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Two new characteristic subgroups

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Abstract. In this paper, we first define two new characteristic subgroups of a group G. Then we identify the relationships of these subgroups with G', $S(G)$, $Ivar(G)$, and some different homomorphisms. Particularly, with one of these two subgroups, we determine the structure of $Ivar(G)$ and a subgroup of it that fixes $Z(G)$ element-wise.

Keywords: IA-group, commutator subgroup, IA-central subgroup, $Ivar(G)$.

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1. Introduction and preliminaries

The center of a group and its subgroups have interesting properties. All kinds of automorphisms are also of very importance, and so these groups have been studied by many researchers. For a group G, let us denote by G', $Z(G)$, $Ker(G)$, $Hom(G, H)$, $Inn(G)$ and $Aut(G)$, the commutator subgroup, the centre, the kernel, the group of homomorphisms of G into an abelian group H, the inner automorphisms and the full automorphism group, respectively. For $g \in G$ and $\alpha \in Aut(G)$, $[g, \alpha] = g^{-1}\alpha(g)$ is the autocommutator of g and α.

In 1965, Bachmuth [1] defined an IA-automorphism as an automorphism of a group G that preserves all cosets of G'. In other words,

$$IA(G) = \left\{\alpha \in Aut(G) \;\middle|\; [g, \alpha] \in G',\ \forall\, g \in G\right\}.$$

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In 1994, Hegarty [3] introduced the absolute center $L(G)$ and autocommutator $K(G)$ subgroups as follows:

$$L(G) = \left\{ g \in G \;\middle|\; [g,\alpha] = 1, \ \forall\, \alpha \in Aut(G) \right\},$$
$$K(G) = \left\langle\, [g,\alpha] \;\middle|\; g \in G, \ \alpha \in Aut(G) \,\right\rangle = [G, Aut(G)].$$

Also, $Aut_c(G) = \left\{ \alpha \in Aut(G) \;\middle|\; [g,\alpha] \in Z(G), \ \forall\, g \in G \right\}$ is the central automorphism group. Since $Aut(G)$ acts on G via automorphisms, we see that $C_G\big(Aut(G)\big) = L(G)$ is the set of fixed points of this action. Also, it is clear that $L(G) \subseteq Z(G)$ and $Inn(G) \subseteq IA(G)$.

On the lines of the results of Schur [4] and Hegarty [3], in 2015, Ghumde and Ghate [2] introduced the $S(G)$ subgroup as follows:

$$\{ g \in G \mid [g,\alpha] = 1, \ \alpha \in IA(G) \}.$$

Also, we can consider $S(G)$ by $S(G) := C_G\big(IA(G)\big)$. Since $Inn(G) \subseteq IA(G)$, we have

$$L(G) = C_G\big(Aut(G)\big) \subseteq S(G) = C_G\big(IA(G)\big) \subseteq C_G\big(Inn(G)\big) = Z(G),$$

whence $L(G) \trianglelefteq S(G) \trianglelefteq Z(G)$. In [2], Ghumde and Ghate showed that if G is a finite p-group, then $S(G)$ is non-trivial. Afterward, they introduced $Ivar(G)$ subgroup as follows:

$$\{ \alpha \in IA(G) \mid [g,\alpha] \in S(G), \ \forall\, g \in G \}.$$

In this paper, by using these definitions, we introduce two subgroups that are related to them. One of these new subgroups is denoted by $\mathcal{E}(G)$. We prove that $Ivar(G)$ acts trivially on $\mathcal{E}(G)$. Then we determine the structure of $Ivar(G)$, where $S(G) \leqslant \mathcal{E}(G)$ or $S(G)$ and $G/\mathcal{E}(G)$ are torsion-free or $Z(G) \leqslant \mathcal{E}(G)$. Also, we determine the structure of the group of automorphisms of $Ivar(G)$ fixing $Z(G)$ element-wise. However, before providing them, we need the following results. We write $H \overset{ch}{\leqslant} G$ if H is a characteristic subgroup of G.

Proposition 1.1 Let G be a group. Then $S(G)$ is a characteristic subgroup of G.

