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Numerical techniques for solving bipolar neutrosophic system of linear equations

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Abstract. In this paper, based on embedding approach three numerical methods namely Richardson, Gauss-Seidel, and successive over relaxation (SOR) have been developed to solve bipolar neutrosophic system of linear equations. To check the accuracy of these newly developed schemes an example with exact and iterative solution is given.

Keywords: Richardson iterative method, Gauss-Seidel method, successive over relaxation (SOR) method, bipolar neutrosophic system of linear equations (BNSLEs).

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1. Introduction and preliminaries

Zadeh [25] presented the concept of fuzzy set to handle uncertainty in real life situation. The basic operations were defined by Bede [7], Dubois and Prade [10, 11], Mizumoto and Tanaka [17]. The parametric form of fuzzy number were proposed by Goetschel and Voxman[15]. Friedman et al. [14], and Amrahov and Askerzade [6] investigated solutions for fuzzy linear system in which coefficient matrix is crisp and right hand side column vector is fuzzy number in parametric form, known as fuzzy system of linear equations (FSLEs). For computing a solution, they used the embedding method and replaced the original $n \times n$ fuzzy linear system into a $2n \times 2n$ crisp linear system. Subsequently many researchers analyzed FSLEs. Allahviranloo [4, 5] used the Jacobi, Gauss-Seidel and SOR numerical techniques to solve FSLEs. Dehgan and Hashmi [12] work on FSLEs

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provided that the coefficient matrix is stricly diagonally dominant and applied many iterative schemes. Furthermore Muzziolia and Reynaerts [19], Abbasbandy and Jafarian [1], Ineirat [16] also investogated the existence of solution of FSLEs. Dubois and Prade [9] started the study of bipolarity. Zhang [26, 27] further proposed and investigated the concept of Ying Yang bipolar fuzzy set. The notion of bipolar fuzzy graphs was developed by Akram [2] and Samanta [2, 21]. Muhammad and Akram [18] studied the solution of bipolar fuzzy system of linear equation (BFSLEs). Akram et al. [3] established the solution of BFSLEs by applying various iterative schemes.

Broumi et al. [8] and Samarandache [22] tried to explore the concept of neutrosophic set (NS) from the philosophical point of view, to represent uncertain, imprecise, incomplete, inconsistence, and indeterminate information. The concept of neutrosophic set is a generalization of the concept of classical set, fuzzy set and intuitionistic fuzzy set. Recently, Shoaib et al. [20] gives the clue of study on bipolar neutrosophic graphs with novel application. There are several methodologies to solve various issues under neutrosophic set and some models are presented to solve linear system with neutrosophic set (see [23, 24]). Edalatpanah [13] studied the existence of solution of a system of neutrosophic linear equations using Embedding method. In continuation of these studies, in this paper we obtain existence of solution of BNSLEs. Main contributions in this study are as follows:

(i) Bipolar neutrosophic system of linear equations (BNSLEs) is developed.
(ii) Embedding technique is used to solve the BNSLEs.
(iii) Three iterative schemes are presented to solve BNSLEs.

The paper is organized as follows: In section 2 some basic notations and definitions are given. In section 3 we study the Richardson, Gauss-Seidel, and SOR methods for solving BNSLEs and obtained necessary theoretical results. Numerical computations of the newly proposed method are illustrated by an example in section 4. Conclusion and comparison analysis is given in section 6. In this section, we give some helping material from the existing literature.

Definition 1.1 [3] A single valued bipolar neutrosophic number "η" in parametric form is a sextuple of the pairs $[(\underline{\eta}_t(r), \overline{\eta}_t(r)), (\underline{\eta}_t(s), \overline{\eta}_t(s)), (\underline{\eta}_i(r), \overline{\eta}_i(r)),$ $(\underline{\eta}_i(s), \overline{\eta}_i(s)), (\underline{\eta}_f(r), \overline{\eta}_f(r)), (\underline{\eta}_f(s), \overline{\eta}_f(s))]$, of the functions $\underline{\eta}_t(r), \overline{\eta}_t(r)$ $, \underline{\eta}_t(s), \overline{\eta}_t(s), \underline{\eta}_i(r), \overline{\eta}_i(r), \underline{\eta}_i(s), \overline{\eta}_i(s), \underline{\eta}_f(r), \overline{\eta}_f(r), \underline{\eta}_f(s), \overline{\eta}_f(s)$ for r $\in [0,1], s \in [-1,0]$ and satisfying the following conditions:

(1) Truth membership function
(a) $\underline{\eta}_t(r)$ is bounded and monotonic increasing and left continuous function.
(b) $\overline{\eta}_t(r)$ is bounded and monotonic decreasing and right continuous function.
(c) $\underline{\eta}_t(s)$ is bounded and monotonic decreasing and right continuous function.
(d)$\overline{\eta}_t(s)$ is bounded and monotonic increasing and left continuous function.
(e) $\underline{\eta}_t(r) \leqslant \overline{\eta}_t(r), \underline{\eta}_t(s) \leqslant \overline{\eta}_t(s)$for r $\in [0,1], s \in [-1,0]$
(2) Indeterminacy membership function
(a) $\underline{\eta}_i(r)$ is bounded and monotonic increasing and left continuous function.
(b) $\overline{\eta}_i(r)$ is bounded and monotonic decreasing and right continuous function.
(c) $\underline{\eta}_i(s)$ is bounded and monotonic decreasing and right continuous function.
(d)$\overline{\eta}_i(s)$ is bounded and monotonic increasing and left continuous function.
(e) $\underline{\eta}_i(r) \leqslant \overline{\eta}_i(r), \underline{\eta}_i(s) \leqslant \overline{\eta}_i(s)$ for $r \in [0,1], s \in [-1,0]$
(3)Falsity membership function
(a) $\underline{\eta}_f(r)$ is bounded and monotonic increasing and left continuous function.
(b) $\overline{\eta}_f(r)$ is bounded and monotonic decreasing and right continuous function.

(c) $\underline{\eta}_f(s)$ is bounded and monotonic decreasing and right continuous function.
(d)$\overline{\eta}_f(s)$ is bounded and monotonic increasing and left continuous function.
(e) $\underline{\eta}_f(r) \leqslant \overline{\eta}_f(r), \underline{\eta}_f(s) \leqslant \overline{\eta}_f(s)$ for $r \in [0,1], s \in [-1,0]$

Definition 1.2 [3] The $n \times n$ linear system of equations is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ \vdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases}, \quad (1)$$

where the coefficient matrix $A = [a_{ij}]$, $1 \leqslant i,j \leqslant n$ is a crisp $n \times n$ matrix and each $b_i \in E^1, 1 \leqslant i \leqslant n$, is a bipolar neutrosophic number is called bipolar neutrosophic system of linear equations (BNSLEs).

Definition 1.3 [3] A bipolar neutrosophic number vector $X = (x_1, x_2, x_3, ...x_n)^t$ given by $\left(x'_t, x''_t, x'_i, x''_i, x'_f, x''_f\right)(r,s) = \begin{bmatrix} (\underline{x}'_t(r), \overline{x}'_t(r), \underline{x}'_i(r), \overline{x}'_i(r), \underline{x}'_f(r), \overline{x}'_f(r), \underline{x}''_t(s), \\ \overline{x}''_t(s), \underline{x}''_i(s), \overline{x}''_i(s), \underline{x}''_f(s), \overline{x}''_f(s). \end{bmatrix}$,
where $1 \leqslant i \leqslant n$, $0 \leqslant r \leqslant 1$ and $-1 \leqslant s \leqslant 0$ is called a solution of a system in definition 1 if

$$\underline{\sum_{j=1}^{n} a_{ij}x'_t} = \sum_{j=1}^{n} a_{ij}\underline{x'_t} = \underline{b}'_t, \ \overline{\sum_{j=1}^{n} a_{ij}x'_t} = \sum_{j=1}^{n} \overline{a_{ij}x'_t} = \overline{b}'_t, \quad (2)$$

$$\underline{\sum_{j=1}^{n} a_{ij}x''_t} = \sum_{j=1}^{n} a_{ij}\underline{x''_t} = \underline{b}''_t, \overline{\sum_{j=1}^{n} a_{ij}x''_t} = \sum_{j=1}^{n} \overline{a_{ij}x''_t} = \overline{b}''_t, \quad (3)$$

$$\underline{\sum_{j=1}^{n} a_{ij}x'_i} = \sum_{j=1}^{n} a_{ij}\underline{x'_i} = \underline{b}'_i, \overline{\sum_{j=1}^{n} a_{ij}x'_i} = \sum_{j=1}^{n} \overline{a_{ij}x'_i} = \overline{b}'_i. \quad (4)$$

$$\underline{\sum_{j=1}^{n} a_{ij}x''_i} = \sum_{j=1}^{n} a_{ij}\underline{x''_i} = \underline{b}''_i, \overline{\sum_{j=1}^{n} a_{ij}x''_i} = \sum_{j=1}^{n} \overline{a_{ij}x''_i} = \overline{b}''_i. \quad (5)$$

$$\underline{\sum_{j=1}^{n} a_{ij}x'_f} = \sum_{j=1}^{n} a_{ij}\underline{x'_f} = \underline{b}'_f, \overline{\sum_{j=1}^{n} a_{ij}x'_f} = \sum_{j=1}^{n} \overline{a_{ij}x'_f} = \overline{b}'_f. \quad (6)$$

$$\underline{\sum_{j=1}^{n} a_{ij}x''_f} = \sum_{j=1}^{n} a_{ij}\underline{x''_f} = \underline{b}''_f, \overline{\sum_{j=1}^{n} a_{ij}x''_f} = \sum_{j=1}^{n} \overline{a_{ij}x''_f} = \overline{b}''_f. \quad (7)$$

for a instant, if $-s = r$, then above define vectors becomes same and it can be written as $6n \times 6n$ instead of $12n \times 12n$ in bipolar neutrosophic environment.

