta_p \in \mathbb{T}\zeta$ such that

$$\alpha d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) + \mu d_c(\zeta, \mathbb{T}\zeta) + \frac{\lambda d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)} \in s(d_c(\xi_{2p+1}, \zeta_p)).$$

Using Definition 2.7, we have

$$d_c(\xi_{2p+1}, \zeta_p) \preceq \alpha d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) + \mu d_c(\zeta, \mathbb{T}\zeta) + \frac{\lambda d_c(\xi_{2p}, \mathbb{S}\xi_{2p}) d_c(\zeta, \mathbb{T}\zeta)}{1 + d_c(\xi_{2p}, \zeta)}.$$

By g.l.b property of $\mathbb{T}$,

$$d_c(\xi_{2p+1}, \zeta_p) \preceq \alpha d_c(\xi_{2p}, \xi_{2p+1}) + \mu d_c(\zeta, \zeta_p) + \frac{\lambda d_c(\xi_{2p}, \xi_{2p+1}) d_c(\zeta, \zeta_p)}{1 + d_c(\xi_{2p}, \zeta)}.$$

Now, $d_c(\zeta, \zeta_p) \preceq \tau[d_c(\zeta, \xi_{2p+1}) + d_c(\xi_{2p+1}, \zeta_p)]$. Therefore,

$$d_c(\zeta, \zeta_p) \preceq \tau d_c(\zeta, \xi_{2p+1}) + \tau\alpha d_c(\xi_{2p}, \xi_{2p+1}) + \tau\mu d_c(\zeta, \zeta_p) + \frac{\tau\lambda d_c(\xi_{2p}, \xi_{2p+1}) d_c(\zeta, \zeta_p)}{1 + d_c(\xi_{2p}, \zeta)}.$$

This gives

$$|d_c(\zeta, \zeta_p)| \leqslant \tau |d_c(\zeta, \xi_{2p+1})| + \tau\alpha |d_c(\xi_{2p}, \xi_{2p+1})| + \tau\mu |d_c(\zeta, \zeta_p)| + \frac{\tau\lambda |d_c(\xi_{2p}, \xi_{2p+1})| |d_c(\zeta, \zeta_p)|}{1 + |d_c(\xi_{2p}, \zeta)|}.$$

As $p \to \infty$, we get $|d_c(\zeta, \zeta_p)| \to 0$. By lemma 2.5, we have $\zeta_p \to \zeta$. Since $\mathbb{T}\zeta$ is closed, so $\zeta \in \mathbb{T}\zeta$. Similarly it follows that $\zeta \in \mathbb{S}\zeta$. Thus $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point in X^o. ■

By setting $\mathbb{S} = \mathbb{T}$ in above Theorem 3.7, we get the following corollary.

Corollary 3.8 Let (X^o, d_c) be a complete complex valued b-metric space and $T : X^o \to CB(X^o)$ be multi-valued mappings satisfying the g.l.b property such that

$$\alpha d_c(\xi, \mathbb{T}\xi) + \mu d_c(\eta, \mathbb{T}\eta) + \frac{\lambda d_c(\xi, \mathbb{T}\zeta) d_c(\eta, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \in s(\mathbb{T}\xi, \mathbb{T}\eta) \tag{13}$$

for all $\xi, \eta \in X^o$ and α, μ, λ are non negative reals with $\tau\alpha + \mu + \lambda < 1$, where $\tau \geqslant 1$. Then $\mathbb{T}$ has a fixed point in X^o.

Theorem 3.9 Let (X^o, d_c) be a complete complex valued b-metric space and $\mathbb{T} : X^o \to CB(X^o)$ be a multi-valued mapping fulfilling the g.l.b property such that

$$\begin{aligned} &\lambda_1 d_c(\xi, \eta) + \lambda_2 \frac{d_c(\xi, \mathbb{T}\xi) d_c(\eta, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} + \lambda_3 \frac{d_c(\eta, \mathbb{T}\xi) d_c(\xi, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \\ &+ \lambda_4 \frac{d_c(\xi, \mathbb{T}\xi) d_c(\xi, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} + \lambda_5 \frac{d_c(\eta, \mathbb{T}\xi) d_c(\eta, \mathbb{T}\eta)}{1 + d_c(\xi, \eta)} \in s(\mathbb{T}\xi, \mathbb{T}\eta) \end{aligned} \tag{14}$$

for all $\xi, \eta \in X^o$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonnegative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + 2\tau\lambda_4 + 2\tau\lambda_5 < 1$. Then $\mathbb{T}$ has a fixed point in X^0.