Proof. As we know, $S(G) \leqslant G$ and $S(G) \leqslant Z(G) \overset{ch}{\leqslant} G$. We prove that $S(G) \overset{ch}{\leqslant} Z(G)$, then $S(G) \overset{ch}{\leqslant} G$ by [5, 2.11.12]. Let $\beta \in Aut\big(Z(G)\big)$ and $s \in S(G)$. We show that $\beta(s) \in S(G)$. By definition $IA(G)$, $[\beta(s), \alpha] = \big(\beta(s)\big)^{-1}\alpha\big(\beta(s)\big) \in G'$ for every $\alpha \in IA(G)$. As $\beta(s) \in Z(G)$, so $S(G) \leqslant Z(G) \leqslant G'$. For abelian group $Z(G)$, $Aut\big(Z(G)\big) = Aut_c\big(Z(G)\big)$, therefore $\beta \in Aut\big(Z(G)\big) = Aut_c\big(Z(G)\big)$. Since $\beta(s) \in Z(G) \leqslant G'$ and the central automorphisms fix G' pointwise, so $\beta(s) = s \in S(G)$. ■

Theorem 1.2 If G is a group, then $Ivar(G)$ is a non-trivial normal subgroup of $Aut(G)$.

Proof. For every arbitrary group G, the identity automorphism is an element of $Ivar(G)$. Therefore, $Ivar(G) \neq \emptyset$. According to the previous proposition, it is clear that $Iver(G)$ is a subgroup of $Aut(G)$, so we only prove that the $Iver(G)$ is normal in $Aut(G)$. Let $\beta \in Aut(G)$ and $\alpha \in Ivar(G)$ be arbitrary. We show that $\beta^{-1}\alpha\beta \in Ivar(G)$. For every $g \in G$, we have $\beta^{-1}(g)\alpha\big(\beta(g)\big) \in S(G)$. Thus, there exists $s_0 \in S(G)$ such that

$\beta^{-1}(g)\alpha\big(\beta(g)\big) = s_0$. Now,

$$\begin{aligned} g^{-1}\big(\beta^{-1}\alpha\beta(g)\big) &= g^{-1}\beta^{-1}\big(\alpha\beta(g)\big) \\ &= g^{-1}\beta^{-1}\big(\beta(g)\beta^{-1}(g)\alpha\beta(g)\big) \\ &= g^{-1}\beta^{-1}\big(\beta(g)s_0\big) \\ &= g^{-1}g\beta^{-1}(s_0) \\ &= \beta^{-1}(s_0) \in \beta^{-1}\big(S(G)\big). \end{aligned}$$

Since $S(G) \overset{ch}{\leqslant} G$, then $\beta^{-1}\big(S(G)\big) = S(G)$. Thus, $g^{-1}\big(\beta^{-1}\alpha\beta(g)\big) \in S(G)$, and the proof ends. ■

Proposition 1.3 Let G be a group. Then

$$Ivar(G) \cong Hom\Big(\frac{G}{S(G)}, S(G) \cap G'\Big).$$

In particular, $Ivar(G)$ is an abelian group.

Proof. Consider the map $\alpha^* :\ G/S(G) \longrightarrow S(G) \cap G'$ defined by $\alpha^*\big(gS(G)\big) = g^{-1}\alpha(g)$ for all $g \in G$ and each $\alpha \in Ivar(G)$. Since every automorphism in $Ivar(G)$ acts trivially on $S(G)$, α^* is a well-defined homomorphism of $G/S(G)$ to $S(G) \cap G'$. Now, it is easy to check that $\psi :\ Ivar(G) \longrightarrow Hom\big(G/S(G), S(G) \cap G'\big)$, defined by $\psi(\alpha) = \alpha^*$ for any $\alpha \in Ivar(G)$, is an isomorphism.
For the second part, we know that $S(G) \cap G' \leqslant S(G)$ is an abelian group, so $\alpha\beta\big(gS(G)\big) = \beta\alpha\big(gS(G)\big)$ for each $\alpha, \beta \in Hom\big(G/S(G), S(G) \cap G'\big)$ and $g \in G$. Thus, $Hom\big(G/S(G), S(G) \cap G'\big)$ is an abelian group. Now, the result follows by the first part.
■

2. Main Results

In this section, we first introduce two new subgroups and investigate their properties and the relations of these subgroups with G', $S(G)$, $Ivar(G)$ and some different homomorphisms. Then we give our main results about the behavior of $Ivar(G)$, and its members that fix $Z(G)$ element-wise.