$$x = \begin{pmatrix} \underline{x}'_{t1}, \underline{x}'_{t2}, ...\underline{x}'_{tn}, \ \underline{x}''_{t1}, \underline{x}''_{t2}, ...\underline{x}''_{tn}, \underline{x}'_{i1}, \underline{x}'_{i2} ... \underline{x}'_{in}, \\ \underline{x}''_{i1}, \underline{x}''_{i2}, ...\underline{x}''_{in}, \underline{x}'_{f1}, \underline{x}'_{f2}, ...\underline{x}'_{fn} \\ \underline{x}'_{f1}, \underline{x}'_{f2}, ...\underline{x}'_{fn}, \ \underline{x}''_{f1}, \underline{x}''_{f2}, ...\underline{x}''_{fn}, \\ -\overline{x}'_{t1}, -\overline{x}'_{t2}, ... - \overline{x}'_{tn}, -\overline{x}''_{t1}, -\overline{x}''_{t2}, ... - \overline{x}''_{tn} \\ -\overline{x}'_{i1}, -\overline{x}'_{i2}, ... - \overline{x}'_{in}, -\overline{x}''_{i1}, -\overline{x}''_{i2}, ... - \overline{x}''_{in}, \\ -\overline{x}'_{f1}, -\overline{x}'_{f2}, ... - \overline{x}'_{fn}, , -\overline{x}''_{f1}, -\overline{x}''_{f2}, ... - \overline{x}''_{fn} \end{pmatrix}^t.$$

$$\begin{cases} b = (\underline{b}'_{t1}, \underline{b}'_{t2}, ...\underline{b}'_{tn}, \underline{b}''_{t1}, \underline{b}''_{t2}, ...\underline{b}''_{tn}, \\ \underline{b}'_{i1}, \underline{b}'_{i2}, ...\underline{b}'_{in}, \underline{b}''_{i1}, \\ \underline{b}''_{i2}, ...\underline{b}''_{in}, \underline{b}'_{f1}, \\ \underline{b}'_{f2}, ...\underline{b}'_{fn}, \underline{b}''_{f1}, \\ \underline{b}''_{f2}, ...\underline{b}''_{fn}, -\overline{b'}_{t1}, -\overline{b'}_{t2} ..... -\overline{b}'_{tn}, \\ \overline{b''}_{t1}, -\overline{b''}_{t2}, ... -\overline{b}''_{tn}, -\overline{b'}_{i1}, -\overline{b'}_{i2}, ... -\overline{b}'_{in} \\ , -\overline{b''}_{i1}, -\overline{b''}_{i2}, ... -\overline{b}''_{in}, \\ -\overline{b'}_{f1}, -\overline{b'}_{f2} ..... -\overline{b}'_{fn} - \overline{b''}_{f1}, -\overline{b''}_{f2}, ... -\overline{b}''_{fn})^t. \end{cases}$$

As we establish that for any " i " we have twelve crisp equation. So BNSLEs extend to $12n \times 12n$ crisp linear system with the R.H.S is the vector function as define above.

We get $12n \times 12n$ linear system as follows:

$w_1, 1\underline{x}'_{t1} + ... + w_1, n\underline{x}'_{tn} + w_1, n+1\left(-\overline{x}_{t1}\right) + ...w_1, 2n\left(-\overline{x}_{tn}\right) + w_1, 2n+1\underline{x}''_{t1} + ...w_1,$
$3n\underline{x}''_{tn} + w_1, 3n+1\left(-\overline{x}''_{t1}\right) + ...w_1, 4n\left(-\overline{x}''_{tn}\right) + ...w_1, 4n+1\underline{x}'_{i1} + ...w_1, 5n\underline{x}'_{in} + w_1, 5n+1\left(-\overline{x}'_{i1}\right) +$
$.....w_1, 6n\left(-\overline{x}'_{in}\right) ....11n\underline{x}''_{fn} + 11n+1\left(-\overline{x}'_{f1}\right) ...12n\left(-\overline{x}''_{t1}\right) = \underline{b}'_{t1}$

$w_2, 1\underline{x}'_{t1} + ... + w_2, n\underline{x}'_{tn} + w_2, n+1\left(-\overline{x}_{t1}\right) + ...w_2, 2n\left(-\overline{x}_{tn}\right) + w_2, 2n+1\underline{x}''_{t1} + ...w_2,$
$3n\underline{x}''_{tn} + w_2, 3n+1\left(-\overline{x}''_{t1}\right) + ...w_2, 4n\left(-\overline{x}'_{in}\right) + ..., 4n+1\underline{x}'_{i1} + ...w_2, 5n\underline{x}''_{in} + w_2, 5n+1\left(-\overline{x}''_{i1}\right) +$
$.....w_2, 6n\left(-\overline{x}''_{in}\right), ...11n\left(-\overline{x}'_{fn}\right) + ...w_2, 11n+1\left(-\overline{x}''_{f1}\right), 12n\left(-\overline{x}''_{f1}\right) = \underline{b}'_{t2}$

: : :

$w_n, 1\underline{x}'_{t1} + ... + w_n, n\underline{x}'_{tn} + w_n, n+1\left(-\overline{x}_{t1}\right) + ...w_n, 2n\left(-\overline{x}_{tn}\right) + w_n, 2n+1\underline{x}''_{t1} + ...w_n, 3n\underline{x}''_{tn} +$
$w_n, 3n+1\left(-\overline{x}''_{t1}\right) +, ... + w_n, 4n\left(-\overline{x}'_{in}\right) + ..., 4n+1\underline{x}'_{i1} + ...w_n, 5n\underline{x}''_{in} + w_n,$
$5n+1\left(-\overline{x}''_{i1}\right) + ...w_n, 6n\left(-\overline{x}''_{in}\right), ...11n\underline{x}''_{fn} + ...11n+1\left(-\overline{x}''_{f1}\right) 12n\left(-\overline{x}''_{f1}\right) = \underline{b}'_{tn}$

: : :

$w_{3n}, 1\underline{x}'_{t1} + ... + w_{3n}, n\underline{x}'_{tn} + w_{3n}, n+1\left(-\overline{x}_{t1}\right) + ...w_{3n}, 2n\left(-\overline{x}_{tn}\right) + w_n, 2n+1\underline{x}''_{t1} + ...w_{3n}, 3n\underline{x}''_{tn} +$
$w_{3n}, 3n+1\left(-\overline{x}''_{t1}\right) + ...w_{3n}4n\left(-\overline{x}''_{tn}\right) + ...w_{3n}, 4n+1\underline{x}'_{i1} + ...w_{3n}, 5n\ \underline{x}'_{in} + w_{3n},$
$5n+1\left(-\overline{x}''_{i1}\right) + ...w_{3n}, 6n\left(-\overline{x}''_{in}\right) ... + 11n\underline{x}''_{f1} + ...11n+1\left(-\overline{x}''_{f1}\right) + ...12n\left(-\overline{x}''_{f1}\right) = \underline{b}''_{tn}$

: :

$w_{3n+1}, 1\underline{x}'_{t1} + ... + w_{3n+1}, n\underline{x}'_{tn} + w_{3n+1}, n+1\left(-\overline{x}_{t1}\right) + ...w_{3n+1}, 2n\left(-\overline{x}_{tn}\right) + w_{n+1}, 2n+1\underline{x}''_{t1}$
$+ ......w_{3n+1}, 3n\underline{x}''_{tn} + w_{3n+1}, 3n+1\left(-\overline{x}''_{t1}\right) + ...w_{3n+1}, 4n\left(-\overline{x}''_{tn}\right) + ...w_{3n+1}, 4n+1\underline{x}'_{i1} +$
$....w_{3n+1}, 5n\underline{x}'_{in} + w_{3n+1}, 5n+1\left(-\overline{x}'_{i1}\right) + ...w_{3n+1}, 6n\left(-\overline{x}''_{in}\right) + ...11n\underline{x}'_{f1} + ...w_n,$
$+ 11n+1\left(-\overline{x}''_{f1}\right) + ...12n\left(-\overline{x}''_{f1}\right) = -\overline{b'}_{t1}$

: : :

$w_{12n}, 1\underline{x}'_{t1} + ... + w_{12n}, n\underline{x}'_{tn} + w_{12n}, n+1\left(-\overline{x}_{t1}\right) + ...w_{12n}, 2n\left(-\overline{x}_{tn}\right) + w_{6n}, 2n+1\underline{x}''_{t1} + ...w_{12n}, 3n\underline{x}''_{tn} +$
$w_n, 3n+1\left(-\overline{x}''_{t1}\right) + ...w_{12n}, 4n\left(-\overline{x}^{//}_{tn}\right) + ...w_{12n}, 4n+1\underline{x}'_{i1} + ...w_{12n}, 5n\underline{x}'_{in} + w_{12n},$
$5n+1\left(-\overline{x}''_{i1}\right) + ...q_{12n}, 6n\left(-\overline{x}''_{in}\right) + ...11n\underline{x}'_{f1} + ...w_n, +11n+1\left(-\overline{x}''_{f1}\right) + ...12n\left(-\overline{x}''_{f1}\right) = -\overline{b}''_{fn}$

$W = (w_{i,j}),\ 1 \leqslant i, j \leqslant 12n$ and $q_{i,j}$ are determined as follows:

$$\begin{cases} a_{ij} \geqslant 0 \Rightarrow\ w_{i,j} = a_{ij}, w_{i+n,j} = a_{ij} \text{ where } i, j = 1, 2, ...n. \\ a_{ij} < 0 \Rightarrow\ w_{i,j+n} = -a_{ij}, w_{i+n,j} = -a_{ij} \text{ where } i, j = 1, 2, ...n. \end{cases}$$

if any $w_{i,j}$ which is not determined by the above system it will considered to be zero.

$$WX = B(r, s). \tag{6}$$

The information about extended matrix W for $w_{ij} \geqslant 0$ as follows

$$W = \begin{bmatrix} F, G \\ G, F \end{bmatrix}, \tag{7}$$

where F contains the the positive entries of $A_{n\times n}$ and G having absolute values of negative elements of the matrix $A_{n\times n}$. Friedman [14] shows that extended matrix Q may be singular even original matrix is non singular.

