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Ring endomorphisms with nil-shifting property

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Abstract. Cohn called a ring R is reversible if whenever $ab = 0$, then $ba = 0$ for $a, b \in R$. The reversible property is an important role in noncommutative ring theory. Recently, Abdul-Jabbar et al. studied the reversible ring property on nilpotent elements, introducing the concept of commutativity of nilpotent elements at zero (simply, a CNZ ring). In this paper, we extend the CNZ property of a ring as follows: Let R be a ring and α an endomorphism of R, we say that R is right (resp., left) α-nil-shifting ring if whenever $a\alpha(b) = 0$ (resp., $\alpha(a)b = 0$) for nilpotents a, b in R, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). The characterization of α-nil-shifting rings and their related properties are investigated.

Keywords: CNZ ring, reversible ring, matrix ring, polynomial ring.

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1. Introduction

Throughout this paper all rings are associative with identity. Let R be a ring. $N^*(R)$ and $N(R)$ denote the upper nilradical (i.e., sum of nil ideals) and the set of all nilpotent elements in R, respectively. Note that $N^*(R) \subseteq N(R)$. Denote the n by n ($n \geqslant 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Denote $\{(a_{ij}) \in U_n(R) \mid \text{ the diagonal entries of } (a_{ij}) \text{ are all equal}\}$ by $D_n(R)$. Use e_{ij}, a matrix unit, for the matrix with (i, j)-entry 1 and elsewhere 0. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}_n$ denotes the ring of integers modulo n.

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A ring is called reduced if it has no non zero nilpotent elements. Cohn [6] called a ring R reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Recently, a ring R is said to satisfy the commutativity of nilpotent elements at zero (simply, R is called a CNZ ring) [1] if $ab = 0$ implies $ba = 0$ for $a, b \in N(R)$. Reversible rings are clearly CNZ, but not conversely as in [1, Example 1.1].

According to Krempa [14], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring R is called α-rigid [9] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by [9, Proposition 5]. Following [8], a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. If R is an α-compatible ring, then the endomorphism α is clearly a monomorphism. The notion of an α-compatible ring is a generalization of α-rigid rings to the more general case where the ring is not assumed to be reduced.

In [4, Definition 2.1], an endomorphism α of a ring R is called right skew reversible if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$, and the ring R is called right α-skew reversible if there exists a right skew reversible endomorphism α of R. Similarly, left α-skew reversible rings are defined. A ring R is called α-skew reversible if it is both left and right α-skew reversible. Note that R is an α-rigid ring if and only if R is semiprime and right α-skew reversible for a monomorphism α of R by [4, Proposition 2.5(iii)]. We change over from "an α-reversible ring" in [4] to "an α-skew reversible ring" to cohere with other related definitions.

An endomorphism α of a ring R is called a right (resp., left) skew CNZ if whenever $ab = 0$ for $a, b \in N(R)$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$), and the ring R is called right (resp., left) α-skew CNZ if there exists a right (resp., left) skew CNZ endomorphism α of R; the ring R is called α-skew CNZ if it is both left and right α-skew CNZ [2, Definition 2.1].

In [5, Definition 2.1], a ring R with an endomorphism α is called right (resp., left) α-shifting if whenever $a\alpha(b) = 0$ (resp., $\alpha(a)b = 0$) for $a, b \in R$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$); and the ring R is called α-shifting if it is both right and left α-shifting. It is shown that R is an α-rigid ring if and only if R is right α-shifting and $aR\alpha(a) = 0$ implies $a = 0$ for any $a \in R$ in [5, Proposition 1.2].

Note that reversible rings are CNZ, and right α-skew reversible rings are right α-skew CNZ, but each converse does not hold by [1, Example 2.2] and [2, Example 2.6]. The concepts of reversible rings and right α-skew reversible rings do not imply each other by [4, Examples 2.2 and 2.3], and the concepts of CNZ rings and right α-skew CNZ rings are independent on each other by [1, Example 2.2] and [2, Example 2.6], using [1, Theorem 2.7]. Furthermore, the right α-skew reversible condition and the right α-shifting condition of a ring do not dependent on each other by [5, Example 1.1].

The following diagram shows all implications among the concepts above.

- {reduced rings} $\longrightarrow$ {reversible rings} $\longrightarrow$ {CNZ rings}

{α-compatible rings}

$\nearrow$

- {α-rigid rings} $\longrightarrow$ {right α-skew reversible rings} $\longrightarrow$ {right α-skew CNZ rings}

$\searrow$

{right α-shifting rings}

Proposition 1.1 Let R be an α-compatible ring. Then

(1) R is reversible if and only if R is right (left) α-skew reversible if and only if R is right (left) α-shifting.

(2) [2, Theorem 2.3(4)] R is CNZ if and only if R is right (left) α-skew CNZ.

Proof. (1) This is routine, noting that $ab = 0 \Leftrightarrow a\alpha(b) = 0 \Leftrightarrow \alpha(a)b = 0$ in R. ■

Based on the arguments above, in this paper, we introduce the notation of an α-nil-shifting ring for an endomorphism α of a ring as a generalization of α-shifting rings and study its related properties. Throughout this paper, α denotes a nonzero endomorphism of a given ring, unless specified otherwise. We denote id_R for the identity endomorphism of a given ring R.

2. Right α-nil-shifting rings

We begin with the following definition.