Definition 2.1 Let G be a group and

$$\begin{aligned} C_{Aut(G)}\big(Ivar(G)\big) &= \{\alpha \in Aut(G) \mid \sigma\alpha = \alpha\sigma,\ \forall\ \sigma \in Ivar(G)\}, \\ C_{IA(G)}\big(Ivar(G)\big) &= \{\alpha \in IA(G) \mid \sigma\alpha = \alpha\sigma,\ \forall\ \sigma \in Ivar(G)\}, \end{aligned}$$

be the centralizers of $Ivar(G)$ in $Aut(G)$ and $IA(G)$, respectively. We define $\xi(G) = [G, C_{Aut(G)}\big(Ivar(G)\big)]$ and $\mathcal{E}(G) = [G, C_{IA(G)}\big(Ivar(G)\big)]$.

It is obvious that $\mathcal{E}(G) \leqslant \xi(G) \leqslant K(G)$. For example, if G is an abelian group, then $\xi(G) = K(G)$ and $\mathcal{E}(G) = 1$.

Proposition 2.2 Let G be a group. Then $G' \leqslant \xi(G) \overset{ch}{\leqslant} G$ and $G' \leqslant \mathcal{E}(G) \overset{ch}{\leqslant} G$.

Proof. Clearly, $\xi(G) \leqslant G$. Let $[g,\alpha] \in \xi(G)$ and $\sigma \in Aut(G)$. Then

$$\begin{aligned}\sigma([g,\alpha]) &= \sigma\big(g^{-1}\alpha(g)\big)\\ &= \sigma(g^{-1})\sigma\big(\alpha(g)\big)\\ &= \sigma(g^{-1})\sigma\alpha\big(\sigma^{-1}\sigma(g)\big)\\ &= \big(\sigma(g)\big)^{-1}\sigma\alpha\sigma^{-1}\big(\sigma(g)\big)\\ &= [\sigma(g), \sigma\alpha\sigma^{-1}].\end{aligned}$$

It will be enough to show that $\sigma\alpha\sigma^{-1} \in C_{Aut(G)}\big(Ivar(G)\big)$. Let $\beta \in Ivar(G)$. We must show that $(\sigma\alpha\sigma^{-1})\beta = \beta(\sigma\alpha\sigma^{-1})$. By Theorem 1.2, $Ivar(G) \trianglelefteq Aut(G)$. Hence, $\sigma^{-1}\beta\sigma \in Ivar(G)$. Since $\alpha \in C_{Aut(G)}\big(Ivar(G)\big)$, we can write

$$(\sigma\alpha\sigma^{-1})\beta = \sigma\alpha\sigma^{-1}\beta\sigma\sigma^{-1} = \sigma\sigma^{-1}\beta\sigma\alpha\sigma^{-1} = \beta(\sigma\alpha\sigma^{-1}).$$

Thus, $\sigma([g,\alpha]) = [\sigma(g), \sigma\alpha\sigma^{-1}] \in \xi(G)$. Now, we show that $G' \leqslant \xi(G)$. Given that $S(G)$ is contained in $Z(G)$, $Ivar(G) \leqslant Aut_c(G)$. As every automorphism in $Aut_c(G)$ commutes with each member of $Inn(G)$, $Inn(G) \leqslant C_{Aut(G)}\big(Ivar(G)\big)$. Now, we have

$$G' = [G, Inn(G)] \subseteq [G, C_{Aut(G)}\big(Ivar(G)\big)] = \xi(G).$$

The second relation $G' \leqslant \mathcal{E}(G) \overset{ch}{\leqslant} G$ follows similarly. ■

Lemma 2.3 Let G be a group. Then $Ivar(G)$ acts trivially on $\mathcal{E}(G)$.