Definition 1.4 [3] If

$$\left(x'_t, x''_t, x'_i, x''_i, x'_f, x''_f\right)(r, s) = \begin{bmatrix} \left(\underline{x}'_t(r), \overline{x}'_t(r), \underline{x}'_i(r), \overline{x}'_i(r), \underline{x}'_f(r), \overline{x}'_f(r),\right. \\ \underline{x}''_t(s), \overline{x}''_t(s), \underline{x}''_i(s), \overline{x}''_i(s), \underline{x}''_f(s), \overline{x}''_f(s) \end{bmatrix}$$

is the solution of the equation for every $1 \leqslant i \leqslant n, 0 \leqslant r \leqslant 1$ and $-1 \leqslant s \leqslant 0$ and also inequalities $\underline{x'}_i \leqslant \overline{x}'_i, \underline{x''}_i \leqslant \overline{x}''_i, \underline{x}'_t \leqslant \overline{x}'_t, \underline{x}''_t \leqslant \overline{x}''_t\ \underline{x'}_f \leqslant \overline{x}'_f, \underline{x''}_f \leqslant \overline{x}''_f$ hold true. Then the solution is called strong solution. Otherwise it is weak.

Definition 1.5 [3] For any two PBNNs m and n with (r, s) cut representation, the distance base on Hausdorff metric in bipolar neutrosophic environment is

$$d = \sup_{0 \leqslant r \leqslant 1, -1 \leqslant s \leqslant 0} \{\max\left(\mid \underline{m}^r - \underline{n}^r \mid, \overline{m}^r - \overline{n}^r\right), \min(\mid \underline{m}^r - \underline{n}^r \mid, \overline{m}^r - \overline{n}^r\} \tag{8}$$

is called Hausdorff distance between m and n.

Definition 1.6 [3] The matrix $A_{n \times n}$ is called diagonally dominant, whenever

$$|a_{i,j}| \geqslant \sum_{j=1,\, i \neq j}^{n} |a_{i,j}|, \qquad j \geqslant 1, 2, 3......n. \tag{15}$$

2. Iterative Methods In Bipolar Neutrosophic Environment

In this section, we consider the solution procedure of BNSLEs by using iterative scheme. A general iterative scheme for BNSLEs

$$Q = \begin{bmatrix} [R'_1 = W'_1, W'_2]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & [R''_1 = W''_1, W''_2] & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & O_{n\times 6n} & [M'''_1 = W'''_1, W'''_2]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & [R'_2 = W'_2, W'_1]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & [R''_2 = W''_2, W''_1]_{n\times 6n} & O_{n\times 2n} \\ O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & [R'''_2 = W'''_2, W'''_1]_{n\times 6n} \\ [R'_3 = W'_2, W'_1]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & [R''_3 = W''_2, W''_1]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & O_{n\times 6n} & [R'''_1 = W'''_2, W'''_1]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & [R'_4 = W'_2, W'_1]_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} \\ O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & [R''_4 = W''_2, W''_1]_{n\times 6n} & O_{n\times 2n} \\ O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & O_{n\times 6n} & [R'''_4 = W'''_2, W'''_1]_{n\times 6n} \end{bmatrix} X = \begin{bmatrix} \underline{X}'_i \\ -\overline{X}'_i \\ \underline{X}''_i \\ -\overline{X}''_i \\ \underline{X}'_f \\ -\overline{X}'_f \\ \underline{X}''_f \\ -\overline{X}''_f \\ \underline{X}'_t \\ -\overline{X}'_t \\ \underline{X}''_t \\ -\overline{X}''_t \end{bmatrix} B = \begin{bmatrix} \underline{B}'_i(r) \\ -\overline{B}'_i(s) \\ \underline{B}''_i(r) \\ -\overline{B}''_i(s) \\ \underline{B}'_f(r) \\ -\overline{B}'_f(s) \\ \underline{B}''_f(r) \\ -\overline{B}''_f(s) \\ \underline{B}'_t(r) \\ -\overline{B}'_t(s) \\ \underline{B}''_t(r) \\ -\overline{B}''_t(s) \end{bmatrix}$$

can be obtain as follows. By using the specific matrix S which is called splitting matrix, E these methods leads to an equivalent form $E(X) = B + (E - W)X$. This equation suggest the following iterative process given by $EX^{(k+1)} = B + (E - W)X^{(k)}, K \geqslant 0$

The choice of initial vector $X^{(0)}$ is arbitrary. A sequence of vector can be obtain by above equation, and our aim to choose S such that

(1) The sequence of function $\left(X^{(k)}\right)$ can easily computed.

(2) The sequence of function $\left(X^{(k)}\right)$ converges to its solution rapidly.

Note: We assume the extended identity identity matrix in bipolar neutrosophic environment is $I_{12n \times 12n}$ where $n = 0, 1, 2....$

2.1 *Richardson Method*

Now we study the Richardson method to solve the BNSLE, in which the choice of "E" is the identity matrix of order $6n$. It is follows that

$$E = (1 - W)X + B. \tag{13}$$

Richardson method in matrix form as given below

$$R=\begin{bmatrix}
I_n-W_1' & -W_2' & -W_3' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_n-W_1'' & -W_2'' & -W_3'' & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n-W_1''' & -W_2''' & -W_3'''\\
I_n-W_1' & -W_2' & -W_3' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_n-W_1'' & -W_2'' & -W_3'' & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n-W_1''' & -W_2''' & -W_3'''\\
-W_3' & -W_2' & I_n-W_1' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -W_3'' & -W_2'' & I_n-W_1'' & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_3''' & -W_2''' & I_n-W_1'''\\
-W_3' & -W_2' & I_n-W_1' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -W_3'' & -W_2'' & I_n-W_1'' & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_3''' & -W_2''' & I_n-W_1'''
\end{bmatrix}.$$

where 0 is the null matrix of order $n\times n$, we write the Richardson method in the following iterative form

$$\begin{cases}
\underline{X}_i^{\prime(k+1)}=\underline{B}_i'(r)+(I_n-W_1')\,\underline{X}_i^{\prime(k)}+W_2'\overline{X}_i^{\prime(k)}+W_3'\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}\\
\quad+0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+0\underline{X}_f^{\prime\prime(k)}+0\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\underline{X}_i^{\prime\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+\underline{B}_i''(s)+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+0\underline{X}_f^{\prime(k)}+(I_n-W_1'')\,\overline{X}_f^{\prime(k)}\\
\quad+W_2''\underline{X}_f^{\prime\prime(k)}+W_3''\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\underline{X}_t^{\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\underline{B}_f'(r)+(I_n-W_1''')\,\underline{X}_f^{\prime(k)}+\\
\quad W_2'''\overline{X}_f^{\prime(k)}+W_3'''\underline{X}_f^{\prime\prime(k)}+0\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\underline{X}_t^{\prime\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\ 0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+\underline{B}_f''(s)\\
\quad+(I_n-W_1')\,\underline{X}_f^{\prime\prime(k)}+W_2'\overline{X}_f^{\prime\prime(k)}+W_3'\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\underline{X}_f^{\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\ 0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+0\underline{X}_f^{\prime\prime(k)}\\
\quad 0\overline{X}_f^{\prime\prime(k)}+\underline{B}_t'(r)+(I_n-W_1'')\,\underline{X}_t^{\prime(k)}+W_2'\overline{X}_t^{\prime(k)}+W_3'\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\underline{X}_f^{\prime\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\ 0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+0\underline{X}_f^{\prime\prime(k)}\\
\quad 0\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+\underline{B}_t''(s)+(I_n-W_1''')\,\overline{X}_t^{\prime(k)}+W_2'''\underline{X}_t^{\prime\prime(k)}+W_3'''\overline{X}_t^{\prime\prime(k)}\\
\overline{X}_i^{\prime(k+1)}=\overline{B}_i'(r)+(I_n-W_1')\,\underline{X}_i^{\prime(k)}+W_2'\overline{X}_i^{\prime(k)}+W_3'\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}\\
\quad+0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+0\underline{X}_f^{\prime\prime(k)}+0\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\overline{X}_i^{\prime\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+\overline{B}_i''(s)+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+0\underline{X}_f^{\prime(k)}+(I_n-W_1'')\,\overline{X}_f^{\prime(k)}\\
\quad+W_2''\underline{X}_f^{\prime\prime(k)}+W_3''\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\overline{X}_t^{\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\overline{B}_t'(r)+(I_n-W_1''')\,\underline{X}_f^{\prime(k)}+\\
\quad W_2'''\overline{X}_f^{\prime(k)}+W_3'''\underline{X}_f^{\prime\prime(k)}+0\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\overline{X}_t^{\prime\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\ 0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+\overline{B}_t''(s)\\
\quad+(I_n-W_1')\,\underline{X}_f^{\prime\prime(k)}+W_2'\overline{X}_f^{\prime\prime(k)}+W_3'\underline{X}_t^{\prime(k)}+0\overline{X}_t^{\prime(k)}+0\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\overline{X}_f^{\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\ 0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+\overline{B}_f'(r)\\
\quad 0\overline{X}_f^{\prime\prime(k)}+\underline{B}_t'(r)+(I_n-W_1'')\,\underline{X}_t^{\prime(k)}+W_2'\overline{X}_t^{\prime(k)}+W_3'\underline{X}_t^{\prime\prime(k)}+0\overline{X}_t^{\prime\prime(k)}\\
\overline{X}_f^{\prime\prime(k+1)}=0\underline{X}_i^{\prime(k)}+0\overline{X}_i^{\prime(k)}+0\underline{X}_i^{\prime\prime(k)}+0\overline{X}_i^{\prime\prime(k)}+\ 0\underline{X}_f^{\prime(k)}+0\overline{X}_f^{\prime(k)}+0\underline{X}_f^{\prime\prime(k)}\\
\quad 0\overline{X}_f^{\prime\prime(k)}+0\underline{X}_t^{\prime(k)}+\overline{B}_f''(s)+(I_n-W_1''')\,\overline{X}_t^{\prime(k)}+W_2'''\underline{X}_t^{\prime\prime(k)}+W_3'''\overline{X}_t^{\prime\prime(k)}
\end{cases}.$$