Definition 2.1 An endomorphism α of a ring R is called right (resp., left) nil-shifting if whenever $a\alpha(b) = 0$ (resp., $\alpha(a)b = 0$) for $a, b \in N(R)$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$), and the ring R is called a right (resp., left) α-nil-shifting if there exists a right (resp., left) nil-shifting endomorphism α of R. A ring R is called α-nil-shifting if it is both left and right α-nil-shifting.

Any right α-shifting ring is clearly right α-nil-shifting but not conversely by next example.

***Example* 2.2** Consider a ring $R = U_2(\mathbb{Z})$ with an endomorphism α defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Then R is obviously right α-nil-shifting, since $N(R) = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}\right\}$. For $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we obtain $A\alpha(B) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, but $B\alpha(a) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$, showing that R is not right α-shifting.

A ring R is a CNZ ring if R is one-sided id_R-nil-shifting. Every subring S with $\alpha(S) \subseteq S$ of a right α-nil-shifting is also right α-nil-shifting. We use this fact freely. It is easily checked that R is CNZ if and only if R is right (left) α-skew CNZ if and only if R is right (left) α-nil-shifting when R is an α-compatible ring, but there exists an α-nil-shifting ring which is not right α-skew CNZ ring as follows.

***Example* 2.3** Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Define an automorphism δ of R by $a \mapsto b$ and $b \mapsto a$. Let I be the ideal of A generated by ab, ba, a^3 and b^3. Set $R = A/I$. Since $\delta(I) \subseteq I$, we can obtain an automorphism α of R by defining $\alpha(s + I) = \delta(s) + I$ for $s \in A$. We identify every element of A with its image in R for simplicity. Then R is not right α-skew CNZ. For $a, b \in N(R)$, $ab = 0$ but $b\alpha(a) = b^2 \neq 0$ by the construction of I.

Now, we show that R is right α-nil-shifting. Note that $N(R) = \{ha + ka^2 + sb + tb^2 \mid h, k, s, t \in K\}$. Let $x\alpha(y) = 0$ for $x = h_1a + k_1a^2 + s_1b + t_1b^2, y = h_2a + k_2a^2 + s_2b + t_2b^2 \in N(R)$ where $h_i, k_j, s_l, t_m \in K$. Then $0 = x\alpha(y) =$

$(h_1a + k_1a^2 + s_1b + t_1b^2)(h_2b + k_2b^2 + s_2a + t_2a^2) = h_1s_2a^2 + s_1h_2b^2$ implies that

$$(h_1 = 0, s_1 = 0), (h_1 = 0, h_2 = 0), (s_2 = 0, s_1 = 0) \text{ or } (s_2 = 0, h_2 = 0).$$

(i)$h_1 = 0, s_1 = 0$: Since $x = k_1a^2 + t_1b^2$ and $y = h_2a + k_2a^2 + s_2b + t_2b^2$, $y\alpha(x) = (h_2a + k_2a^2 + s_2b + t_2b^2)(k_1\delta(a^2) + t_1\delta(b^2)) = (h_2a + k_2a^2 + s_2b + t_2b^2)(k_1b^2 + t_1a^2) = 0$.
(ii) $h_1 = 0, h_2 = 0$: Since $x = k_1a^2 + s_1b + t_1b^2$ and $y = k_2a^2 + s_2b + t_2b^2$, $y\alpha(x) = (k_2a^2 + s_2b + t_2b^2)(k_1\delta(a^2) + s_1\delta(b) + t_1\delta(b^2)) = (k_2a^2 + s_2b + t_2b^2)(k_1b^2 + s_1a + t_1a^2) = 0$.
(iii) $s_2 = 0, s_1 = 0$: Since $x = h_1a + k_1a^2 + t_1b^2$ and $y = h_2a + k_2a^2 + t_2b^2$, $y\alpha(x) = (h_2a + k_2a^2 + t_2b^2)(h_1\delta(a) + k_1\delta(a^2) + t_1\delta(b^2)) = (h_2a + k_2a^2 + t_2b^2)(h_1b + k_1b^2 + t_1a^2) = 0$
(iv)$s_2 = 0, h_2 = 0$: Since $x = h_1a + k_1a^2 + s_1b + t_1b^2$ and $y = k_2a^2 + t_2b^2$, $y\alpha(x) = (k_2a^2 + t_2b^2)(h_1\delta(a) + k_1\delta(a^2) + s_1\delta(b) + t_1\delta(b^2)) = (k_2a^2 + t_2b^2)(h_1b + k_1b^2 + s_1a + t_1a^2) = 0$.
Therefore R is right α-nil-shifting. The proof for the left case is similar.

The next example shows that the concept of an α-nil-shifting ring is not left-right symmetric.

***Example* 2.4** We adapt [2, Example 2.2]. Consider a ring $R = U_2(\mathbb{Z}_4)$ and an endomorphism of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Note that $N(R) = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in \{0, 2\}, b \in \mathbb{Z}_4\right\}$. Let $A\alpha(B) = 0$ for $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in N(R)$. Then $aa' = 0$ and it implies that $B\alpha(A) = 0$. Hence R is a right α-nil-shifting ring. Next, we show that R is not a left α-nil-shifting ring. To see this, take $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in N(R)$. Then $\alpha(A)B = 0$, but $\alpha(B)A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$.

Note that any domain with a monomorphism α is obviously α-nil-shifting, but the converse is not true by Example 2.4.

