Journal of
Linear and Topological Algebra
Vol. 08*, No.* 03*,* 2019*,* 183- 189

Application of DJ method to Itô stochastic differential equations

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Received 8 December 2018; Revised 1 July 2019; Accepted 2 July 2019.
Communicated by Mohammad Sadegh Asgari

Abstract. This paper develops iterative method described by [V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl. 316 (2006) 753-763] to solve Itô stochastic differential equations. The convergence of the method for Itô stochastic differential equations is assessed. To verify efficiency of method, some examples are expressed.

Keywords: Stochastic differential equations, iterative methods, Itô calculus.

2010 AMS Subject Classification: 60H10, 47J25.

1. Introduction

The Itô calculus is known as a stochastic generalization of the Riemann-Stieltjes integral and the Itô stochastic differential equations have been employed for modelling in different phenomena. Some applications may be seen in the biological systems [1], medicine [15], heat equation [18], and finance mathematics [29].

In recent years, it has been a great concentration on solutions of Itô stochastic differential and integral equations. The following methods are related to this fact: Bernstein polynomials operational matrix [2], random Euler method [5], random Euler difference scheme [6], Block Pulse stochastic operational matrix [14, 20], triangular functions [13], stochastic operational matrix of generalized hat basis functions [16], Euler-Maruyama and Milstein methods [17], linear analytic approximation [19], Chebyshev wavelet operational method [21, 23], Legendre wavelet operational method [22], Haar wavelets method [24], Second kind Chebyshev wavelet Galerkin scheme [25], successive approximation

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

[26], Adomian decomposition [27]. However, many of suggested methods have tedious calculations.

In 2006, Daftardar and Jafari proposed an iterative method, which is abbreviated by "DJ" method, for solving various types of functional equations [9]. In [3], this method was applied to partial differential equations of integer and fractional order. The authors concluded DJ method produces better results than those obtained by Adomian decomposition method, Homotopy perturbation method, and Variational iteration method. The method of [3] was also extended for solutions of fractional boundary value problems with Dirichlet boundary conditions [8]. Other utilizations of DJ method can be found in [4, 7, 10–12, 30, 31]. The objective of this work is to derive DJ method for solving

$$\begin{cases} dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), & 0 \leqslant t \leqslant T, \\ X(0) = c, \end{cases} \tag{1}$$

where f, g are known real functions; $X(t)$ is a random variable; $B(t)$ is a Brownian motion; T and c are real constants.

The paper is organized as follows. Section 2 is devoted to methodology. In Section 3, two test problems are evaluated. A conclusion is drawn in Section 4.

2. The structure of DJ method for Itô SDE

The DJ method considers a functional equation as a sum of linear and nonlinear terms. Then, by using an iterative procedure, the solution of equation is identified in the form of an infinite series. Obviously, (1) may be rewritten as follows

$$X(t) = c + \mathcal{N}(X(t)), \tag{2}$$

so that $\mathcal{N}$ refers to the nonlinear operator with the form of

$$\mathcal{N}(X(t)) = \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dB(s).$$

In this technique, $X(t)$ is decomposed as the following infinite series

$$X(t) = \sum_{i=0}^{+\infty} X_i(t). \tag{3}$$

Also, the nonlinear operator of (2) is decomposed in the form below

$$\mathcal{N}\left(\sum_{i=0}^{+\infty} X_i(t)\right) = \mathcal{N}(c) + \sum_{i=1}^{+\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} X_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} X_j(t)\right)\right\}. \tag{4}$$

Placing (3) and (4) into (2) entails

$$\sum_{i=0}^{+\infty} X_i(t) = c + \mathcal{N}(X_0) + \sum_{i=1}^{+\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} X_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} X_j(t)\right)\right\}.$$

To determine the components of X_i in (3), the recurrence formula below can be announced

$$\begin{cases} X_0 = c, \\ X_1 = \mathcal{N}(X_0), \\ X_{m+1} = \mathcal{N}(X_0 + \cdots + X_m) - \mathcal{N}(X_0 + \cdots + X_{m-1}), \;\; m = 1, 2, \ldots \; . \end{cases} \tag{5}$$

In the following theorems, the presented method is analyzed.