The elements of

$$X^{(K+1)}=\begin{pmatrix}\underline{X}_t^{\prime(K+1)},\underline{X}_t^{\prime\prime(K+1)},\underline{X}_i^{\prime(K+1)},\underline{X}_i^{\prime\prime(K+1)},\underline{X}_f^{\prime(K+1)},\underline{X}_f^{\prime\prime(K+1)},\overline{X}_t^{\prime(K+1)},\\ \overline{X}_t^{\prime\prime(K+1)},\overline{X}_i^{\prime(K+1)},\overline{X}_i^{\prime\prime(K+1)},\overline{X}_f^{\prime(K+1)},\overline{X}_f^{\prime\prime(K+1)}.\end{pmatrix}$$

is

$$\begin{cases} \underline{X_t}'^{(k+1)} = \underline{B_t}'(r) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\underline{X_j}'^{(k)}(r) + \sum_{j=1}^{n} w'_{i,n+j}\overline{X_j}'^{(k)}(r) \\ \underline{X_t}''^{(K+1)} = \underline{B_t}''(s) + \left(1-\sum_{j=1}^{n} w''_{i,j}\right)\underline{X_j}''^{(K)}(s) + \sum_{j=1}^{n} w''_{i,n+j}\overline{X_j}''^{(k)}(s) \\ \underline{X_i}'^{(K+1)} = \underline{B_i}'(r) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\underline{X_j}'^{(K)}(r) + \sum_{j=1}^{n} w'_{i,n+j}\overline{X_j}'^{(k)}(r) \\ \underline{X_i}''^{(K+1)} = \underline{B_i}''(s) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\underline{X_j}''^{(K)}(s) + \sum_{j=1}^{n} w''_{i,n+j}\overline{X_j}''^{(k)}(s) \\ \underline{X_f}'^{(K+1)} = \underline{B_f}'(r) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\underline{X_j}'^{(K)}(r) + \sum_{j=1}^{n} w'_{i,n+j}\overline{X_j}'^{(k)}(s) \\ \underline{X_f}''^{(K+1)} = \underline{B_f}''(s) + \left(1-\sum_{j=1}^{n} w''_{i,j}\right)\underline{X_j}''^{(K)}(s) + \sum_{j=1}^{n} w''_{i,n+j}\overline{X_j}''^{(k)}(s) \\ \overline{X_t}'^{(K+1)} = \overline{B}'_t(r) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\overline{X_j}'^{(K)}(r) + \sum_{j=1}^{n} w'_{i,n+j}\underline{X_j}'^{(K)}(r) \\ \overline{X_t}''^{(K+1)} = \overline{B}''_t(s) + \left(1-\sum_{j=1}^{n} w''_{i,j}\right)\overline{X_j}''^{(K)}(s) + \sum_{j=1}^{n} w''_{i,n+j}\underline{X_j}''^{(K)}(s) \\ \overline{X_i}'^{(K+1)} = \overline{B}'_i(r) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\overline{X_j}'^{(K)}(r) + \sum_{j=1}^{n} w'_{i,n+j}\underline{X_j}'^{(K)}(r) \\ \overline{X_i}''^{(K+1)} = \overline{B}''_i(s) + \left(1-\sum_{j=1}^{n} w''_{i,j}\right)\overline{X_j}''^{(K)}(s) + \sum_{j=1}^{n} w''_{i,n+j}\underline{X_j}''^{(K)}(s) \\ \overline{X_f}'^{(K+1)} = \overline{B}'_f(r) + \left(1-\sum_{j=1}^{n} w'_{i,j}\right)\overline{X_j}'^{(K)}(r) + \sum_{j=1}^{n} w'_{i,n+j}\underline{X_j}'^{(K)}(r) \\ \overline{X_f}''^{(K+1)} = \overline{B}''_f(s) + \left(1-\sum_{j=1}^{n} w''_{i,j}\right)\overline{X_j}''^{(K)}(s) + \sum_{j=1}^{n} w''_{i,n+j}\underline{X_j}''^{(K)}(s) \end{cases} \tag{14}$$

By this above procedure, the sequence $X^{(K+1)}$ can easily computed.

2.2 *Gauss-Seidel iterative method*

Without loss of generality, we assume $W_{i,j} \geqslant 0$ for every $i, j = 1, 2, 3......12n$. We decompose the matrix W as $W = D_L + U_M + L_N$, where

$$D_L = D'_i + D'_t + D'_f + D''_i + D''_t + D''_f,$$
$$L_N = L'_i + L'_t + L'_f + L''_i + L''_t + L''_f.$$

Similar extension for U_M. D_L, U_M, L_N are diagonal, upper triangular, lower triangular matrices in bipolar neutrosophic environment respectively.

Consider the matrices

$$\left[D'_i + D'_t + D'_f + D''_i + D''_t + D''_f + L'_i + L'_t + L'_f + L''_i + L''_t + L''_f\right]\begin{bmatrix} \underline{X}'_i \\ \overline{X_i}' \\ \underline{X}''_i \\ \overline{X}''_i \\ \underline{X}'_f \\ \overline{X_f}' \\ \underline{X}''_f \\ \overline{X_f}'' \\ \underline{X}'_t \\ \overline{X_t}' \\ \underline{X}''_t \\ \overline{X_t}'' \end{bmatrix} + \left[U'_i + U'_t + U'_f + U''_i + U''_t + U''_f\right]\begin{bmatrix} \underline{X}'_i \\ -\overline{X_i}' \\ \underline{X}''_i \\ -\overline{X}''_i \\ \underline{X}'_f \\ -\overline{X_f}' \\ \underline{X}''_f \\ -\overline{X_f}'' \\ \underline{X}'_t \\ -\overline{X_t}' \\ \underline{X}''_t \\ -\overline{X_t}'' \end{bmatrix} = \begin{bmatrix} \underline{B}'_i\,(r) \\ \overline{B}'_i\,(s) \\ \underline{B}''_i\,(r) \\ \overline{B}''_i\,(s) \\ \underline{B}'_f\,(r) \\ \overline{B}'_f\,(s) \\ \underline{B}''_f\,(r) \\ \overline{B}''_f\,(s) \\ \underline{B}'_t\,(r) \\ \overline{B}'_t\,(s) \\ \underline{B}''_t\,(r) \\ \overline{B}''_t\,(s) \end{bmatrix}.$$

Then

$$\begin{cases}
\underline{X'_i} = (D'_i + L'_i)^{-1}\underline{B'_i} - (D'_i + L'_i)^{-1}U'_i\underline{X'_i} - (D'_i + L'_i)^{-1}W'\overline{X'_i} \\
\overline{X'_i} = (D'_i + L'_i)^{-1}\overline{B}'_i - (D'_i + L'_i)^{-1}U'_i\overline{X'_i} - (D'_i + L'_i)^{-1}W'\underline{X'_i} \\
\underline{X''_i} = (D''_i + L''_i)^{-1}\underline{B_i}'' - (D''_i + L''_i)^{-1}U''_i\underline{X''_i} - (D''_i + L''_i)^{-1}W''\overline{X''_i} \\
\overline{X''_i} = (D''_i + L''_i)^{-1}\overline{B_i}'' - (D''_i + L''_i)^{-1}U''_i\overline{X''_i} - (D''_i + L''_i)^{-1}W''\underline{X''_i} \\
\underline{X'_t} = (D'_t + L'_t)^{-1}\underline{B'_t} - (D'_t + L'_t)^{-1}U'_t\underline{X'_t} - (D'_t + L'_t)^{-1}W'\overline{X'_t} \\
\overline{X'_t} = (D'_t + L'_t)^{-1}\overline{B}'_t - (D'_t + L'_t)^{-1}U'_t\overline{X'_t} - (D'_t + L'_t)^{-1}W'\underline{X'_t} \\
\underline{X''_t} = (D''_t + L''_t)^{-1}\underline{B_t}'' - (D''_t + L''_t)^{-1}U''_t\underline{X''_t} - (D''_t + L''_t)^{-1}W''\overline{X''_t} \\
\overline{X''_t} = (D''_t + L''_t)^{-1}\overline{B_t}'' - (D''_t + L''_t)^{-1}U''_t\overline{X''_t} - (D''_t + L''_t)^{-1}W''\underline{X''_t} \\
\underline{X'_f} = (D'_f + L'_f)^{-1}\underline{B'_f} - (D'_f + L'_f)^{-1}U'_f\underline{X'_f} - (D'_f + L'_f)^{-1}W'\overline{X'_f} \\
\overline{X'_f} = (D'_f + L'_f)^{-1}\overline{B}'_f - (D'_f + L'_f)^{-1}U'_f\overline{X'_f} - (D'_f + L'_f)^{-1}W'\underline{X'_f} \\
\underline{X''_f} = (D''_f + L''_f)^{-1}\underline{B_f}'' - (D''_f + L''_f)^{-1}U''_f\underline{X''_f} - (D''_f + L''_f)^{-1}W''\overline{X''_f} \\
\overline{X''_f} = (D''_f + L''_f)^{-1}\overline{B_f}'' - (D''_f + L''_f)^{-1}U''_f\overline{X''_f} - (D''_f + L''_f)^{-1}W''\underline{X''_f}
\end{cases}$$