Proof. Let $\xi_o \in X^o$. Then $T\xi_o$ is non-empty, so we take $\xi_1 \in T\xi_o$. Thus, from (14), we have

$$\lambda_1 d_c(\xi_o, \xi_1) + \lambda_2 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_3 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)}$$
$$+ \lambda_4 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_5 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(\mathbb{T}\xi_o, \mathbb{T}\xi_1).$$

This implies

$$\lambda_1 d_c(\xi_o, \xi_1) + \lambda_2 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_3 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)}$$
$$+ \lambda_4 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_5 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in \bigcap_{a \in \mathbb{T}\xi_o} s(a, \mathbb{T}\xi_1).$$

Thus,

$$\lambda_1 d_c(\xi_o, \xi_1) + \lambda_2 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_3 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)}$$
$$+ \lambda_4 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_5 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(a, \mathbb{T}\xi_1)$$

for all $a \in \mathbb{T}\xi_o$. Since $\xi_1 \in \mathbb{T}\xi_o$, we get

$$\lambda_1 d_c(\xi_o, \xi_1) + \lambda_2 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_3 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)}$$
$$+ \lambda_4 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_5 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(\xi_1, \mathbb{T}\xi_1).$$

Hence,

$$\lambda_1 d_c(\xi_o, \xi_1) + \lambda_2 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_3 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)}$$
$$+ \lambda_4 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_5 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in \bigcup_{b \in \mathbb{T}x_1} s(d_c(\xi_1, b)).$$

Thus, there exists some $\xi_2 \in \mathbb{T}\xi_1$ such that

$$\lambda_1 d_c(\xi_o, \xi_1) + \lambda_2 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_3 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_0, \xi_1)}$$
$$+ \lambda_4 \frac{d_c(\xi_o, \mathbb{T}\xi_o) d_c(\xi_o, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} + \lambda_5 \frac{d_c(\xi_1, \mathbb{T}\xi_o) d_c(\xi_1, \mathbb{T}\xi_1)}{1 + d_c(\xi_o, \xi_1)} \in s(d_c(\xi_1, \xi_2)).$$

By Definition 2.7, we get

$$d_c(\xi_1,\xi_2) \preceq \lambda_1 d_c(\xi_o,\xi_1) + \lambda_2 \frac{d_c(\xi_o,\mathbb{T}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)}{1+d_c(\xi_o,\xi_1)} + \lambda_3 \frac{d_c(\xi_1,\mathbb{T}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_0,\xi_1)} + \lambda_4 \frac{d_c(\xi_o,\mathbb{T}\xi_o)d_c(\xi_o,\mathbb{T}\xi_1)}{1+d_c(\xi_o,\xi_1)} + \lambda_5 \frac{d_c(\xi_1,\mathbb{T}\xi_o)d_c(\xi_1,\mathbb{T}\xi_1)}{1+d_c(\xi_o,\xi_1)}.$$

Using the g.l.b property of $\mathbb{T}$, we have

$$d_c(\xi_1,\xi_2) \preceq \lambda_1 d_c(\xi_o,\xi_1) + \lambda_2 \frac{d_c(\xi_o,\xi_1)d_c(\xi_1,\xi_2)}{1+d_c(\xi_o,\xi_1)} + \lambda_3 \frac{d_c(\xi_1,\xi_1)d_c(\xi_o,\xi_2)}{1+d_c(\xi_0,\xi_1)} + \lambda_4 \frac{d_c(\xi_o,\xi_1)d_c(\xi_o,\xi_2)}{1+d_c(\xi_o,\xi_1)} + \lambda_5 \frac{d_c(\xi_1,\xi_1)d_c(\xi_1,\xi_2)}{1+d_c(\xi_o,\xi_1)}$$