Proposition 2.5 For a ring R with an endomorphism α, we have the following statements.
(1) If $N(R)^2 = 0$, then R is an α-nil-shifting ring.
(2) Let R be a CNZ ring. Then (i) R is right α-nil-shifting if and only if R is α-nil-shifting; and (ii) if R is right α-skew CNZ and α is a monomorphism, then R is right α-nil-shifting.
(3) Let R be a right α-nil-shifting ring with a monomorphism α. Then $ab = 0$ if and only if $b\alpha^2(a) = 0$.
(4) Let $\alpha^2 = id_R$. Then R is right α-nil-shifting if and only if R is CNZ.

Proof. (1) It follows from the fact that $\alpha(N(R)) \subseteq N(R)$.
(2) (i) Suppose that R is a right α-nil-shifting ring and let $\alpha(a)b = 0$ for $a, b \in N(R)$. Then $b\alpha(a) = 0$ and so $a\alpha(b) = 0$. Thus $\alpha(b)a = 0$ since R is CNZ and $\alpha(b) \in N(R)$. Hence R is left α-nil-shifting, entailing that R is α-nil-shifting.
(ii) Suppose that R is right α-skew CNZ with an monomorphism α and let $a\alpha(b) = 0$ for $a, b \in N(R)$. Then $0 = \alpha(b)\alpha(a) = \alpha(ba)$ and so $ba = 0$ since α is a monomorphism. So $ab = 0$ and hence $b\alpha(a) = 0$ by hypothesis, showing that R is right α-nil-shifting.
(3) For $a, b \in N(R)$, $ab = 0$ if and only if $\alpha(a)\alpha(b) = 0$ if and only if $b\alpha^2(a) = 0$.
(4) Suppose that R is right α-nil-shifting and let $ab = 0$ for $a, b \in N(R)$. By (3), we have $b\alpha^2(a) = 0$ and so $ba = 0$. Thus, R is CNZ.
Conversely, assume that R is CNZ and let $a\alpha(b) = 0$ for $a, b \in N(R)$. Then $\alpha(b)a = 0$,

since $\alpha(b) \in N(R)$. Hence, $0 = \alpha(\alpha(b))a = \alpha^2(b)\alpha(a) = b\alpha(a)$ showing that R is right α-nil-shifting. ■

The converse of Proposition 2.5(1) does not hold by Example 2.4(1). Example 2.4 shows that the condition "R is a CNZ ring" in Proposition 2.5(2) cannot be dropped. In fact, the ring $R = U_2(\mathbb{Z}_4)$ is right α-skew CNZ but not CNZ by [2, Example 2.4].

Theorem 2.6 (1) Let α_γ be an endomorphism of a ring R_γ for each $\gamma \in \Gamma$. Then the following are equivalent:

(i) R_γ is a right α_γ-nil-shifting ring for each $\gamma \in \Gamma$.

(ii) The direct sum $\bigoplus_{\gamma\in\Gamma} R_\gamma$ of R_γ is right $\bar{\alpha}$-nil-shifting for the endomorphism $\bar{\alpha} : \bigoplus_{\gamma\in\Gamma} R_\gamma \to \bigoplus_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma\in\Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma\in\Gamma}$.

(iii) The direct product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ is right $\bar{\alpha}$-nil-shifting for the endomorphism $\bar{\alpha} : \prod_{\gamma\in\Gamma} R_\gamma \to \prod_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma\in\Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma\in\Gamma}$.

(2) Let S be a ring and $\sigma : R \to S$ a ring isomorphism. Then R is a right α-nil-shifting if and only if S is a right $\sigma\alpha\sigma^{-1}$-nil-shifting.

Proof. (1) It is enough to show (i))⇒(iii), since the class of α-nil-shifting rings is closed under subrings. Note that $N(\prod_{\gamma\in\Gamma} R_\gamma) \subseteq \prod_{\gamma\in\Gamma} N(R_\gamma)$ and $\alpha_\gamma(R_\gamma) \subseteq R_\gamma$ for each $\gamma \in \Gamma$. Suppose that R_γ is right α-nil-shifting for each $\gamma \in \Gamma$ and let $A\bar{\alpha}(B) = 0$ where $A = (a_\gamma)_{\gamma\in\Gamma}, B = (b_\gamma)_{\gamma\in\Gamma} \in N(\prod_{\gamma\in\Gamma} R_\gamma)$. Then $a_\gamma\alpha(b_\gamma) = 0$ for each $\gamma \in \Gamma$ and $b_\gamma\alpha(a_\gamma) = 0$ since R_γ is right α-nil-shifting and $a_\gamma, b_\gamma \in N(R_\gamma)$. Thus $B\bar{\alpha}(A) = 0$, entailing that the direct product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ is right $\bar{\alpha}$-nil-shifting.
(2) Clearly, $N(S) = \sigma(N(R))$. Then $a, b \in N(R)$ if and only if $a' = \sigma(a)$, $b' = \sigma(b) \in N(S)$. So, $a\alpha(b) = 0 \Leftrightarrow \sigma(a\alpha(b)) = 0 \Leftrightarrow 0 = \sigma(a)\sigma\alpha(b)) = \sigma(a)\sigma\alpha\sigma^{-1}(\sigma(b)) \Leftrightarrow a'\sigma\alpha\sigma^{-1}(b') = 0$. The proof is complete. ■

Corollary 2.7 Let R be a ring with an endomorphism α. If e is a central idempotent of a ring R with $\alpha(e) = e$ and $\alpha(1-e) = 1-e$, then eR and $(1-e)R$ are right α-nil-shifting if and only if R is right α-nil-shifting.

