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Fixed point results for Su-type contractive mappings with an application

A. Ali[a], H. Işık[b,*], F. Uddin[a], M. Arshad[a]

[a] *Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan.*
[b] *Department of Mathematics, Faculty of Science and Arts, Muş Alparslan University, Muş 49250, Turkey.*

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Abstract. In this paper, we introduce the concept of Su-type contractive mapping and establish fixed point theorems for such mappings in the setting of ordered extended partial b-metric space. We also develop an application for Fredholm type integral equations to validate our main result and a non-trivial example is given to elucidate our work.

Keywords: Altering distance function, Su-type contraction, extended partial b-metric space, integral equation.

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1. Introduction

In the recent past, a lot of generalization of the famous Banach fixed point theorem has appeared in different framework. So far numerous authors have studied this classical result to establish the existence and uniqueness of a fixed point for different kind of contractive shapes, (see, [1–35]).

Khan et al. [21] introduced and employed the notion of altering distance function to obtain some interesting fixed point results in metric spaces. Note that altering distance

*Corresponding author.
E-mail address: amjad.phdma98@iiu.edu.pk (A. Ali); isikhuseyin76@gmail.com (H. Işık); fahamiiu@gmail.com (F. Uddin); marshadzia@iiu.edu.pk (M. Arshad).

functions are continuous whereas Su [31] defined generalized altering distance function, not necessarily continuous.

In 1993, Czerwik [8] introduced the idea of b-metric space. and in 1994, Matthews [24] investigated the concept of partial metric space. Firstly, Shukla [29] presented the notion of partial b-metric space. Then, Mustafa et al. [25] modified the idea of partial b-metric space and also discussed for the sequel some basic lemma for the convergence of sequences in such spaces. Recently, Parvaneh and Kadelburg [27] introduced the concept of extended partial b-metric space.

The aim of this article goes in this framework, we generalize the results based on [27] in the context of extended partial b-metric space via Su-type contractive mapping. Finally, we have an application for the Fredholm type integral equations.

2. Preliminaries

Before the main results of this paper, We recall some basic concept to make it helpful for the main sequel.

Definition 2.1 [24] A partial metric space on a non-empty set G is a function $p : G \times G \to [0, +\infty)$ such that, for all $g_1, g_2, g_3 \in G$, the following conditions hold:

($p1$) $g_1 = g_2$ if and only if $p(g_1, g_1) = p(g_2, g_2) = p(g_1, g_2)$,
($p2$) $p(g_1, g_1) \leqslant p(g_1, g_2)$,
($p3$) $p(g_1, g_2) = p(g_2, g_1)$,
($p4$) $p(g_1, g_2) \leqslant p(g_1, g_3) + p(g_3, g_2) - p(g_3, g_3)$.

The pair (G, p) is called a partial metric space.

Definition 2.2 [29] A partial b-metric space on a non-empty set G is a function $p_b : G \times G \to [0, +\infty)$, such that for each $g_1, g_2, g_3 \in G$ with $s \geqslant 1$, the following conditions hold:

(p_b1) $g_1 = g_2$ if and only if $p_b(g_1, g_1) = p_b(g_1, g_2) = p_b(g_2, g_2)$,
(p_b2) $p_b(g_1, g_1) \leqslant p_b(g_1, g_2)$,
(p_b3) $p_b(g_1, g_2) = p_b(g_2, g_1)$,
(p_b4) $p_b(g_1, g_2) \leqslant s[p_b(g_1, g_3) + p_b(g_3, g_2) - p_b(g_3, g_3)] + \frac{(1-s)}{2}(p_b(g_1, g_1) + p_b(g_2, g_2))$.

The pair (G, p_b) is called a partial b-metric space.

Remark 1 *Every partial metric space is a partial b-metric space with the setting $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and the behavior of zero self distance. However, the converses do not hold in general.*

Definition 2.3 [27] Let G be a non-empty set. A function $d : G \times G \to [0, +\infty)$ is a p-metric if there exists a strictly increasing continuous function $\Omega : [0, +\infty) \to [0, +\infty)$ with $k \leqslant \Omega(k)$ for $k \in [0, +\infty)$ such that for each $g_1, g_2, g_3 \in G$, the following conditions hold:

(1) $d(g_1, g_2) = 0$ if and only if $g_1 = g_2$,
(2) $d(g_1, g_2) = d(g_2, g_1)$,
(3) $d(g_1, g_2) \leqslant \Omega(d(g_1, g_3) + d(g_3, g_2))$.

The pair (G, d) is called a p-metric space or an extended b-metric space.

Remark 2 *It should be noted that the class of p-metric spaces is considerably larger than the class of b-metric spaces, since a b-metric is a p-metric with the setting $\Omega(k) = sk$*

and a metric is a p-metric with the setting $\Omega(k) = k$.