Theorem 2.1 Let $t \in [0, T]$ and f and g in (1) be Lipschitz functions with real positive constants λ and μ. Then, infinite series (3) is absolutely convergent if $q(T + \sqrt{T}) < 1$, where $q = \max\{\lambda, \mu\}$.

Proof. Since f and g satisfy in the Lipschitz condition, for two arbitrary random variables $X(t)$ and $Y(t)$, there exist real positive constants λ and μ such that

$$\begin{cases} \|f(t, X(t)) - f(t, Y(t))\| \leqslant \lambda \|X(t) - Y(t)\|, \\ \|g(t, X(t)) - g(t, Y(t))\| \leqslant \mu \|X(t) - Y(t)\|. \end{cases}$$

Subsequently, one can write

$$\|\mathcal{N}(X(t)) - \mathcal{N}(Y(t))\| \leqslant \left(\int_0^t \lambda ds + \int_0^t \mu dB(s) \right) \|X(t) - Y(t)\|. \tag{6}$$

Now, by introducing $q = \max\{\lambda, \mu\}$, (6) results in

$$\|\mathcal{N}(X(t)) - \mathcal{N}(Y(t))\| \leqslant q(T + \sqrt{T}) \|X(t) - Y(t)\|. \tag{7}$$

On the other hand, according to (5) and (7), we have

$$\begin{aligned} \|X_{m+1}\| &= \|\mathcal{N}(X_0 + \cdots + X_m) - \mathcal{N}(X_0 + \cdots + X_{m-1})\| \\ &\leqslant q(T + \sqrt{T}) \|X_m\| \\ &\;\vdots \\ &\leqslant q^{m+1}(T + \sqrt{T})^{m+1} \|X_0\|, \;\; m = 0, 1, 2, \ldots \; . \end{aligned}$$

It is clear that if $q(T + \sqrt{T}) < 1$, then $\sum_{i=0}^{+\infty} X_i$ is absolutely convergent. ■

Theorem 2.2 Under the assumptions of Theorem 2.1, the infinite series (3) provides exact solution of the problem (1).

Proof. Employing (5) implies that

$$X_0 + X_1 + \cdots + X_{m+1} = c + \mathcal{N}(X_0 + X_1 + \cdots + X_m).$$

As m tends to infinity, we obtain

$$\begin{aligned} X &= \lim_{m \to +\infty} \sum_{i=0}^{m+1} X_i = \lim_{m \to +\infty} \left(c + \mathcal{N}\left(\sum_{i=0}^{m} X_i \right) \right) = c + \mathcal{N}\left(\lim_{m \to +\infty} \sum_{i=0}^{m} X_i \right) \\ &= c + \mathcal{N}(X), \end{aligned}$$

that is, the convergent series (3) satisfies in (2). This means (3) discloses exact solution of (1). ■

3. Numerical examples

This section is assigned to implementation of two test problems in order to clarify the validity of the proposed method. In these examples, Itô's formula is used frequently. For the convenience of reader, 1-dimensional Itô's formula is reviewed as follows [28]:

Let $h(t,x)$ be a twice continuously differentiable function on $[0,+\infty)\times\mathbb{R}$, and $Y(t) = h(t,X(t))$. Then,

$$dY(t) = \frac{\partial h}{\partial t}(t,X(t))dt + \frac{\partial h}{\partial x}(t,X(t))dX(t) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,X(t))(dX(t))^2,$$

in which $(dX(t))^2 = (dX(t))\cdot(dX(t))$ is computed by the rules

$$dt\cdot dt = dt\cdot dB(t) = dB(t)\cdot dt = 0, \quad dB(t)\cdot dB(t) = dt.$$

***Example* 3.1** [28] Consider the following Itô stochastic differential equation

$$\begin{cases} dX(t) = \frac{1}{2}X(t)dt + X(t)dB(t), \\ X(0) = 1. \end{cases} \tag{8}$$

The exact solution of (8) is $X(t) = e^{B(t)}$.