Proof. It comes from Theorem 2.6(1), since $R \cong eR \oplus (1-e)R$ and the class of right α-nil-shifting rings is closed under subrings. ■

Recall that for a ring R with an endomorphism α and an ideal I of R, if I is an α-ideal (i.e., $\alpha(I) \subseteq I$) of R, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a)+I$ for $a \in R$ is an endomorphism of a factor ring R/I.

***Example* 2.8** (1) Let K be a field and $R = K\langle a, b\rangle$ be the free algebra with non-commuting indeterminates a, b over K. Then R is a domain. Define an automorphism α of R by $a \mapsto b$ and $b \mapsto a$. Then R is obviously an α-nil-shifting ring. Now, let I be the ideal of R generated by ab, a^2 and b^3. For $a + I, b + I \in N(R/I)$, we get $(a+I)\bar{\alpha}((b+I)) = (a+I)(\alpha(b)+I)) = a^2+I = I$, but $(b+I)\bar{\alpha}((a+I)) = b^2+I \neq I$ by the construction of I. Thus, R/I is not right $\bar{\alpha}$-nil-shifting. This concludes that the class of right α-nil-shifting rings is not closed under homomorphic images.

(2) We refer to [2, Example 2.8]. Let A be a reduced ring and consider a ring $R = U_3(A)$ with an endomorphism α defined by

$$\alpha\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

Then $N(R) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d \in A \right\}$. For $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in N(R)$, we obtain $x\alpha(y) = 0$, but $y\alpha(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. Thus, R is not right α-nil-shifting. Now, for a nonzero proper ideal $I = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & A \\ 0 & 0 & A \end{pmatrix}$ of R, $R/I \cong U_2(A)$ is right $\bar{\alpha}$-nil-shifting by Proposition 2.5(1) and obviously, $\alpha(I) \subseteq I$.

Theorem 2.9 Let R be a ring with an endomorphism α. For an α-ideal I of R, let R/I be a right $\bar{\alpha}$-nil-shifting ring for some ideal I of a ring R with $\alpha(I) \subseteq I$. If I is an α-rigid as a ring without identity, then R is a right α-nil-shifting ring.

Proof. Let $a\alpha(b) = 0$ for $a, b \in N(R)$. Then $b\alpha(a) \in I$ since R/I is a right $\bar{\alpha}$-nil-shifting ring. Then $b\alpha(a)\alpha(b\alpha(a)) = 0$ and so $b\alpha(a) = 0$, since $b\alpha(a) \in I$ and I is an α-rigid (and so reduced). Thus, R is right α-nil-shifting. ■

The condition "I is α-rigid as a ring without identity" in Theorem 2.9 is not superfluous by Example 2.8(2): In fact, $(x + I)\bar{\alpha}(x + I) = I$, where $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin I$.

For a ring R with an endomorphism α and $n \geqslant 2$, the corresponding $(a_{ij}) \to (\alpha(a_{ij}))$ induces an endomorphism of $Mat_n(R)$, $U_n(R)$ and $D_n(R)$, respectively. We denote them by $\bar{\alpha}$. Notice that for a reduced ring R, both $U_2(R)$ and $D_2(R)$ are $\bar{\alpha}$-nil-shifting for any endomorphism α of R by Proposition 2.5(1). We will freely use these facts without reference.

However, there exists a reduced ring A with an endomorphism α such that $Mat_2(A)$ is not right $\bar{\alpha}$-nil-shifting as follows.

***Example* 2.10** Define an automorphism α of $\mathbb{Z}_2$ by $0 \mapsto 1$ and $1 \mapsto 0$. Consider $R = Mat_2(\mathbb{Z}_2)$. For $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in N(R)$, we have

$$a\bar{\alpha}(b) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bar{\alpha}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha(1) & \alpha(1) \\ \alpha(1) & \alpha(1) \end{pmatrix} = 0,$$

but

$$b\bar{\alpha}(a) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha(0) & \alpha(1) \\ \alpha(0) & \alpha(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0.$$

Thus, $Mat_2(A)$ is not right $\bar{\alpha}$-nil-shifting.

***Remark* 1** *Note that $Mat_n(R)$, $D_n(R)$ and $U_n(R)$, for $n \geqslant 3$ are not right $\bar{\alpha}$-nil-shifting for any ring R with an endomorphism α such that $\alpha(1) \neq 0$ (e.g., α is a monomorphism). Let R be a ring with an endomorphism α such that $\alpha(1) \neq 0$. For the ring $D_3(R)$, consider $e_{12}, e_{23} \in N(D_3((R)))$. Then $e_{23}\bar{\alpha}(e_{12}) = 0$, but $e_{12}\bar{\alpha}(e_{23}) = \begin{pmatrix} 0 & 0 & \alpha(1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ showing that $D_3(R)$ is not right $\bar{\alpha}$-nil-shifting.*

Similarly, we can show that $D_m(R)$ for $m \geqslant 4$ is not right $\bar{\alpha}$-nil-shifting. Consequently, it can be obtained that $Mat_n(R)$ and $U_n(R)$ for $n \geqslant 3$ are not right $\bar{\alpha}$-nil-shifting, since the class of α-nil-shifting rings is closed under subrings S with $\alpha(S) \subseteq S$.

One may ask whether both $D_2(R)$ and $U_2(R)$ are right $\bar{\alpha}$-nil-shifting when either R is a reversible ring or R is a right α-nil-shifting ring with an endomorphism α. However the answer is negative by the following example.