We can write the Gauss-Seidel method in the following iterative form as

$$\left[\begin{cases}
\underline{X_i'^{(k+1)}} = (D'_i + L'_i)^{-1}\underline{B'_i} - (D'_i + L'_i)^{-1}U'_i\underline{X_i'^{(k)}} - (D'_i + L'_i)^{-1}W'\overline{X_i'^{(k)}} \\
\overline{X_i'^{(k+1)}} = (D'_i + L'_i)^{-1}\overline{B}'_i - (D'_i + L'_i)^{-1}U'_i\overline{X_i'^{(k)}} - (D'_i + L'_i)^{-1}W'\underline{X_i'^{(k)}} \\
\underline{X_i''^{(k+1)}} = (D''_i + L''_i)^{-1}\underline{B_i}'' - (D''_i + L''_i)^{-1}U''_i\underline{X_i''^{(k)}} - (D''_i + L''_i)^{-1}W''\overline{X_i''^{(k)}} \\
\overline{X_i''^{(k+1)}} = (D''_i + L''_i)^{-1}\overline{B_i}'' - (D''_i + L''_i)^{-1}U''_i\overline{X_i''^{(k)}} - (D''_i + L''_i)^{-1}W''\underline{X_i''^{(k)}} \\
\underline{X_t'^{(k+1)}} = (D'_t + L'_t)^{-1}\underline{B'_t} - (D'_t + L'_t)^{-1}U'_t\underline{X_t'^{(k)}} - (D'_t + L'_t)^{-1}W'\overline{X_t'^{(k)}} \\
\overline{X_t'^{(k+1)}} = (D'_t + L'_t)^{-1}\overline{B}'_t - (D'_t + L'_t)^{-1}U'_t\overline{X_t'^{(k)}} - (D'_t + L'_t)^{-1}W'\underline{X_t'^{(k)}} \\
\underline{X_t''^{(k+1)}} = (D''_t + L''_t)^{-1}\underline{B_t}'' - (D''_t + L''_t)^{-1}U''_t\underline{X_t''^{(k)}} - (D''_t + L''_t)^{-1}W''\overline{X_t''^{(k)}} \\
\overline{X_t''^{(k+1)}} = (D''_t + L''_t)^{-1}\overline{B_t}'' - (D''_t + L''_t)^{-1}U''_t\overline{X_t''^{(k)}} - (D''_t + L''_t)^{-1}W''\underline{X_t''^{(k)}} \\
\underline{X_f'^{(k+1)}} = (D'_f + L'_f)^{-1}\underline{B'_f} - (D'_f + L'_f)^{-1}U'_f\underline{X_f'^{(k)}} - (D'_f + L'_f)^{-1}W'\overline{X_f'^{(k)}} \\
\overline{X_f'^{(k+1)}} = (D'_f + L'_f)^{-1}\overline{B}'_f - (D'_f + L'_f)^{-1}U'_f\overline{X_f'^{(k)}} - (D'_f + L'_f)^{-1}W'\underline{X_f'^{(k)}} \\
\underline{X_f''^{(k+1)}} = (D''_f + L''_f)^{-1}\underline{B_f}'' - (D''_f + L''_f)^{-1}U''_f\underline{X_f''^{(k)}} - (D''_f + L''_f)^{-1}W''\overline{X_f''^{(k)}} \\
\overline{X_f''^{(k+1)}} = (D''_f + L''_f)^{-1}\overline{B_f}'' - (D''_f + L''_f)^{-1}U''_f\overline{X_f''^{(k)}} - (D''_f + L''_f)^{-1}W''\underline{X_f''^{(k)}}
\end{cases}\right].$$

So the elements of

$$X^{(k+1)} = \begin{pmatrix} \underline{X}_i'^{(k+1)}, \overline{X_i}'^{(k+1)}, \underline{X}_i''^{(k+1)}, \overline{X_i}''^{(k+1)}, \underline{X}_f'^{(k+1)}, \overline{X_f}'^{(k+1)}, \underline{X}_f''^{(k+1)} \\ , \overline{X_f}''^{(k+1)}, \underline{X}_t'^{(k+1)}, \overline{X_t}'^{(k+1)}, \underline{X}_t''^{(k+1)}, \overline{X_t}''^{(k+1)} \end{pmatrix}$$

is

$$\begin{cases}
\underline{x}_i'^{(k+1)} = \frac{1}{w'_{i,i}}[\underline{B}'_i(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,j}\underline{x}_j'^{(k+1)}(r) - \sum\limits_{\substack{j=1+i\\ i\neq j}}^{n} w'_{i,j}\overline{x}_j'^{(k)}(r) - \sum\limits_{j=1}^{n} w'_{i,n+j}\overline{x}_j'^{(k)}(r)]] \\
\underline{x}_i''^{(k+1)} = \frac{1}{w''_{i,i}}[\underline{B}'_i(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,j}\underline{x}_j''^{(k+1)}(s) - \sum\limits_{\substack{j=1+i\\ i\neq j}}^{n} w''_{i,j}\overline{x}_j''^{(k)}(s) - \sum\limits_{j=1}^{n} w''_{i,n+j}\overline{x}_j''^{(k)}(s)] \\
\underline{x}_f'^{(k+1)} = \frac{1}{w'_{i,i}}[\underline{B}'_f(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,j}\underline{x}_j'^{(k+1)}(r) - \sum\limits_{\substack{j=1+i\\ i\neq j}}^{n} w'_{i,j}\overline{x}_j'^{(k)}(r) - \sum\limits_{j=1}^{n} w'_{i,n+j}\overline{x}_j'^{(k)}(r)] \\
\underline{x}_f''^{(k+1)} = \frac{1}{w'_{i,i}}[\underline{B}''_f(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,j}\underline{x}_j''^{(k+1)}(s) - \sum\limits_{\substack{j=1+i\\ i\neq j}}^{n} w''_{i,j}\overline{x}_j'^{(k)}(s) - \sum\limits_{j=1}^{n} w''_{i,n+j}\overline{x}_j''^{(k)}(s)] \\
\underline{x}_t'^{(k+1)} = \frac{1}{w'_{i,i}}[\underline{B}'_t(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,j}\underline{x}_j'^{(k+1)}(r) - \sum\limits_{\substack{j=1+i\\ i\neq j}}^{n} w'_{i,j}\overline{x}_j'^{(k)}(r) - \sum\limits_{j=1}^{n} w'_{i,n+j}\overline{x}_j'^{(k)}(r)] \\
\underline{x}_t''^{(k+1)} = \frac{1}{w'_{i,i}}[\underline{B}''_t(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,j}\underline{x}_j''^{(k+1)}(s) - \sum\limits_{\substack{j=1+i\\ i\neq j}}^{n} w''_{i,j}\overline{x}_j'^{(k)}(s) - \sum\limits_{j=1}^{n} w''_{i,n+j}\overline{x}_j''^{(k)}(s)] \\
\overline{x}_i'^{(k+1)} = \frac{1}{w'_{i,i}}[\overline{B'_i}(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,j}\overline{x}_j'^{(k+1)}(r) - \sum\limits_{\substack{j=i+1\\ i\neq j}}^{n} w'_{i,j}\underline{x}_j'^{(k)}(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,n+j}\underline{x}_j'^{(k)}(r)] \\
\overline{x''}_i^{(k+1)} = \frac{1}{w'_{i,i}}[\overline{B''_i}(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,j}\overline{x}_j''^{(k+1)}(s) - \sum\limits_{\substack{j=i+1\\ i\neq j}}^{n} w''_{i,j}\underline{x}_j''^{(k)}(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,n+j}\underline{x}_j''^{(k)}(s)] \\
\overline{x}_f'^{(k+1)} = \frac{1}{w'_{i,i}}[\overline{B'_f}(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,j}\overline{x}_j'^{(k+1)}(r) - \sum\limits_{\substack{j=i+1\\ i\neq j}}^{n} w'_{i,j}\underline{x}_j'^{(k)}(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,n+j}\underline{x}_j'^{(k)}(r)] \\
\overline{x}_f''^{(k+1)} = \frac{1}{w''_{i,i}}[\overline{B''_f}(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,j}\overline{x}_j''^{(k+1)}(s) - \sum\limits_{\substack{j=i+1\\ i\neq j}}^{n} w''_{i,j}\underline{x}_j''^{(k)}(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,n+j}\underline{x}_j''^{(k)}(s)] \\
\overline{x}_t'^{(k+1)} = \frac{1}{w'_{i,i}}[\overline{B'_t}(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,j}\overline{x}_j'^{(k+1)}(r) - \sum\limits_{\substack{j=i+1\\ i\neq j}}^{n} w'_{i,j}\underline{x}_j'^{(k)}(r) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w'_{i,n+j}\underline{x}_j'^{(k)}(r)] \\
\overline{x}_t''^{(k+1)} = \frac{1}{w''_{i,i}}[\overline{B''_t}(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,j}\overline{x}_j''^{(k+1)}(s) - \sum\limits_{\substack{j=i+1\\ i\neq j}}^{n} w''_{i,j}\underline{x}_j''^{(k)}(s) - \sum\limits_{\substack{j=1\\ i\neq j}}^{n} w''_{i,n+j}\underline{x}_j''^{(k)}(s)]
\end{cases},$$

where $k = 0, 1, 2, 3, \ldots, n$. So the Gauss-Seidel method in matrix notation is $X^{(k+1)} = T_{GS}X^{(k)} + C$, where T_{GS} is 12×12 matrix containing entries $-(D'_i + L'_i)W'_i, -(D'_i + L'_i)U'_i, -(D''_i + L''_i)W''_i, -(D''_i + L''_i)U''_i, -(D'_f + L'_f)W'_f, -(D'_f + L'_f)U'_f, -(D''_f + L''_f)W''_f, -(D''_f + L''_f)U''_f, -(D'_t + L'_t)W'_t, -(D'_t + L'_t)U'_t, -(D''_t + L''_t)W''_t, -(D''_t + L''_t)U''_t$, and all other are zeros.

$$C = \begin{bmatrix} (D'_i + L'_i)\underline{B}'_i \\ (D''_i + L''_i)\underline{B}''_i \\ \left(D'_f + L'_f\right)\underline{B}'_f \\ \left(D''_f + L''_f\right)\underline{B}''_f \\ (D'_t + L'_t)\underline{B}'_t \\ (D''_t + L''_t)\underline{B}''_t \\ (D'_i + L'_i)\overline{B'_i} \\ (D''_i + L''_i)\overline{B''_i} \\ (D'_f + L'_f)\overline{B'_f} \\ (D''_f + L''_f)\overline{B''_f} \\ (D'_t + L'_t)\overline{B'_t} \\ (D''_t + L''_t)\overline{B''_t} \end{bmatrix} \text{ and } X = \begin{bmatrix} \underline{X}'_i \\ \underline{X}''_i \\ \underline{X}'_f \\ \underline{X}''_f \\ \underline{X}'_t \\ \underline{X}''_t \\ \overline{X_i}' \\ \overline{X_i}'' \\ \overline{X_f}' \\ \overline{X_f}'' \\ \overline{X_t}' \\ \overline{X_t}'' \end{bmatrix}.$$