Hence,

$$|d_c(\xi_1,\xi_2)| \leqslant \lambda_1 |d_c(\xi_o,\xi_1)| + \lambda_2)|d_c(\xi_1,\xi_2)| + \lambda_4 |d_c(\xi_o,\xi_2)|$$

Thus,

$$(1-\lambda_2-\lambda_4)|d_c(\xi_1,\xi_2)| \leqslant (\lambda_1+\lambda_4)|d_c(\xi_o,\xi_1)|.$$

and $|d_c(\xi_1,\xi_2)| \leqslant \kappa |d_c(\xi_o,\xi_1)|$, where $\kappa = \frac{\lambda_1+\lambda_4}{1-\lambda_2-\lambda_4}$. Inductively, we develop a sequence $\{\xi_p\}$ in X^o such that $|d_c(\xi_p,\xi_{p+q})| \leqslant \kappa^p |d_c(\xi_o,\xi_1)|$. On similar steps as in the previous theorems , we conclude that $\{\xi_p\}$ is a Cauchy sequence in X^o. So, by completeness of X^o, there exists some $\zeta \in X^o$, such that $\lim\limits_{p\to\infty} \xi_p = \zeta$. We show that $\zeta \in \mathbb{T}\zeta$. From (14), we have

$$\lambda_1 d_c(\xi_{2p},\zeta) + \lambda_2 \frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_3 \frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_4 \frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_5 \frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} \in s(\mathbb{T}\xi_{2p},\mathbb{T}\zeta).$$

This implies

$$\lambda_1 d_c(\xi_{2p},\zeta) + \lambda_2 \frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+_c d(\xi_{2p},\zeta)} + \lambda_3 \frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_4 \frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_5 \frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} \in \bigcap_{a\in\mathbb{T}\xi_{2p}} s(a,\mathbb{T}\zeta).$$

Since $\xi_{2p+1} \in \mathbb{T}\xi_{2p}$, we get

$$\lambda_1 d_c(\xi_{2p},\zeta) + \lambda_2 \frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_3 \frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_4 \frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} + \lambda_5 \frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)} \in s(\xi_{2p+1},\mathbb{T}\zeta)$$

or

$$\lambda_1 d_c(\xi_{2p},\zeta)+\lambda_2\frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}+\lambda_3\frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}$$
$$+\lambda_4\frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}+\lambda_5\frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}\in\bigcup_{b\in\mathbb{T}\zeta}s(d_c(\xi_{2p+1},b)).$$

So, there exists some $\zeta_p\in\mathbb{T}\zeta$ such that

$$\lambda_1 d_c(\xi_{2p},\zeta)+\lambda_2\frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}+\lambda_3\frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}$$
$$+\lambda_4\frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}+\lambda_5\frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}\in s(d_c(\xi_{2p+1},\zeta_p)).$$

Therefore, by Definition 2.7,

$$d_c(\xi_{2p+1},\zeta_p)\preceq\lambda_1 d_c(\xi_{2p},\zeta)+\lambda_2\frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}+\lambda_3\frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}$$
$$+\lambda_4\frac{d_c(\xi_{2p},\mathbb{T}\xi_{2p})d_c(\xi_{2p},\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}+\lambda_5\frac{d_c(\zeta,\mathbb{T}\xi_{2p})d_c(\zeta,\mathbb{T}\zeta)}{1+d_c(\xi_{2p},\zeta)}.$$

Using the g.l.b property, we have

$$d_c(\xi_{2p+1},\zeta_p)\preceq\lambda_1 d_c(\xi_{2p},\zeta)+\lambda_2\frac{d_c(\xi_{2p},\xi_{2p+1})d_c(\zeta,\zeta_p)}{1+d_c(\xi_{2p},\zeta)}+\lambda_3\frac{d_c(\zeta,\xi_{2p+1})d_(\xi_{2p},\zeta_P)}{1+d_c(\xi_{2p},\zeta)}$$
$$+\lambda_4\frac{d_c(\xi_{2p},\xi_{2p+1})d_c(\xi_{2p},\zeta_p)}{1+d_c(\xi_{2p},\zeta)}+\lambda_5\frac{d_c(\zeta,\xi_{2p+1})d_c(\zeta,\zeta_p)}{1+d_c(\xi_{2p},\zeta)}.$$