***Example* 2.11** (1) We apply the ring construction and argument in [13, Example 2.1]. Consider the free algebra $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Define an automorphism δ of A by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c,$$

respectively. Let B be the set of all polynomials with zero constant terms in A and consider the ideal I of A generated by

$$\begin{aligned}&a_0a_0, a_0a_1 + a_1a_0, a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\ &b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\ &(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2) \text{ and } r_1r_2r_3r_4,\end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \subseteq I$. Set $R = A/I$. Since $\delta(I) \subseteq I$, we can obtain an automorphism α of R by defining $\alpha(s + I) = \delta(s) + I$ for $s \in A$. We identify every element of A with its image in R for simplicity. Then R is a reversible ring by the argument in [13, Example 2.1]. Note that R is not right α-nil-shifting, since $a_0\alpha(b_0) = a_0\delta(b_0) = a_0a_0 = 0$ for $a_0, b_0 \in N(R)$, but $b_0\alpha(a_0) = b_0\delta(a_0) = b_0^2 \neq 0$.

Now, we show that $D_2(R)$ is not right $\bar{\alpha}$-nil-shifting. For $x = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}$ and $y = \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix}$ in $N(D_2(R))$, we have $x\bar{\alpha}(y) = 0$ by the construction of I. But

$$y\bar{\alpha}(x) = \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} \bar{\alpha}\left(\begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}\right) = \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} \begin{pmatrix} \delta(a_0) & \delta(a_1) \\ 0 & \delta(a_0) \end{pmatrix} = \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix}^2 \neq 0,$$

entailing that $D_2(R)$ is not right $\bar{\alpha}$-nil-shifting. Therefore, we conclude that both $D_n(R)$ and $U_n(R)$ for $n \geqslant 2$ need not be right $\bar{\alpha}$-nil-shifting.

(2) We use [2, Example 3.5]. Consider a ring $R = U_2(A)$ over a reduced ring A and an endomorphism α of R is defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

Then R is right α-nil-shifting by Proposition 2.5(1). Clearly R is not reversible. For

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(D_2(R))$$

with $A^3 = 0$ and $B^2 = 0$, we have $A\bar{\alpha}(B) = 0$ but

$$B\bar{\alpha}(A) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) & \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \\ \alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) & \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Thus, $D_2(R)$ is not right $\bar{\alpha}$-nil-shifting, and it implies that $D_n(R)$ and $U_n(R)$ for $n \geqslant 2$ need not be right $\bar{\alpha}$-nil-shifting, when R is right α-nil-shifting with an endomorphism α.

Theorem 2.12 For a ring R with an endomorphism α, the following are equivalent:
(1) R is α-rigid;
(2) $U_2(R)$ is $\bar{\alpha}$-nil-shifting;
(3) $U_2(R)$ is right $\bar{\alpha}$-nil-shifting.

Proof. Recall that if R is an α-rigid ring, then R is reduced and $\alpha(1) = 1$ by [9, Proposition 5]. So, it is enough to show that (3)⇒(1). Let $U_2(R)$ be right $\bar{\alpha}$-nil-shifting and assume on the contrary that R is not α-rigid. Then there exists $0 \neq a \in R$ with $a\alpha(a) = 0$. For $A = \begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in N(U_2(R))$, we have $A\bar{\alpha}(B) = 0$ but $B\bar{\alpha}(A) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq 0$, entailing that $U_2(R)$ is not right $\bar{\alpha}$-shifting. This induces a contradiction, and so such a cannot exist. Thus R is α-rigid. ■

As a corollary of Proposition 2.5(4) and Theorem 2.12, we get the following.

Corollary 2.13 [1, Theorem 2.7] A ring R is reduced if and only if $U_2(R)$ is a CNZ ring.

The ring "$U_2(R)$" in Theorem 2.12 cannot be replaced by the ring "$D_2(R)$" as follows.

***Example* 2.14** Consider the direct sum $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the endomorphism defined by $\alpha((a, b)) = (b, a)$. Then R is a commutative reduced ring, and so $D_2(R)$ is $\bar{\alpha}$-nil-shifting. But R is not α-rigid, since $(1, 0)\alpha((1, 0)) = (0, 0)$ for $(1, 0) \in R$.

For a ring R and $n \geqslant 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$. Note that $V_n(R)$ over an α-rigid ring R is $\bar{\alpha}$-shifting by [5, Theorem 3.13(2)] and hence $\bar{\alpha}$-nil-shifting.

3. Extensions of right α-nil-shifting rings

For a ring R with an endomorphism α, we denote $R[x; \alpha]$ a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i, a_i \in R$, where the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. The set $\{x^j\}_{j \geqslant 0}$ is easily seen to be a left Ore subset of $R[x; \alpha]$, so that one can localize $R[x; \alpha]$ and form the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Elements of $R[x, x^{-1}; \alpha]$ are finite sums of elements of the form $x^{-j} a x^i$ where $a \in R$ and i and j are nonnegative integers. The skew power series ring is denoted by $R[[x; \alpha]]$, whose elements are the series $\sum_{i=0}^{\infty} a_i x^i$ for some $a_i \in R$ and nonnegative integers i. The *skew Laurent power series* ring $R[[x, x^{-1}; \alpha]]$ which contains