Now, the proposed method is executed step by step. The initial value problem (8) is equivalent to

$$X(t) = 1 + \tfrac{1}{2}\textstyle\int_0^t X(s)ds + \int_0^t X(s)dB(s).$$

By defining

$$\mathcal{N}(X(t)) = \frac{1}{2}\int_0^t X(s)ds + \int_0^t X(s)dB(s),$$

and using Itô's formula, it follows

$$\begin{aligned} X_1 &= \mathcal{N}(1) = \frac{1}{2}\int_0^t ds + \int_0^t dB(s) = \frac{1}{2}t + B(t), \\ X_2 &= \mathcal{N}(1+X_1) - \mathcal{N}(1) = \frac{1}{2}\int_0^t \left(1 + \frac{1}{2}s + B(s)\right)ds \\ &+ \int_0^t \left(1 + \frac{1}{2}s + B(s)\right)dB(s) - \left(\frac{1}{2}t + B(t)\right) \\ &= \frac{1}{2}\left(\int_0^t B(s)ds + \int_0^t sdB(s)\right) + \int_0^t B(s)dB(s) + \frac{1}{8}t^2 \\ &= \frac{1}{2}tB(t) + \frac{1}{2}(B^2(t) - t) + \frac{1}{8}t^2. \end{aligned}$$

After some calculations, it would be acquired

$$
\begin{aligned}
X_3 &= \mathcal{N}(1+X_1+X_2)-\mathcal{N}(1+X_1)\\
&= \mathcal{N}(1+B(t)+\frac{1}{2}tB(t)+\frac{1}{2}B^2(t)+\frac{1}{8}t^2)-\mathcal{N}(1+\frac{1}{2}t+B(t))\\
&= \frac{1}{6}B^3(t)+\frac{1}{8}t^2B(t)+\frac{1}{4}tB^2(t)-\frac{1}{2}tB(t)-\frac{1}{4}t^2+\frac{1}{48}t^3.
\end{aligned}
$$

Now, if S_m be m-th partial sum of random variables, then one can deduce that

$$
\begin{cases}
S_0 = 1,\\
S_1 = 1+B(t)+\zeta_1(t),\\
S_2 = 1+B(t)+\frac{1}{2}B^2(t)+\zeta_2(t),\\
S_3 = 1+B(t)+\frac{1}{2}B^2(t)+\frac{1}{6}B^3(t)+\zeta_3(t),\\
\vdots \qquad \vdots,
\end{cases}
$$

where

$$
\begin{cases}
\zeta_1(t) = \frac{1}{2}t,\\
\zeta_2(t) = \frac{1}{2}tB(t)+\frac{1}{8}t^2,\\
\zeta_3(t) = \frac{1}{8}t^2B(t)+\frac{1}{4}tB^2(t)-\frac{1}{8}t^2+\frac{1}{48}t^3
\end{cases}
$$

are the noise terms. By obtaining S_m for sufficiently large value of m, obviously $X(t) = e^{B(t)}$, which is the exact solution.

***Example* 3.2** [28] Consider the system of Itô stochastic differential equations in the form of

$$
\begin{cases}
dX_1(t) = -\frac{1}{2}X_1(t)dt - X_2(t)dB(t),\\
dX_2(t) = -\frac{1}{2}X_2(t)dt + X_1(t)dB(t),
\end{cases}
\tag{9}
$$

under the initial conditions $X_1(0) = 1$ and $X_2(0) = 0$. The exact solution of (9) is $X_1(t) = \cos(B(t))$ and $X_2(t) = \sin(B(t))$.

The initial value problem (9) is equivalent to

$$
\begin{cases}
X_1(t) = 1-\frac{1}{2}\int_0^t X_1(s)ds - \int_0^t X_2(s)dB(s),\\
X_2(t) = -\frac{1}{2}\int_0^t X_2(s)ds + \int_0^t X_1(s)dB(s).
\end{cases}
$$

By considering the following operators

$$
\begin{cases}
\mathcal{N}(X_1(t),X_2(t)) = -\frac{1}{2}\int_0^t X_1(s)ds - \int_0^t X_2(s)dB(s),\\
\mathcal{M}(X_1(t),X_2(t)) = -\frac{1}{2}\int_0^t X_2(s)ds + \int_0^t X_1(s)dB(s),
\end{cases}
$$

and using Itô's formula, if $X_1^{(j)}$ and $X_2^{(j)}$ denote to the solution of problem at the iteration