By triangle inequality, $d_c(\zeta,\zeta_p)\preceq\tau[d_c(\zeta,\xi_{2p+1})+d_c(\xi_{2p+1},\zeta_p)]$. Thus,

$$d_c(\xi_{2p+1},\zeta_p)\preceq\tau d_c(\zeta,\xi_{2p+1})+\tau\lambda_1 d_c(\xi_{2p},\zeta)+\tau\lambda_2\frac{d_c(\xi_{2p},\xi_{2p+1})d_c(\zeta,\zeta_p)}{1+d_c(\xi_{2p},\zeta)}$$
$$+\tau\lambda_3\frac{d_c(\zeta,\xi_{2p+1})d_c(\xi_{2p},\zeta_P)}{1+d_c(\xi_{2p},\zeta)}+\tau\lambda_4\frac{d_c(\xi_{2p},\xi_{2p+1})d_c(\xi_{2p},\zeta_p)}{1+d_c(\xi_{2p},\zeta)}$$
$$+\tau\lambda_5\frac{d_c(\zeta,\xi_{2p+1})d_c(\zeta,\zeta_p)}{1+d_c(\xi_{2p},\zeta)},$$

which implies

$$|d_c(\xi_{2p+1},\zeta_p)|\leqslant\tau|d_c(\zeta,\xi_{2p+1})|+\tau\lambda_1|d_c(\xi_{2p},\zeta)|+\tau\lambda_2\left|\frac{d_c(\xi_{2p},\xi_{2p+1})d_c(\zeta,\zeta_p)}{1+d_c(\xi_{2p},\zeta)}\right|$$
$$+\tau\lambda_3\left|\frac{d_c(\zeta,\xi_{2p+1})d_c(\xi_{2p},\zeta_P)}{1+d_c(\xi_{2p},\zeta)}\right|+\tau\lambda_4\left|\frac{d_c(\xi_{2p},\xi_{2p+1})d_c(\xi_{2p},\zeta_p)}{1+d_c(\xi_{2p},\zeta)}\right|$$
$$+\tau\lambda_5\left|\frac{d_c(\zeta,\xi_{2p+1})d_c(\zeta,\zeta_p)}{1+d_c(\xi_{2p},\zeta)}\right|.$$

As $p \to \infty$, we get $|d_c(\xi_{2p+1}, \zeta_p)| \to 0$. By lemma 2.5, $\zeta_p \to \zeta$ as $p \to \infty$, also since $\mathbb{T}\zeta$ is closed then $\zeta \in \mathbb{T}\zeta$. Thus $\mathbb{T}$ has a fixed point in X^o. ■

4. Application to Homotopy Result

In this section, we apply Corollary 3.6 to prove a homotopy result. First, for convenience, we recall the following familiar definitions.

Definition 4.1 A relation $\leqslant$ is a total order on a set G if for all $s, t, u \in G$, the following conditions hold:

(i) Reflexivity: $s \leqslant s$;
(ii) Antisymmetry: if $s \leqslant t$ and $t \leqslant s$, then $s = t$;
(iii) Transitivity: if $s \leqslant t$ and $t \leqslant u$, then $s \leqslant u$;
(iv) Comparability: for every $s, t \in G$, either $s \leqslant t$ or $t \leqslant s$.

Recall that if the set G satisfies only the axioms $(i)-(iii)$, then it is said to be partially ordered. In what follows, we shall call a totally ordered set a chain.

Lemma 4.2 (Kuratowski-Zorn's Lemma) If G is any nonempty partially ordered set in which every chain has an upper bound, then G has a maximal element.

Definition 4.3 Let X^o and Y^o be any two topological spaces and $\pi, \omega : X^o \longrightarrow Y^o$ continuous functions. A function $H : X^o \times [0,1] \longrightarrow Y^o$ such that if $x \in X^o$, then $H(x,0) = \pi(x)$ and $H(x,1) = \omega(x)$, is called a homotopy between π and ω.

We shall denote the boundary of a set G by $Bd(G)$.