2.3 *Successive over relaxation iterative method(SOR)*

We now modify GS iterative method which is known as SOR method. For SOR method, the splitting matrix W is to be chosen such that $W = \frac{1}{\omega}[D'_i + D''_i + D'_f + D''_f + D'_f + D''_f +$

$\omega(L'_i + L''_i + L'_f + L''_f + L'_f + L''_f)]$ and ω is some real parameter, so the SOR method in following iterative form as

$$\begin{cases} \underline{X}'^{(k+1)}_i = \omega(D'_i + \omega L'_i)^{-1}\underline{B}'_i + (D'_i + \omega L'_i)^{-1}[(1-\omega)D'_i - \omega U'_i]\underline{X}'^{(k)}_i - \omega(D'_i + \omega L'_i)^{-1}W'\overline{X}'^{(k)}_i \\ \underline{X}''^{(k+1)}_i = \omega(D''_i + \omega L''_i)^{-1}\underline{B}''_i + (D''_i + \omega L''_i)^{-1}[(1-\omega)D''_i - \omega U''_i]\underline{X}''^{(k)}_i - \omega(D''_i + \omega L''_i)^{-1}W''\overline{X}''^{(k)}_i \\ \underline{X}'^{(k+1)}_f = \omega(D'_f + \omega L'_f)^{-1}\underline{B}'_f + (D'_f + \omega L'_f)^{-1}[(1-\omega)D'_f - \omega U'_f]\underline{X}'^{(k)}_f - \omega(D'_f + \omega L'_f)^{-1}W'\overline{X}'^{(k)}_f \\ \underline{X}''^{(k+1)}_f = \omega(D''_f + \omega L''_f)^{-1}\underline{B}''_f + (D''_f + \omega L''_f)^{-1}[(1-\omega)D''_f - \omega U''_f]\underline{X}''^{(k)}_f - \omega(D''_f + \omega L''_f)^{-1}W''\overline{X}''^{(k)}_f \\ \underline{X}'^{(k+1)}_t = \omega(D'_t + \omega L'_t)^{-1}\underline{B}'_t + (D'_t + \omega L'_t)^{-1}[(1-\omega)D'_t - \omega U'_t]\underline{X}'^{(k)}_t - \omega(D'_t + \omega L'_t)^{-1}W'\overline{X}'^{(k)}_t \\ \underline{X}''^{(k+1)}_t = \omega(D''_t + \omega L''_t)^{-1}\underline{B}''_t + (D''_t + \omega L''_t)^{-1}[(1-\omega)D''_t - \omega U''_t]\underline{X}''^{(k)}_t - \omega(D''_t + \omega L''_t)^{-1}W''\overline{X}''^{(k)}_t \\ \overline{X}'^{(k+1)}_i = \omega(D'_i + \omega L'_i)^{-1}\overline{B}'_i + (D'_i + \omega L'_i)^{-1}[(1-\omega)D'_i - \omega U'_i]\overline{X'}^{(k)}_i - \omega(D'_i + \omega L'_i)^{-1}W'\underline{X}'^{(k)}_i \\ \overline{X}''^{(k+1)}_i = \omega(D''_i + \omega L''_i)^{-1}\overline{B}''_i + (D''_i + \omega L''_i)^{-1}[(1-\omega)D''_i - \omega U''_i]\overline{X''}^{(k)}_i - \omega(D''_i + \omega L''_i)^{-1}W''\underline{X}'^{(k)}_i \\ \overline{X}'^{(k+1)}_f = \omega(D'_f + \omega L'_f)^{-1}\overline{B}'_f + (D'_f + \omega L'_f)^{-1}[(1-\omega)D'_f - \omega U'_f]\overline{X'}^{(k)}_f - \omega(D'_f + \omega L'_f)^{-1}W'\underline{X}'^{(k)}_f \\ \overline{X}''^{(k+1)}_f = \omega(D''_f + \omega L''_f)^{-1}\overline{B}''_f + (D''_f + \omega L''_f)^{-1}[(1-\omega)D''_f - \omega U''_f]\overline{X''}^{(k)}_f - \omega(D''_f + \omega L''_f)^{-1}W''\underline{X}''^{(k)}_f \\ \overline{X}'^{(k+1)}_t = \omega(D'_t + \omega L'_t)^{-1}\overline{B}'_t + (D'_t + \omega L'_t)^{-1}[(1-\omega)D'_t - \omega U'_t]\overline{X'}^{(k)}_t - \omega(D'_t + \omega L'_t)^{-1}W'\underline{X}'^{(k)}_t \\ \overline{X}''^{(k+1)}_t = \omega(D''_t + \omega L''_t)^{-1}\overline{B}''_t + (D''_t + \omega L''_t)^{-1}[(1-\omega)D''_t - \omega U''_t]\overline{X''}^{(k)}_t - \omega(D''_t + \omega L''_t)^{-1}W''\underline{X}''^{(k)}_t \end{cases}$$

We can write the SOR iterative technique in matrix form as $X^{(k+1)} = T_{SOR}X^{(k)} + C$, where

$$T_{SOR} = \begin{bmatrix} \tau'_i & \tau'_f & \tau'_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau'_i & \tau'_f & \tau'_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau'_i & \tau'_f & \tau'_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau'_i & \tau'_f & \tau'_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau'_i & \tau'_f & \tau'_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau'_i & \tau'_f & \tau'_t \\ \tau''_i & \tau''_f & \tau''_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau''_i & \tau''_f & \tau''_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau''_i & \tau''_f & \tau''_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau''_i & \tau''_f & \tau''_t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau''_i & \tau''_f & \tau''_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau''_i & \tau''_f & \tau''_t \end{bmatrix} \quad C = \begin{bmatrix} \omega(D'_i + \omega L'_i)^{-1}\underline{B}'_i \\ \omega(D''_i + \omega L''_i)^{-1}\underline{B}''_i \\ \omega(D'_f + \omega L'_f)^{-1}\underline{B}'_f \\ \omega(D''_f + \omega L''_f)^{-1}\underline{B}''_f \\ \omega(D'_t + \omega L'_t)^{-1}\underline{B}'_t \\ \omega(D''_t + \omega L''_t)^{-1}\underline{B}''_t \\ \omega(D'_i + \omega L'_i)^{-1}\overline{B}'_i \\ \omega(D''_i + \omega L''_i)^{-1}\overline{B}''_i \\ \omega(D'_f + \omega L'_f)^{-1}\overline{B}'_f \\ \omega(D''_f + \omega L''_f)^{-1}\overline{B}''_f \\ \omega(D'_t + \omega L'_t)^{-1}\overline{B}'_t \\ \omega(D''_t + \omega L''_t)^{-1}\overline{B}''_t \end{bmatrix} \text{ and } X = \begin{bmatrix} \underline{X}'_i \\ \underline{X}''_i \\ \underline{X}'_f \\ \underline{X}''_f \\ \underline{X}'_t \\ \underline{X}''_t \\ \overline{X_i}' \\ \overline{X_i}'' \\ \overline{X_f}' \\ \overline{X_f}'' \\ \overline{X_t}' \\ \overline{X_t}'' \end{bmatrix}$$

Similarly the backward SOR method can be written by following modification $W = \frac{1}{\omega}[D'_i + D''_i + D'_f + D''_f + D'_f + D''_f + \omega(U'_i + U''_i + U'_f + U''_f + U'_f + U''_f)]$. We choose "$\omega$" between 0 and 1, if the system does not converge by GS method than we use SOR for desired convergence. This scheme is also apply for accelerating the convergence rate.

3. Numerical Example

In this section we discuss the efficiency of our defined numerical methods. The following numerical example is considered.

***Example* 3.1** Consider 2×2 non-symmetric bipolar neutrosophic linear system

$$\begin{aligned} 4x_1 + 2x_2 &= \langle [1+3r, 8-4r], [2+r, 9-6r], [3+r, 10-6r], [-2s+3, 2s+7], \\ &\quad [-3s+4, 3s+10], [-4s+4, 4s+12] \rangle. \\ x_1 - 3x_2 &= \langle [4+2r, 14-8r], [6+r, 10-3r], [7+2r, 13-4r], [-3s+3, 3s+9], \\ &\quad [-5s+2, 5s+12], [-2s+3, 2s+7] \rangle. \end{aligned}$$

The extended 24×24 matrix is

$$A = \begin{bmatrix} 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 3\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 3 \\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 3\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 3\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 2 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 1\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 1+3r \\ 4+2r \\ -2s+3 \\ -3s+3 \\ 2+r \\ 6+r \\ -3s+4 \\ -5s+2 \\ 3+r \\ 7+2r \\ -4s+4 \\ -2s+3 \\ 8-4r \\ 14-8r \\ 2s+7 \\ 3s+9 \\ 9-6r \\ 10-3r \\ 3s+10 \\ 5s+12 \\ 10-6r \\ 13-4r \\ 4s+12 \\ 2s+7 \end{bmatrix} \quad X = \begin{bmatrix} \underline{X}'_{1i} \\ \underline{X}''_{1i} \\ \underline{X}'_{1f} \\ \underline{X}''_{1f} \\ \underline{X}'_{1t} \\ \underline{X}''_{1t} \\ \underline{X}'_{2i} \\ \underline{X}''_{2i} \\ \underline{X}'_{2f} \\ \underline{X}''_{2f} \\ \underline{X}'_{2t} \\ \underline{X}''_{2t} \\ \overline{X_{1i}}' \\ \overline{X_{1i}}'' \\ \overline{X_{1f}}' \\ \overline{X_{1f}}'' \\ \overline{X_{1t}}' \\ \overline{X_{1t}}'' \\ \overline{X_{2i}}' \\ \overline{X_{2i}}'' \\ \overline{X_{2f}}' \\ \overline{X_{2f}}'' \\ \overline{X_{2t}}' \\ \overline{X_{2t}}'' \end{bmatrix}$$