$j \in \mathbb{N}$ then one gets

$$\begin{cases} X_1^{(1)} = \mathcal{N}(1,0) = -\frac{1}{2}t, \\ X_2^{(1)} = \mathcal{M}(1,0) = B(t). \end{cases}$$

$$\begin{cases} X_1^{(2)} = \mathcal{N}\left(1-\frac{1}{2}t, B(t)\right) - \mathcal{N}(1,0) = -\frac{1}{2}B^2(t) + \frac{1}{8}t^2 + \frac{1}{2}t, \\ X_2^{(2)} = \mathcal{M}\left(1-\frac{1}{2}t, B(t)\right) - \mathcal{M}(1,0) = -\frac{1}{2}tB(t). \end{cases}$$

$$\begin{cases} X_1^{(3)} = \mathcal{N}\left(1+\frac{1}{8}t^2-\frac{1}{2}B^2(t), B(t)-\frac{1}{2}tB(t)\right) - \mathcal{N}\left(1-\frac{1}{2}t, B(t)\right) \\ \qquad = \frac{1}{4}tB(t) - \frac{1}{96}t^3 - \frac{1}{4}t^2, \\ X_2^{(3)} = \mathcal{M}\left(1+\frac{1}{8}t^2-\frac{1}{2}B^2(t), B(t)-\frac{1}{2}tB(t)\right) - \mathcal{M}\left(1-\frac{1}{2}t, B(t)\right) \\ \qquad = -\frac{1}{6}B^3(t) + \frac{1}{8}t^2B(t) + \frac{1}{2}tB(t). \end{cases}$$

If $S_1^{(3)}$ and $S_2^{(3)}$ be sum of initial values and three first iterations, then one can imply

$$\begin{cases} S_1^{(3)} = 1 - \frac{1}{2}B^2(t) + \zeta_1(t), \\ S_2^{(3)} = B(t) - \frac{1}{6}B^3(t) + \zeta_2(t), \end{cases}$$

in which $\zeta_1(t) = \frac{1}{4}tB(t) - \frac{1}{8}t^2 - \frac{1}{96}t^3$ and $\zeta_2(t) = \frac{1}{8}t^2B(t)$ are the noise terms. By determining partial sums for sufficiently large iterations, it is visible

$$\begin{cases} X_1(t) = \cos(B(t)), \\ X_2(t) = \sin(B(t)), \end{cases}$$

which is the exact solution.

4. conclusion

The DJ method has been understood as a contributory method for solving diverse types of functional equations. Throughout this paper, a stochastic version of this method was exhibited to gain the solution of Itô stochastic differential equations. The convergence analysis was also performed and two examples were investigated through the presented version. It is notable that the procedure of this paper may be employed to stochastic Itô Volterra-Fredholm integral equations, but some modifications must be done. This subject seems to be appropriate for those who want to continue this research.

References

[1] M. Abundo, A stochastic model for predator-pray systems: basic properties, stability and computer simulation, J. Math. Biol. 29 (1991), 495-511.

[2] M. Asgari, E. Hasheminejad, M. Khodabin, K. Maleknejad, Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials, Bull. Math. Soc. Sci. Math. Roumanie. 57 (2014), 3-12.

[3] S. Bhalekar, V. Daftardar-Gejji, New iterative method: application to partial differntial equations, Appl. Math. Comput. 203 (2008), 778-783.