Theorem 4.4 Let (X^0, d_c) be a complex-valued b-metric space with G an open subset of X^o. Let $H : [0,1] \times \overline{G} \longrightarrow CB(X)$ be multi-valued mapping having the g.l.b property. Assume that there exists $\dot{a} \in X^o$ and $0 \prec r \in \mathbb{C}$ such that the following conditions are satisfied:

(i) $a \notin [H(t,a)]$, for each $a \in Bd(G)$ and each $t \in [0,1]$;
(ii) $H(t,.) : \overline{G} \longrightarrow CB(X)$ be a multi-valued mapping satisfying

$$\alpha d_c(a,b) + \frac{\mu d_c(a, H(t,a)) d_c(b, H(\acute{t}, b)) + \lambda d_c(b, H(t,a)) d_c(a, H(\acute{t}), b)}{1 + d_c(a,b)} \in s\left(H(t,a), H(\acute{t}, b)\right)$$

and

$$\frac{(1-\kappa)r}{\tau} \in s(\dot{a}, H(\dot{t}, \dot{a})), \tag{15}$$

where $\kappa = \frac{\alpha}{1-\mu} < 1$, for any $\tau \geqslant 1$;
(iii) there exists a continuous nondecreasing function $g : (0,1] \longrightarrow A \cup \{0\}$ such that

$$g(s) - g(t) \in s\left(H(s,a), H(t,b)\right), \quad g(s) \in g(t)$$

for all $s, t \in [0,1]$ and each $a \in \overline{G}$, where $A = \{z \in \mathbb{C} : 0 \prec z\}$. Then $H(0,.)$ has a fixed point if and only if $H(1,.)$ has a fixed point.

Proof. Assume $H(0,.)$ has a fixed point u, so $u \in H(0,u)$. From (i), $u \in G$. Define

$$\Omega := \{(t,a) \in [0,1] \times G : a \in H(a,t)\}.$$

Obviously, $\Omega \neq \emptyset$. Define the partial ordering in Ω as

$$(t,a) \preceq (s,b) \Leftrightarrow t \leqslant s \quad \text{and} \quad d_c(a,b) \preceq \frac{2}{1-\kappa}(g(s)-g(t)).$$

Let Z be a chain in Ω and $\dot{t} = \sup\{t : (t,a) \in Z\}$. Also, let $\{(t_n,a_n)\}_{n\in\mathbb{N}}$ be a sequence in Z such that $(t_n,a_n) \preceq (t_{n+1},a_{n+1})$ and $t_n \longrightarrow \dot{t}$ as $n \longrightarrow \infty$. Then for $n < m$, we have

$$d_c(a_m,a_n) \preceq \frac{2}{1-\kappa}\left(g(t_m)-g(t_n)\right) \longrightarrow 0 \quad \text{as} \quad n,m \longrightarrow \infty,$$

which implies that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X^o. By the completeness of the complex-valued b-metric space X^o, there exists $\dot{a} \in X^o$ such that $a_n \longrightarrow \dot{a}$ as $n \longrightarrow \infty$. By condition (iii), we have

$$\begin{aligned}\alpha d_c(a_n,\dot{a}) + \frac{\mu d_c(a_n,H(t_n,a_n))d_c(\dot{a},H(\dot{t},\dot{a})) + \lambda d_c(\dot{a},H(t_n,a_n))d_c(a_n,H(\dot{t},\dot{a}))}{1+d_c(a_n,\dot{a})} \\ \in s\left(H(t_n,a_n),H(\dot{t},\dot{a})\right).\end{aligned}$$

Since $a_n \in H(t_n,a_a)$, then we have

$$\begin{aligned}\alpha d_c(a_n,\dot{a}) + \frac{\mu d_c(a_n,H(t_n,a_n))d_c(\dot{a},H(\acute{t},\dot{a})) + \lambda d_c(\dot{a},H(t_n,a_n))d_c(a_n,H(\dot{t},\dot{a}))}{1+d_c(a_n,\dot{a})} \\ \in s\left(a_n,H(\dot{t},\dot{a})\right).\end{aligned}$$