The exact solution is (By using Matlab software "R2014a(8.3.0.532)")

$$\begin{bmatrix} \underline{X}'_{1i} \\ \underline{X}''_{1i} \\ \underline{X}'_{1f} \\ \underline{X}''_{1f} \\ \underline{X}'_{1t} \\ \underline{X}''_{1t} \\ \underline{X}'_{2i} \\ \underline{X}''_{2i} \\ \underline{X}'_{2f} \\ \underline{X}''_{2f} \\ \underline{X}'_{2t} \\ \underline{X}''_{2t} \\ \overline{X_{1i}}' \\ \overline{X_{1i}}'' \\ \overline{X_{1f}}' \\ \overline{X_{1f}}'' \\ \overline{X_{1t}}' \\ \overline{X_{1t}}'' \\ \overline{X_{2i}}' \\ \overline{X_{2i}}'' \\ \overline{X_{2f}}' \\ \overline{X_{2f}}'' \\ \overline{X_{2t}}' \\ \overline{X_{2t}}'' \end{bmatrix} = \begin{bmatrix} 1.9142857142857142857142857142857 \times r - 1.9142857142857142857142857142857 \\ 4.3285714285714285714285714285714 - 2.3285714285714285714285714285714 \times r \\ 1.0142857142857142857142857142857 - 1.0142857142857142857142857142857 \times r \\ 0.028571428571428571428571428571429 \times r + 1.9714285714285714285714285714286 \\ 0.47142857142857142857142857142857 - 1.8857142857142857142857142857143 \times s \\ 2.7714285714285714285714285714286 \times s + 0.55714285714285714285714285714286 \\ 0.68571428571428571428571428571429 \times s + 1.3285714285714285714285714285714 \\ 0.84285714285714285714285714285714 - 0.37142857142857142857142857142857 \times s \\ 0.48571428571428571428571428571429 \times r - 0.98571428571428571428571428571429 \\ 2.9714285714285714285714285714286 - 0.47142857142857142857142857142857 \times r \\ 1.0857142857142857142857142857143 - 1.5857142857142857142857142857143 \times r \\ 0.17142857142857142857142857142857 \times r + 2.3285714285714285714285714285714 \\ 0.54285714285714285714285714285714 - 2.5571428571428571428571428571429 \times s \\ 3.6142857142857142857142857142857 \times s + 0.91428571428571428571428571428571 \\ 1.1571428571428571428571428571429 \times s + 2.2571428571428571428571428571429 \\ 0.48571428571428571428571428571429 - 0.81428571428571428571428571428571 \times s \\ 0.62857142857142857142857142857143 \times r - 1.2285714285714285714285714285714 \\ 3.9571428571428571428571428571429 - 0.75714285714285714285714285714286 \times r \\ 1.1285714285714285714285714285714 - 1.7285714285714285714285714285714 \times r \\ 0.45714285714285714285714285714286 \times r + 2.7428571428571428571428571428571 \\ 1.1142857142857142857142857142857 - 1.9571428571428571428571428571429 \times s \\ 1.9142857142857142857142857142857 \times s - 0.22857142857142857142857142857143 \\ 1.2571428571428571428571428571429 \times s + 2.6857142857142857142857142857143 \\ 0.62857142857142857142857142857143 - 0.014285714285714285714285714285714 \times s \end{bmatrix}$$

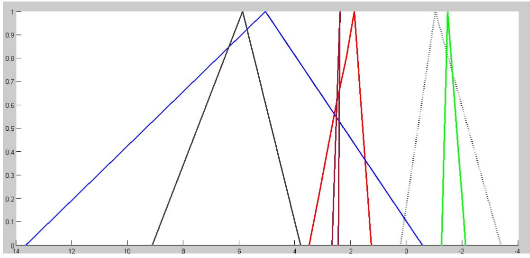

Figure 1. The exact solution of positive membership of the 2×2 bipolar neutrosophic system of linear equations.

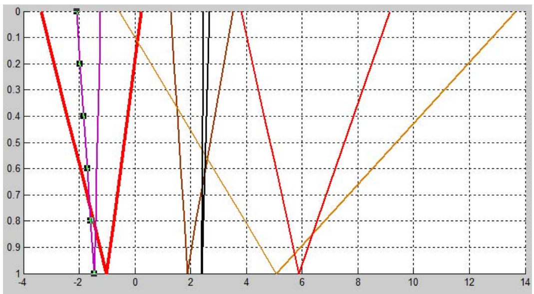

Figure 2. Exact solution of negative membership of 2×2 bipolar neutrosophic system of linear equation

By using MATLAB software we obtain the approximate solution by using Richardson iterative method in 25 iteration

$$\begin{bmatrix} X'_{\mathbf{1}i} \\ X''_{\mathbf{1}i} \\ X'_{\mathbf{1}f} \\ X''_{\mathbf{1}f} \\ X'_{\mathbf{1}t} \\ X''_{\mathbf{1}t} \\ X'_{\mathbf{2}i} \\ X''_{\mathbf{2}i} \\ X'_{\mathbf{2}f} \\ X''_{\mathbf{2}f} \\ X'_{\mathbf{2}t} \\ X''_{\mathbf{2}t} \\ \overline{X_{\mathbf{1}i}}' \\ \overline{X_{\mathbf{1}i}}'' \\ \overline{X_{\mathbf{1}f}}' \\ \overline{X_{\mathbf{1}f}}'' \\ \overline{X_{\mathbf{1}t}}' \\ \overline{X_{\mathbf{1}t}}'' \\ \overline{X_{\mathbf{2}i}}' \\ \overline{X_{\mathbf{2}i}}'' \\ \overline{X_{\mathbf{2}f}}' \\ \overline{X_{\mathbf{2}f}}'' \\ \overline{X_{\mathbf{2}t}}' \\ \overline{X_{\mathbf{2}t}}'' \end{bmatrix}^{25} = \begin{bmatrix} 1.9142857142857142857111727374\times r-1.9142857142857142857142857724649 \\ 4.3285714285714285714285273746-2.3285714285714285714285713760278\times r \\ 1.0142857142857142857142826839\,5-1.0142857142857142857142827465\,8\times r \\ 0.0285714285714285714285712748438\times r+1.9714285714285714285712745829 \\ 0.4714285714285714285714275943\,75-1.8857142857142857142857127364654\times s \\ 2.7714285714285714285714253647645\times s+0.5571428571428571428571428374\,64 \\ 0.6857142857142857142857142857142\,9\times s+1.3285714285714285714285714285714 \\ 0.8428571428571428571428571428571\,4-0.3714285714285714285714285714285\,7\times s \\ 0.4857142857142857142857142857142\,9\times r-0.9857142857142857142857142857142\,9 \\ 2.9714285714285714285714285714286-0.4714285714285714285714285714285\,7\times r \\ 1.0857142857142857142857142857143-1.5857142857142857142857142857143\times r \\ 0.1714285714285714285714285714285\,7\times r+2.3285714285714285714285714285714 \\ 0.5428571428571428571428571428571\,4-2.5571428571428571428571428571429\times s \\ 3.6142857142857142857142857142857\times s+0.9142857142857142857142857142857\,1 \\ 1.1571428571428571428571428571429\times s+2.2571428571428571428571428571429 \\ 0.4857142857142857142857142857142\,9-0.8142857142857142857142857142857\,1\times s \\ 0.6285714285714285714285714285714\,3\times r-1.2285714285714285714285714285714 \\ 3.9571428571428571428571428571429-0.7571428571428571428571428571428\,6\times r \\ 1.1285714285714285714285714285714-1.7285714285714285714285714285714\times r \\ 0.4571428571428571428571428571428\,6\times r+2.7428571428571428571428571428571 \\ 1.1142857142857142857142857142857-1.9571428571428571428571428571429\times s \\ 1.9142857142857142857142857142857\times s-0.2285714285714285714285714285714\,3 \\ 1.2571428571428571428571428571429\times s+2.6857142857142857142857142857143 \\ 0.6285714285714285714285714285714\,3-0.0142857142857142857142857142857\,14\times s \end{bmatrix}$$

By using MATLAB software we obtain the approximate solution by using GS iterative method in 20 iterations

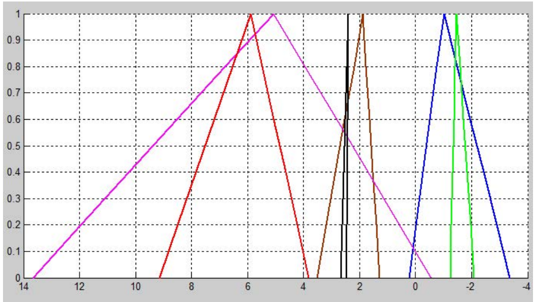

Figure 3. The approximated solution of positive membership by using Richardson iterative method after 25 iterations of 2×2 bipolar neutrosophic system of linear equations.