$R[[x;\alpha]]$ as a subring, arises as the localization of $R[[x;\alpha]]$ with respect to the Ore set $\{x^j\}_{j\geqslant 0}$, and when α is an automorphism of R, it consists elements of the form $a_s x^s + a_{s+1}x^{s+1} + \cdots + b_0 + b_1 x + \cdots$, for some $a_i, b_j \in R$ and integers s, i, j, where the addition is defined as usual and the multiplication is defined by the rule $xa = \alpha(a)x$ for any $a \in R$. Note that $\alpha(1) = 1$ for any skew Laurent power series (skew Laurent polynomial) ring $R[[x, x^{-1};\alpha]](R[x, x^{-1};\alpha])$, since $1x^n = x^n = x1x^{n-1} = \alpha(1)x^n$ for any $n \geqslant 1$ where 1 is the identity of R. For a ring R with endomorphism α, the corresponding $\sum a_i x^i \to \sum \alpha(a_i)x^i$ induces an endomorphism of $R[x;\alpha]$, $R[x, x^{-1};\alpha]$, $R[[x;\alpha]]$ and $R[[x, x^{-1};\alpha]]$, respectively. We denote them by $\bar{\alpha}$.

The concept of a right α-nil-shifting ring does not go up to skew polynomial rings (skew power series rings) by next example.

***Example* 3.1** We adapt the ring in [12, Example 2.8], based on [13, Example 2.1]. Take the same A and the automorphism δ of A as in Example 2.11. Let C be the set of all polynomials with zero constant terms in A and consider the ideal I of A generated by

$$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,$$

$$b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,$$

$$(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2),$$

$$a_0a_0, a_2a_2, a_0ra_0, a_2ra_2, b_0b_0, b_2b_2, b_0rb_0, b_2rb_2, r_1r_2r_3r_4,$$

$$a_0a_1 + a_1a_0, a_0a_2 + a_1a_1 + a_2a_0, a_1a_2 + a_2a_1, b_0b_1 + b_1b_0, b_0b_2 + b_1b_1 + b_2b_0, b_1b_2 + b_2b_1,$$

$$(a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2),$$

where $r, r_1, r_2, r_3, r_4 \in C$. Then clearly $C^4 \subseteq I$. Set $R = A/I$. Since $\delta(I) \subseteq I$, we can obtain an automorphism α of R by defining $\alpha(s + I) = \delta(s) + I$ for $s \in A$. We identify every element of A with its image in R for simplicity. For $p(x) = a_0 + a_1x^2 + a_2x^4, q(x) = b_0c + b_1cx^2 + b_2cx^4 \in N(R[x;\alpha])$, since $C^4 \subseteq I$ we have

$$p(x)\bar{\alpha}(q(x)) = (a_0 + a_1x^2 + a_2x^4)(a_0c + a_1cx^2 + a_2cx^4) = 0$$

but, since $b_0cb_1 + b_1cb_0 \neq 0$ we have

$$q(x)\bar{\alpha}(p(x)) = (b_0c + b_1cx^2 + b_2cx^4)(b_0 + b_1x^2 + b_2x^4) \neq 0.$$

Thus $R[x;\alpha]$ is not right $\bar{\alpha}$-nil-shifting ring. Notice that R is reversible and right α-skew CNZ by [13, Example 2.1] and [2, Example 3.6], respectively. Thus R is a right α-nil-shifting ring by Proposition 2.5(2-ii), since α is an automorphism of R.

Following [3], a ring R is called skew power-serieswise α-Armendariz if $a_ib_j = 0$ for all i and j whenever $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{\infty} a_ix^i, q(x) = \sum_{j=0}^{\infty} b_jx^j \in R[[x;\alpha]]$. It is shown that R is a α-rigid ring if and only if R is reduced and skew power-serieswise α-Armendariz in [3, Theorem 3.3(1)]. It is obvious that skew power-serieswise α-Armendariz property of a ring is inherited to its subrings, and α is clearly a monomorphism by help of [3, Theorem 3.3(3)]. (We also change over from "a skew power series Armendariz ring with the endomorphism α" in [3] to "a skew power-serieswise α-Armendariz ring".)

Note that every skew power-serieswise α-Armendariz ring is α-compatible by help of [12, Proposition 3.14], and thus the concepts of CNZ rings, right α-skew CNZ rings and right α-nil-shifting rings are coincided in skew power-serieswise α-Armendariz rings.

Lemma 3.2 [16, Theorem 2.13] Let R be a skew power-serieswise α-Armendariz ring and α an automorphism of R. If we let S is one of symbols $R[x;\alpha], R[x, x^{-1};\alpha], R[[x;\alpha]]$

or $R[[x, x^{-1}; \alpha]]$, then $N(RS) = N(R)S$.

Theorem 3.3 Let R be a skew power-serieswise α-Armendariz ring and α an automorphism of R. Then the following are equivalent:
(1) R is right α-nil-shifting.
(2) $R[x; \alpha]$ is a right $\bar{\alpha}$-nil-shifting.
(3) $R[x, x^{-1}; \alpha]$ is a right $\bar{\alpha}$-nil-shifting.
(4) $R[[x; \alpha]]$ is a right $\bar{\alpha}$-nil-shifting.
(5) $R[[x, x^{-1}; \alpha]]$ is a right $\bar{\alpha}$-nil-shifting.