Therefore, there exists $a_k \in H(\dot{t},\dot{a})$ such that

$$\begin{aligned}d_c(a_n,a_k) \preceq \alpha d_c(a_n,\dot{a}) + \frac{\mu d_c(a_n,H(t_n,a_n))d_c(\dot{a},H(\dot{t},\dot{a})) + \lambda d_c(\dot{a},H(t_n,a_n))d_c(a_n,H(\dot{t},\dot{a}))}{1+d_c(a_n,\dot{a})} \\ \in s\left(a_n,H(\dot{t},\dot{a})\right).\end{aligned}$$

Since H has the g.l.b property, then

$$d_c(a_n,a_k) \preceq \alpha d_c(a_n,\dot{a}) + \frac{\lambda d_c(\dot{a},a_n)d_c(a_n,a_k)}{1+d_c(a_n,\dot{a})}.$$

Therefore,

$$|d_c(a_n,a_k)| \leqslant \alpha|d_c(a_n,\dot{a})| + \frac{\lambda|d_c(\dot{a},a_n)||d_c(a_n,a_k)|}{|1+d_c(a_n,\dot{a})|}.$$

Using the fact that $|1+d_c(a_n,\dot{a})| > |d_c(a_n,\dot{a})|$, we obtain $|d_c(a_n,a_k)| \leqslant \alpha|d_c(a_n,\dot{a})| +$

$\lambda|d_c(a_n, a_k)|$. Therefore, $|d_c(a_n, a_k)| \leqslant \frac{\alpha}{1-\lambda}|d_c(a_n, \dot{a})|$. Notice that

$$\begin{aligned}|d_c(\dot{a}, a_k)| &\leqslant \tau|d_c(\dot{a}, a_n)| + \tau|d_c(a_n, a_k)| \\ &\leqslant \tau|d_c(\dot{a}, a_n)| + \frac{\tau\alpha}{1-\lambda}|d_c(a_n, \dot{a})| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned}$$

Therefore, $a_k \longrightarrow \dot{a} \in H(\dot{t}, \dot{a})$ and thus $\dot{a} \in G$ implies $(\dot{t}, \dot{a}) \in \Omega$. It follows that $(t, a) \preceq (\dot{t}, \dot{a})$ for all $(t, a) \in Z$, which yields that $(\dot{t}, \dot{a})$ is an upper bound of Z. Consequently, by Kuratowski-Zorn's Lemma, Z has a maximal element $(\dot{t}, \dot{a})$. We claim that $\dot{t} = 1$. But suppose $\dot{t} \leqslant 1$ and choose $0 \prec r \in \mathbb{C}$, $\dot{t} \leqslant t$ such that $\overline{B}(\dot{t}, r) \subset G$. By condition (iii), we have $g(t) - g(\dot{t}) \in s\left(H(t, a), H(\dot{t}, \dot{a})\right)$, where $g(t) - g(\dot{t}) \in s\left(\dot{a}, H(t, a)\right)$ for all $\dot{a} \in H(\dot{t}, \dot{a})$. Hence, there exists $a \in H(t, a)$ such that $g(t) - g(\dot{t}) \in s(d_c(\dot{a}, a))$ and $d_c(a, \dot{a}) \preceq g(t) - g(\dot{t}) \prec (1 - \kappa)r$ for $r = \frac{2}{1-\kappa}(g(t) - g(\dot{a}))$. It follows that $|d_c(a, \dot{a})| \leqslant (1 - \kappa)|r|$. Thus, by condition (ii), we deduce that the mapping $H(t, .) : \overline{B}(\dot{a}, r) \longrightarrow CB(X)$ satisfies all the hypotheses of Corollary 3.6. Therefore, for all $t \in [0, 1]$, there exists $a \in \overline{B}(\dot{a}, r)$ such that $a \in H(t, a)$. Hence $(a, t) \in \Omega$. Since $d_c(a, \dot{a}) \prec r = \frac{2}{1-\kappa}(g(t) - g(\dot{t}))$, then we have $(\dot{t}, \dot{a}) \preceq (t, a)$, a contradiction. So, $\dot{t} = 1$. This shows that $H(., 1)$ has a fixed point. Conversely, if $H(1, .)$ has a fixed point, then on similar steps, one can prove that $H(0, .)$ has a fixed point. ■

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