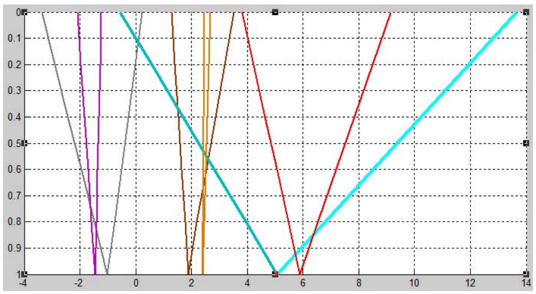

Figure 4. The approximated solution of negativemembership by using Richardson iterative method after 25 iterations of 2×2 bipolar neutrosophic system of linear equations.

$$\begin{bmatrix} \underline{X}'_{\mathbf{1}i} \\ \underline{X}''_{\mathbf{1}i} \\ \underline{X}'_{\mathbf{1}f} \\ \underline{X}''_{\mathbf{1}f} \\ \underline{X}'_{\mathbf{1}t} \\ \underline{X}''_{\mathbf{1}t} \\ \underline{X}'_{\mathbf{2}i} \\ \underline{X}''_{\mathbf{2}i} \\ \underline{X}'_{\mathbf{2}f} \\ \underline{X}''_{\mathbf{2}f} \\ \underline{X}'_{\mathbf{2}t} \\ \underline{X}''_{\mathbf{2}t} \\ \overline{X_{\mathbf{1}i}}' \\ \overline{X_{\mathbf{1}i}}'' \\ \overline{X_{\mathbf{1}f}}' \\ \overline{X_{\mathbf{1}f}}'' \\ \overline{X_{\mathbf{1}t}}' \\ \overline{X_{\mathbf{1}t}}'' \\ \overline{X_{\mathbf{2}i}}' \\ \overline{X_{\mathbf{2}i}}'' \\ \overline{X_{\mathbf{2}f}}' \\ \overline{X_{\mathbf{2}f}}'' \\ \overline{X_{\mathbf{2}t}}' \\ \overline{X_{\mathbf{2}t}}'' \end{bmatrix}^{20} = \begin{bmatrix} 1.91428571428571428571428527364 \times r - 1.91428571428571428571428527364 \\ 4.32857142857142857142857142857427374 - 2.32857142857142857142857142857142837577 \times r \\ 1.014285714285714285714285714280362 - 1.01428571428571428571428571428570666622 \times r \\ 0.02857142857142857142857142857142273747 4 \times r + 1.97142857142857142857142827364 \\ 0.47142857142857142857142857142853747 64 - 1.88571428571428571428571448457 57 \times s \\ 2.77142857142857142857142857142273 7 \times s + 0.55714285714285714285714285714285674 83 \\ 0.68571428571428571428571428571428273 74 \times s + 1.32857142857142857142857142857475 65 \\ 0.84285714285714285714285714285273747 - 0.37142857142857142857142857142854 7465 \times s \\ 0.48571428571428571428571428571428521 0673 \times r - 0.98571428571428571428571428571428375942 \\ 2.97142857142857142857142857142838 5875 - 0.47142857142857142857142857142853 9485 \times r \\ 1.08571428571428571428571428571428571 43 - 1.58571428571428571428571428571428571 43 \times r \\ 0.171428571428571428571428571428571428 57 \times r + 2.328571428571428571428571428571428571 4 \\ 0.542857142857142857142857142857142857 14 - 2.557142857142857142857142857142857142 9 \times s \\ 3.614285714285714285714285714285714285 7 \times s + 0.914285714285714285714285714285714285 71 \\ 1.157142857142857142857142857142857142 9 \times s + 2.257142857142857142857142857142857142 9 \\ 0.12542857142857142857142857142857187 6 - 0.81428571428571428571428571428571487 65 \times s \\ 0.62857142857142857142857142857142857 143 \times r - 1.22857142857142857142857142857142857 14 \\ 3.957142857142857142857142857142857142 9 - 0.757142857142857142857142857142857142 86 \times r \\ 1.287373142857142857142857142857142857 14 - 1.728571428571428571428571428571428571 4 \times r \\ 0.374642857142857142857142857142857273 63 \times r + 2.742857142857142857142857142857374 64 \\ 1.114285714285714285714285714285714285 7 - 1.957142857142857142857142857142857142 9 \times s \\ 1.914285714285714285714285714285727376 43 \times s - 0.228571428571428571428571428571428343 \\ 1.257142857142857142857142857142857142 9 \times s + 2.685714285714285714285714285714285714 3 \\ 0.628571428571428571428571428571426 36 - 0.014285714285714285714285714285714364 75 \times s \end{bmatrix}$$

By using MATLAB software approximated solution by using SOR iterative method after 15 iterations with $\omega = 1.30$ gives following solution.

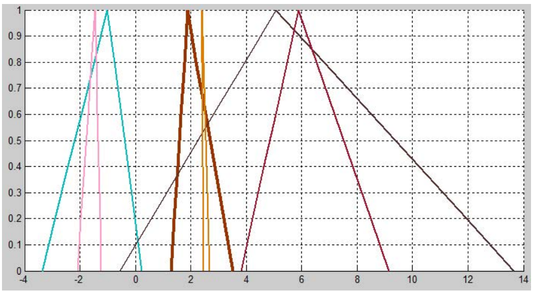

Figure 5. The approximated solution of positivemembership by using GS iterative method after 20 iterations of 2×2 bipolar neutrosophic system of linear equations.

Figure 6. The approximated solution of negativemembership by using GS iterative method after 20 iterations of 2×2 bipolar neutrosophic system of linear equations.

$$\begin{bmatrix} \underline{X}'_{\mathbf{1}i} \\ \underline{X}''_{\mathbf{1}i} \\ \underline{X}'_{\mathbf{1}f} \\ \underline{X}''_{\mathbf{1}f} \\ \underline{X}'_{\mathbf{1}t} \\ \underline{X}''_{\mathbf{1}t} \\ \underline{X}'_{\mathbf{2}i} \\ \underline{X}''_{\mathbf{2}i} \\ \underline{X}'_{\mathbf{2}f} \\ \underline{X}''_{\mathbf{2}f} \\ \underline{X}'_{\mathbf{2}t} \\ \underline{X}''_{\mathbf{2}t} \\ \overline{X_{\mathbf{1}i}}' \\ \overline{X_{\mathbf{1}i}}'' \\ \overline{X_{\mathbf{1}f}}' \\ \overline{X_{\mathbf{1}f}}'' \\ \overline{X_{\mathbf{1}t}}' \\ \overline{X_{\mathbf{1}t}}'' \\ \overline{X_{\mathbf{2}i}}' \\ \overline{X_{\mathbf{2}i}}'' \\ \overline{X_{\mathbf{2}f}}' \\ \overline{X_{\mathbf{2}f}}'' \\ \overline{X_{\mathbf{2}t}}' \\ \overline{X_{\mathbf{2}t}}'' \end{bmatrix}^{15} = \begin{bmatrix} 1.27364714285714285714285714285571273764 \times r - 1.274364857142857142857142857127487 \\ 4.3285714285714285714285714285743648 - 2.3285714285714285714285714238674 \times r \\ 1.2345714285714285714285714285714285 7 - 1.2842857142857142857142857142857423601 \times r \\ 0.0285714285714285714285714285714 29 \times r + 1.9714285714285714285714285714286 \\ 0.4714285714285714285714285714285 7 - 1.8857142857142857142857142857143 \times s \\ 2.7714285714285714285714285714286 \times s + 0.5571428571428571428571428571428 6 \\ 0.6857142857142857142857142857142 9 \times s + 1.3285714285714285714285714285714 \\ 0.8428571428571428571428571428571 4 - 0.3714285714285714285714285714285 7 \times s \\ 0.4857142857142857142857142857142 9 \times r - 0.9857142857142857142857142857142 9 \\ 2.9714285714285714285714285714286 - 0.4714285714285714285714285714285 7 \times r \\ 1.0857142857142857142857142857143 - 1.5857142857142857142857142857143 \times r \\ 0.1714285714285714285714285714285 7 \times r + 2.3285714285714285714285714285714 \\ 0.2746571428571428571428571428571 4 - 2.2646428571428571428571428571429 \times s \\ 3.9763857142857142857142857142857 \times s + 0.8532871428571428571428571428571 \\ 1.9743285714285714285714285714285 71429 \times s + 2.8647142857142857142857142857142 85864 \\ 0.4857142857142857142857142857532 - 0.8142857142857142857142857142857464264 \times s \\ 0.8753714285714285714285714285714 3 \times r - 1.0987571428571428571428571428575309 \\ 3.9651428571428571428571428632 - 0.6743857142857142857142857142857 7453 \times r \\ 1.5637142857142857142857142857142 85654 - 1.9823142857142857142857142857142 85643 \times r \\ 0.4571428571428571428571428571765 \times r + 2.9758571428571428571428571428575 \\ 1.7538571428571428571428571428571 48654 - 1.5643428571428571428571428571429 \times s \\ 1.5432857142857142857142857142857 17653 \times s - 0.9753571428571428571428571428874 \\ 1.2571428571428571428571428571429 \times s + 2.6857142857142857142857142857143 \\ 0.9865714285714285714285714285714 28876 - 0.0753857142857142857142857142857 1428985 \times s \end{bmatrix}$$

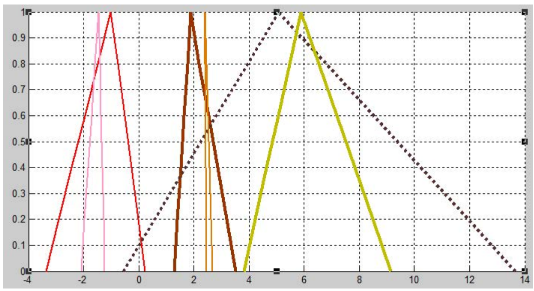

Figure 7. The approximated solution of positive membership by using SOR iterative method after 15 iterations of 2×2 bipolar neutrosophic system of linear equations.

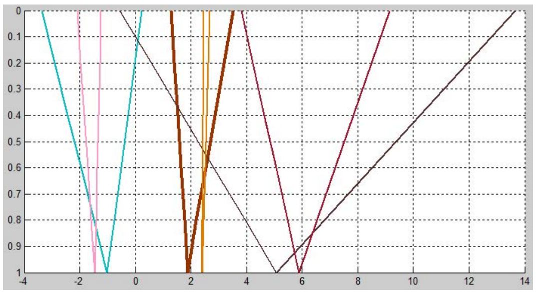

Figure 8. The approximated solution of negative membership by using SOR iterative method after 15 iterations of 2×2 bipolar neutrosophic system of linear equations.

4. Conclusion and comparison analysis

Iterative methods such as, Richardson, Gauss-Seidel, and successive over relaxation are proposed to solve the bipolar neutrosophic system of linear equations (BNSLEs), where L.H.S coefficient are crisp numbers, the R.H.S column vector is bipolar neutrosophic numbers in parametric form. The unknowns of the system are also bipolar neutrosophic numbers. To show the accuracy of these proposed methods numerical example has been developed, display Hausdroff distance. Clearly if we use good SOR parameter then SOR method work better.

In the future, we plan to solve bipolar neutrosophic system of linear equations (BNSLEs) with polynomial parametric numbers and plithogenic parametric numbers.

A comparison analysis of proposed methods

Method	Number of iterations	Distance based on Housdorff Metric
Richardson	25	1.33678×10^{-6}
Gauss-Seidel	20	2.56437×10^{-6}
SOR	15	2.99988×10^{-7}

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