Proof. It suffices to show that (1)⇒(5): Assume that (1) holds R is right α-nil-shifting. Let $p(x)\bar{\alpha}(q(x)) = 0$ for $p(x) = \sum_{i=0}^{\infty} a_i x^i, q(x) = \sum_{j=0}^{\infty} b_j x^j \in N(R[[x, x^{-1}; \alpha]])$. Then $a_i, b_j \in N(R)$ by Lemma 3.2 and so $a_i\alpha(b_j) = 0$ for all i, j. Thus, $b_j\alpha(a_i) = 0$ by (1) and $b_j\alpha^n(a_i) = 0$ for any non negative integer n, since R is α-compatible as noted above. This yields $q(x)\bar{\alpha}(p(x)) = 0$, and thus, $R[[x, x^{-1}; \alpha]]$ is right $\bar{\alpha}$-nil-shifting. ■

Let R be a ring and α a monomorphism of R. Now, we consider the Jordan's construction of an over-ring of R by α (see [11] for more details). Let $A(R, \alpha)$ be the subset $\{x^{-i}rx^i \mid r \in R \text{ and } i \geqslant 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Note that for $j \geqslant 0$, $x^j r = \alpha^j(r)x^j$ implies $rx^{-j} = x^{-j}\alpha^j(r)$ for $r \in R$. This yields that for each $j \geqslant 0$ we have $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{i+j}$. It follows that $A(R, \alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following natural operations: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$ for $r, s \in R$ and $i, j \geqslant 0$. Note that $A(R, \alpha)$ is an over-ring of R, and the map $\bar{\alpha} : A(R, \alpha) \to A(R, \alpha)$ defined by $\bar{\alpha}(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$ is an automorphism of $A(R, \alpha)$. Jordan showed, with the use of left localization of the skew polynomial $R[x; \alpha]$ with respect to the set of powers of x, that for any pair (R, α), such an extension $A(R, \alpha)$ always exists in [11]. This ring $A(R, \alpha)$ is usually said to be the Jordan extension of R by α.

Proposition 3.4 For a ring R with a monomorphism α, R is right α-nil-shifting if and only if the Jordan extension $A = A(R, \alpha)$ of R by α is right $\bar{\alpha}$-nil-shifting.

Proof. It is sufficient to show the necessity. Suppose that R is right α-nil-shifting and $c\bar{\alpha}(d) = 0$ for $c = x^{-i}rx^i, d = x^{-j}sx^j \in N(A)$ for $i, j \geqslant 0$. Then $r, s \in N(R)$ obviously. From $c\bar{\alpha}(d) = 0$, we get $\alpha^j(r)\alpha^{i+1}(s) = 0$ and so $0 = \alpha^i(s)\alpha(\alpha^j(r)) = \alpha^i(s)\alpha^{j+1}(r)$ by hypothesis. Hence,

$$\begin{aligned} d\bar{\alpha}(c) &= (x^{-j}sx^j)\bar{\alpha}(x^{-i}rx^i) = (x^{-j}sx^j)(x^{-i}\alpha(r)x^i) \\ &= x^{-(j+i)}\alpha^i(s)\alpha^j(\alpha(r))x^{i+j} = x^{-(j+i)}\alpha^i(s)\alpha^{j+1}(r))x^{i+j} = 0. \end{aligned}$$

Therefore, the Jordan extension A is right $\bar{\alpha}$-nil-shifting. ■

Let R be an algebra over a commutative ring S. Due to Dorroh [7], the Dorroh extension of R by S is the Abelian group $R \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$. We use D to denote the Dorroh extension of R by S. For an S-endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

Theorem 3.5 Let R be an algebra over a commutative reduced ring S with an S-endomorphism α. Then R is a right α-nil-shifting ring if and only if the Dorroh extension D of R by S is a right $\bar{\alpha}$-nil-shifting.

Proof. It can be easily checked that $N(D) = (N(R), 0)$ since S is a commutative reduced ring. Then every nilpotent element D is of the form $(r, 0)$ for some nilpotent element r of R. Thus, $(r_1, 0)\bar{\alpha}((r_2, 0)) = (0, 0)$ if and only if $r_1\alpha(r_2) = 0$. This implies that R is right α-nil-shifting if and only if the Dorroh extension D is $\bar{\alpha}$-nil-shifting. ■

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* is defined, and *regular* means if it is both left and right regular (and hence not a zero divisor). Assume that M is a multiplicatively closed subset of R consisting of central regular elements. Let α be an automorphism of R and assume $\alpha(m) = m$ for every $m \in M$. Then $\alpha(m^{-1}) = m^{-1}$ in $M^{-1}R$ and the induced map $\bar{\alpha}m : M^{-1}R \to M^{-1}R$ defined by $\bar{\alpha}(u^{-1}a) = u^{-1}\alpha(a)$ is also an automorphism.

Proposition 3.6 Let R be a ring with an automorphism α and assume that there exists a multiplicatively closed subset M of R consisting of central regular elements and $\alpha(m) = m$ for every $m \in M$. Then R is a right α-nil-shifting ring if and only if $M^{-1}R$ is a right $\bar{\alpha}$-nil-shifting ring.