Proof. Let $\alpha \in Ivar(G)$ be an arbitrary automorphism. Then $g^{-1}\alpha(g) \in S(G)$ for all $g \in G$ and hence, $\alpha(g) = gs$ for some $s \in S(G)$. Now, let $\beta \in C_{IA(G)}\big(Ivar(G)\big)$ be arbitrary. Then using the property of β and $[g,\beta] \in \mathcal{E}(G)$, we have

$$\begin{aligned}\alpha([g,\beta]) &= \alpha\big(g^{-1}\beta(g)\big)\\ &= \big(\alpha(g)\big)^{-1}\alpha\big(\beta(g)\big)\\ &= s^{-1}g^{-1}\beta\big(\alpha(g)\big)\\ &= s^{-1}g^{-1}\beta(gs)\\ &= s^{-1}g^{-1}\beta(g)\beta(s)\\ &= s^{-1}g^{-1}\beta(g)s\\ &= g^{-1}\beta(g)\\ &= [g,\beta]\end{aligned}$$

for all $g \in G$, which gives the result. ■

The next theorem provides the properties of $Ivar(G)$when $S(G)$ is torsion-free.

Theorem 2.4 Let G be a group with $S(G)$ torsion-free. Then

(1) $Ivar(G)$ is torsion-free.
(2) If $G/\mathcal{E}(G)$ is torsion, then $Ivar(G) = \langle 1 \rangle$.

Proof. For part (1), it will be enough to prove by Proposition 1.3 that $Hom\big(G/S(G), S(G)\cap G'\big)$ is torsion-free. Let $\alpha \in Hom\big(G/S(G), S(G)\cap G'\big)$ be arbitrary and non-trivial. Then $\alpha\big(gS(G)\big) \neq 1$ for some $gS(G) \in G/S(G)$. By the assumption, $S(G)$ is a torsion-free group and so $\alpha^n\big(gS(G)\big) \neq 1$ for every positive integer n. Thus, $\alpha^n \neq 1$, which implies $Hom\big(G/S(G), S(G)\cap G'\big)$ is torsion-free, and this gives the result.

(2) We prove that $\alpha(g) = g$ for every $\alpha \in Ivar(G)$ and each $g \in G$. As $G/\mathcal{E}(G)$ is torsion, $g^n \in \mathcal{E}(G)$ for some positive integer n. By Lemma 2.3, we have $\alpha(g)^n = \alpha(g^n) = g^n$. Hence, $g^{-n}\alpha(g)^n = 1$. Since $g^{-1}\alpha(g) \in S(G)$, we have $\big(g^{-1}\alpha(g)\big)^n = 1$. Because $S(G)$ is torsion-free, $g^{-1}\alpha(g) = 1$. Hence, $\alpha(g) = g$ for all $\alpha \in Ivar(G)$ and $g \in G$. Therefore, $Ivar(G) = \langle 1\rangle$. ∎

The following theorem determines the structure of $Ivar(G)$ while $S(G)$ is a subgroup of $\mathcal{E}(G)$.

Theorem 2.5 Let G be a group and $S(G) \leqslant \mathcal{E}(G)$. Then

$$Ivar(G) \cong Hom\big(\frac{G}{\mathcal{E}(G)}, S(G)\cap G'\big).$$

Proof. Since $S(G)\mathcal{E}(G) = \mathcal{E}(G)$, we prove that

$$Ivar(G) \cong Hom\big(\frac{G}{S(G)\mathcal{E}(G)}, S(G)\cap G'\big).$$

We define

$$\begin{aligned}\psi :\ Ivar(G) &\longrightarrow Hom\big(\frac{G}{S(G)\mathcal{E}(G)}, S(G)\cap G'\big)\\ \alpha &\longmapsto \alpha^*,\end{aligned}$$

where

$$\begin{aligned}\alpha^* :\ \frac{G}{S(G)\mathcal{E}(G)} &\longrightarrow S(G)\cap G'\\ gS(G)\mathcal{E}(G) &\longmapsto g^{-1}\alpha(g), \quad \text{for every } g \in G.\end{aligned}$$

Obviously, α^* is a well-defined homomorphism, because for every g_1 and g_2 in G, if $g_1S(G)\mathcal{E}(G) = g_2S(G)\mathcal{E}(G)$, then $g_1^{-1}g_2 \in S(G)\mathcal{E}(G)$. By the definition of $S(G)$ and Lemma 2.3, $\alpha(g_1^{-1}g_2) = g_1^{-1}g_2$ and so $g_1^{-1}\alpha(g_1) = g_2^{-1}\alpha(g_2)$. Moreover, α^* is a homomorphism, because

$$\begin{aligned}\alpha^*\big(g_1S(G)\mathcal{E}(G)g_2S(G)\mathcal{E}(G)\big) &= \alpha^*\big(g_1g_2S(G)\mathcal{E}(G)\big)\\ &= (g_1g_2)^{-1}\alpha(g_1g_2)\\ &= g_2^{-1}g_1^{-1}\alpha(g_1)\alpha(g_2)\\ &= g_1^{-1}\alpha(g_1)g_2^{-1}\alpha(g_2)\\ &= \alpha^*\big(g_1S(G)\mathcal{E}(G)\big)\alpha^*\big(g_2S(G)\mathcal{E}(G)\big).\end{aligned}$$