[4] S. Bhalekar, V. Daftardar-Gejji, Solving evolution equations using a new iterative method, Numer. Methods Partial Differ. Equat. 26 (2010), 906-916.
[5] J. C. Cortes, L. Jodar, L. Villafuerte, Mean square numerical solution of random differential equations: facts and possibilities, Comput. Math. Appl. 53 (2007), 1098-1106.
[6] J. C. Cortes, L. Jodar, L. Villafuerte, Numerical solution of random differential equations: a mean square approach, Math. Comput. Model. 45 (2007), 757-765.
[7] V. Daftardar-Gejji, S. Bhalekar, An iterative method for solving fractional differential equations, Proc. Appl. Math. Mech. 7 (2008), 2050017-2050018.
[8] V. Daftardar-Gejji, S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Compt. Appl. Math. 59 (2010), 1801-1809.
[9] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl. 316 (2006), 753-763.
[10] M. R. Yaghouti, H. Deilami, Numerical solution of singular differential-difference equations, World Appl. Program. 3 (2013), 182-189.
[11] H. Jafari, S. J. Johnston, S. M. Sani, D. Baleanu, A decomposition method for solving q-difference equations, Appl. Math. Inf. Sci. 9 (2015), 2917-2920.
[12] H. Jafari, S. Seifi, A. Alipoor, M. Zabihi, An iterative method for solving linear and nonlinear fractional diffusion-wave equation, Int. J. Numer. Anal. Relat. Topics. 3 (2009), 20-32.
[13] M. Khodabin, K. Maleknejad, F. Hosseini Shekarabi, Application of triangular functions to numerical solution of stochastic Volterra integral equations, Int. J. Appl. Math. 43 (2011), 1-9.
[14] M. Khodabin, K. Maleknejad, M. Rostami, M. Nouri, Numerical approach for solving stochastic Volterra-Fredholm integral equations by stochastic operational matrix, Comput. Math. Appl. 64 (2012), 1903-1913.
[15] Y. Liang, D. Greenhalgh, X. Mao, A stochastic differential equation model for the spread of HIV amongst people who inject drugs, Comput. Math. Methods. Med. (2016), 14 pages.
[16] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek, C. Cattani, A computational method for solving stochastic Itô-Volterra integral equations based on stochastic operational matrix for generalized hat basis functions, J. Comput. Phys. 270 (2014) 402-415.
[17] D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, Siam. Rev. 43 (2001), 525-546.
[18] Y. Hu, D. Nualart, J. Song, A nonlinear stochastic heat equation: Holder continuity and smoothness of the density of the solution, Stoch. Proc. Appl. 123 (2013), 1083-1103.
[19] S. Janković, D. Ilic, One linear analytic approximation for stochastic integro-differential equations, Acta Math. Sci. 30 (2010), 1073-1085.
[20] K. Maleknejad, M. Khodabin, M. Rostami, Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions, Math. Comput. Model. 55 (2012), 791-800.
[21] F. Mohammadi, A Chebyshev wavelet operational method for solving stochastic Volterra-Fredholm integral equations, Int. J. Appl. Math. Res. 4 (2015), 217-227.
[22] F. Mohammadi, A computational wavelet method for numerical solution of stochastic Volterra-Fredholm integral equations, Wavelets. Linear. Algebr. 3 (2016), 13-25.
[23] F. Mohammadi, A wavelet-based computational method for solving stochastic Ito-Volterra equations, J. Comput. Math. 298 (2015), 254-265.
[24] F. Mohammadi, Numerical solution of stochastic Volterra-Fredholm integral equations using Haar wavelets, U.P.B. Sci. Bull. (Series A). 78 (2016), 111-126.
[25] F. Mohammadi, Second kind Chebyshev wavelet Galerkin method for stochastic Itô-Volterra integral equations, Mediterr. J. Math. 13 (2016), 2613-2631.
[26] M. G. Murge, B. G. Pachpatte, Successive approximations for solutions of second order stochastic integro-differential equations of Itô type, Indian J. Pure Appl. Math. 21 (1990), 260-274.
[27] K. Nouri, Study on efficiency of Adomian decomposition method for stochastic differential equations, Int. J. Nonlinear Anal. Appl. 8 (2017), 61-68.
[28] B. Oksendal, Stochastic differential equations: An introduction with applications, Springer, 1998.
[29] K. P. Sharp, Stochastic differential equations in finance, Appl. Math. Comput. 39 (1990), 207-224.
[30] M. Yaseen, M. Samraiz, A modified new iterative method for solving linear and nonlinear Klein-Gordon Equations, Appl. Math. Sci. 6 (2012), 2979-2987.
[31] M. Yaseen, M. Samraiz, S. Naheed, Exact solutions of Laplace equation by DJ method, Results Phys. 3 (2013), 38-40.