Proof. It suffices to prove the necessary condition. First, note that $N(M^{-1}R) = M^{-1}N(R)$. Suppose that R is right α-nil-shifting. Let $A\bar{\alpha}(B) = 0$ with $A = u^{-1}a$, $B = v^{-1}b \in N(M^{-1}R)$ where $u, v \in M$ and $a, b \in N(R)$. Then $a\alpha(b) = 0$ and so $b\alpha(a) = 0$ by assumption. Thus,

$$B\bar{\alpha}(A) = v^{-1}b\bar{\alpha}(u^{-1}a) = v^{-1}u^{-1}b\alpha(a) = 0$$

showing that $M^{-1}R$ is a right $\bar{\alpha}$-nil-shifting ring. ■

Let R be a ring with an endomorphism α. Recall that the map $R[x] \to R[x]$ (resp., $R[x, x^{-1}] \to R[x, x^{-1}]$) defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i)x^i$ (resp., $\sum_{i=0}^{\infty} a_i x^i \mapsto \sum_{i=0}^{\infty} \alpha(a_i)x^i$) is an endomorphism of $R[x]$ (resp., $R[x, x^{-1}]$), and clearly the map extends α. We still denote the extended maps $R[x] \to R[x]$ and $R[x, x^{-1}] \to R[x, x^{-1}]$ by $\bar{\alpha}$.

Corollary 3.7 Let R be a ring with an endomorphism α such that $\alpha(1) = 1$. Then $R[x]$ is a right $\bar{\alpha}$-nil-shifting if and only if $R[x; x^{-1}]$ is a right $\bar{\alpha}$-nil-shifting.

Proof. It directly follows from Proposition 3.6. For, letting $M = \{1, x, x^2, \cdots\}$, M is a multiplicatively closed subset of $R[x]$ such that $R[x, x^{-1}] = M^{-1}R[x]$ and $\bar{\alpha}(x) = x$ since $\alpha(1) = 1$. ■

A ring R is called right Ore if for given $a, b \in R$ with b is regular, there exists $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known fact that R is a right Ore ring if and only if the classical right quotient ring $Q(R)$ of R exists. Let R be a ring with the classical right quotient ring $Q(R)$. Then each automorphism α of R extends to $Q(R)$ by setting $\bar{\alpha}(ab^{-1}) = \alpha(a)(\alpha(b))^{-1}$ for $a, b \in R$, assuming that $\alpha(b)$ is regular for each regular element $b \in R$.

Recall that a ring R is called *NI* [15] if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced.

Theorem 3.8 Let R be a right Ore ring with the classical right quotient ring $Q(R)$ of R and α an automorphism of R. If $Q(R)$ is an NI ring, then R is a right α-nil-shifting ring if and only if $Q(R)$ is a right $\bar{\alpha}$-nil-shifting ring.

Proof. It suffices to establish the necessity. Let $Q(R)$ be an NI ring and R be a right α-nil-shifting. Then R is NI by [10, Lemma 2.1]. We freely use these assumption without reference in the following procedure. Let $A\bar{\alpha}(B) = 0$ for $A = ab^{-1}$, $B = cd^{-1} \in N(Q(R))$, where $a, b, c, d \in R$ with b, d regular. Set I and J be the ideals of $Q(R)$ generated by A and $\bar{\alpha}(B)$, respectively. Then both I and J are nil with $a = Ab \in I$ and $\alpha(c) =$

$B\alpha(d) \in J$, and so $a, \alpha(c) \in N(R)$ and moreover $c \in N(R)$. Since R is right Ore, there exist $c_1, b_1 \in R$ with b_1 regular such that $bc_1 = \alpha(c)b_1$ and $c_1 b_1^{-1} = b^{-1}\alpha(c)$. Here note that $c_1 \in N(R)$. Indeed, $bc_1 = \alpha(c)b_1 \in J$ and so $c_1 = b^{-1}(bc_1) \in J$. From $0 = A\bar{\alpha}(B) = ab^{-1}\bar{\alpha}(cd^{-1}) = ab^{-1}\alpha(c)\alpha(d)^{-1} = ac_1 b_1^{-1}\alpha(d^{-1})$, we have $0 = ac_1 = a\alpha(c')$ putting $c_1 = \alpha(c')$ for some $c' \in N(R)$ and so $c'\alpha(a) = 0$ implies $c'\alpha(a)\alpha(b) = 0$. Thus $c'\alpha(ab) = 0 \Rightarrow ab\alpha(c') = 0 \Rightarrow abc_1 = 0 \Rightarrow a\alpha(c)b_1 = 0 \Rightarrow a\alpha(c) = 0$ and $c\alpha(a) = 0$.

Now for $a \in N(R)$, $d \in R$ with d regular, there exist $a_1 \in N(R)$, $d_1 \in R$ with d_1 regular such that $da_1 = \alpha(a)d_1$ where $\alpha(a) \in N(R)$ and $a_1 d_1^{-1} = d^{-1}\alpha(a)$ by the same computation as above. Then $a_1 = d^{-1}\alpha(a)d_1 \in N(R)$ because $\alpha(a) \in N(R)$. Put $a_1 = \alpha(a')$ for some $a' \in N(R)$. Then, we have

$$\begin{aligned} & 0 = c\alpha(a) = c\alpha(a)d_1 = cda_1 = cd\alpha(a') \\ \Rightarrow\ & 0 = a'\alpha(c)\alpha(d) \ \Rightarrow\ a'\alpha(c) = 0, \text{ since } \alpha(d) \text{ is regular} \\ \Rightarrow\ & 0 = c\alpha(a') = ca_1. \end{aligned}$$

Thus,

$$B\bar{\alpha}(A) = cd^{-1}\bar{\alpha}(ab^{-1}) = c(d^{-1}\alpha(a))\alpha(b)^{-1} = ca_1 d_1^{-1}\alpha(b)^{-1} = 0,$$

concluding that $Q(R)$ is right $\bar{\alpha}$-nil-shifting. ■

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