It is obvious that the map ψ is a well-defined monomorphism. Now, we show that ψ is surjective. Let

$$\beta \in Hom\big(\frac{G}{S(G)\mathcal{E}(G)}, S(G) \cap G'\big).$$

We define the map

$$\begin{aligned} \alpha :\ G \longrightarrow & G \\ g \longmapsto & g\beta\big(gS(G)\mathcal{E}(G)\big). \end{aligned}$$

We prove that $\alpha \in Ivar(G)$. Obviously, α is a well-defined homomorphism. Also, it is an injective map, because if $x \in Ker(\alpha)$, then $1 = \alpha(x) = x\beta\big(xS(G)\mathcal{E}(G)\big)$. Therefore,

$$x^{-1} = \beta\big(xS(G)\mathcal{E}(G)\big) \in S(G) \leqslant S(G)\mathcal{E}(G)$$

and $1 = \alpha(x) = x$, so $Ker(\alpha) = \langle 1 \rangle$. To prove that α is surjective, we first show that $Im(\beta) \subseteq Im(\alpha)$. Let $s \in Im(\beta)$. Then $\beta\big(gS(G)\mathcal{E}(G)\big) = s \in S(G)$ for some $g \in G$. Since $S(G) \leqslant S(G)\mathcal{E}(G)$, we have $\alpha(s) = s\beta\big(sS(G)\mathcal{E}(G)\big) = s$. Hence, $s \in Im(\alpha)$. For every $g \in G$, $g = \alpha(g)\beta\big(gS(G)\mathcal{E}(G)\big)^{-1} \in Im(\alpha)$. Therefore, $G = Im(\alpha)$ and α is surjective. Thus, $\alpha \in Ivar(G)$ and $\alpha^* = \beta$ which means ψ is an automorphism and this completes the proof. ■

We use notation $C_{Ivar(G)}\big(Z(G)\big)$ for the group of automorphisms of $Ivar(G)$ fixing $Z(G)$ element-wise. Thus,

$$C_{Ivar(G)}\big(Z(G)\big) = \{\alpha \in Ivar(G) \mid \alpha(z) = z,\ \forall\ z \in Z(G)\}.$$

The following statements give some conditions in which $Ivar(G) = C_{Ivar(G)}\big(Z(G)\big) = \langle 1 \rangle$.

1) G be an abelian group,
2) $S(G) = \langle 1 \rangle$,
3) $Z(G) \leqslant \mathcal{E}(G)$.

Lastly, in the following theorem, we give the structure of the group of automorphisms of $Ivar(G)$ fixing $Z(G)$ element-wise.

Theorem 2.6 Let G be a group. Then

$$C_{Ivar(G)}\big(Z(G)\big) \cong Hom\big(\frac{G}{\mathcal{E}(G)Z(G)}, S(G) \cap G'\big).$$

Proof. We consider the map

$$\begin{aligned} \psi :\ C_{Ivar(G)}\big(Z(G)\big) \longrightarrow & Hom\big(\frac{G}{\mathcal{E}(G)Z(G)}, S(G) \cap G'\big) \\ \alpha \longmapsto & \sigma_\alpha, \end{aligned}$$

where

$$\sigma_\alpha : \frac{G}{\mathcal{E}(G)Z(G)} \longrightarrow S(G) \cap G'$$
$$g\mathcal{E}(G)Z(G) \longmapsto g^{-1}\alpha(g), \quad \forall\, g \in G.$$

By Lemma 2.3, every automorphism $\alpha \in Ivar(G)$ acts trivially on $\mathcal{E}(G)$. On the other hand, by definition, α acts trivially on $Z(G)$ which shows that σ_α is well-defined. The remainder of this argument is done with the same interpretation of Theorem 2.5. ■

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