Definition 2.4 [27] Let G be a non-empty set and $\Omega : [0, +\infty) \to [0, +\infty)$ be a strictly increasing continuous function with $\Omega^{-1}(k) \leqslant k \leqslant \Omega(k)$ for $k \in [0, +\infty)$. A function $p_p : G \times G \to [0, +\infty)$ is called an extended partial b-metric, or a partial p-metric if, for each $g_1, g_2, g_3 \in G$, the following conditions are satisfied:

(p_p1) $g_1 = g_2$ if and only if $p_p(g_1, g_1) = p_p(g_1, g_2) = p_p(g_2, g_2)$,
(p_p2) $p_p(g_1, g_1) \leqslant p_p(g_1, g_2)$,
(p_p3) $p_p(g_1, g_2) = p_p(g_2, g_1)$,
(p_p4) $p_p(g_1, g_2) - p_p(g_1, g_1) \leqslant \Omega(p_p(g_1, g_3) + p_p(g_3, g_2) - p_p(g_3, g_3) - p_p(g_1, g_1))$.

The pair (G, p_p) is called a partial p-metric space, or an extended partial b-metric space.

Remark 3 *Note that condition (p_p4) together with (p_p3) implies that also the following holds for all* $g_1, g_2, g_3 \in G$:

$$p_p(g_1, g_2) - p_p(g_2, g_2) \leqslant \Omega(p_p(g_1, g_3) + p_p(g_3, g_2) - p_p(g_3, g_3) - p_p(g_2, g_2)).$$

It should be noted that the class of partial p-metric spaces is considerably larger than the class of partial b-metric spaces, since a partial b-metric is a partial p-metric with $\Omega(k) = sk$ *and a partial metric is a partial p-metric with* $\Omega(k) = k$.

***Example* 2.5** [27] Let (G, d) be a metric space and $p_p(g_1, g_2) = 1 + \zeta(d(g_1, g_2))$ where $\zeta : [0, +\infty) \to [0, +\infty)$ is a strictly increasing continuous function with $k \leqslant \zeta(k)$ for $k \in [0, +\infty)$ and $\zeta(0) = 0$. Then, it is easy to see that p_p is a partial p-metric with $\Omega(k) = \zeta(k)$. In particular, one can take $\zeta(k) = e^k - 1$, then $p_p(g_1, g_2) = e^{d(g_1, g_2)}$ is a partial p-metric with $\Omega(k) = e^k - 1$.

***Example* 2.6** [27] Let (G, d) be a metric space and $p_p(g_1, g_2) = 1 + \sinh(d(g_1, g_2)^2)$. Then, it is easy to see that p_p is a partial p-metric with $\Omega(k) = 2\cosh k \sinh k = \sinh 2k$.

Note that (G, p_p) is not necessarily a partial metric space. For example, if $G = \mathbb{R}$ is the set of real numbers with usual metric, then $p_p(g_1, g_2) = 1 + \sinh(g_1 - g_2)^2$ is a partial p-metric on G with $\Omega(k) = \sinh 2k$. But it is not a partial metric on G. Indeed, the ordinary (partial) triangle inequality does not hold. To see this, let $g_1 = 2$, $g_2 = 5$ and $g_3 = \frac{5}{2}$. Then, $p_p(2, 5) \approx 4052.54$, $p_p(2, \frac{5}{2}) \approx 1.25$ and $p_p(\frac{5}{2}, 5) \approx 260.01$. Thus, $p_p(2, 5) \nleq p_p(2, \frac{5}{2}) + p_p(\frac{5}{2}, 5) - p_p(\frac{5}{2}, \frac{5}{2})$.

Also, p_p is not a partial b-metric. Indeed, if p_p were partial b-metric, then there would exist fixed $s \geq 1$ for which

$$p_p(g_1, g_2) \leqslant s[p_p(g_1, g_3) + p_p(g_3, g_2) - p_p(g_3, g_3)] + \frac{(1-s)}{2}(p_p(g_1, g_1) + p_p(g_2, g_2)),$$

for all $g_1, g_2, g_3 \geq 0$. However, taking $g_2 = 0$ and $g_3 = 1$, we would have $p_p(g_1, 0) \leqslant s[p_p(g_1, 1) + 1 + \sinh 1 - 1] + \frac{(1-s)}{2}(1+1)$, i.e., $\sinh {g_1}^2 \leq s(1 + \sinh(g_1 - 1)^2 + \sinh 1) - s$ which can not hold for fixed s when $g_1 \to +\infty$.

Recall that a real function T is called super-additive, if $T(r_1 + r_2) \geqslant T(r_1) + T(r_2)$ for each $r_1, r_2 \in D(T)$. If T is a super-additive function, and if $0 \in D(T)$, then $T(0) \leqslant 0$. Indeed, super-additivity of T yields that $T(r_1) \leqslant T(r_1 + r_2) - T(r_2)$, for each $r_1, r_2 \in D(T)$. Setting $r_1 = 0$, we get $T(0) \leqslant T(0 + r_2) - T(r_2) = 0$. Morever, it is easy to see that $2T(r) \leqslant T(2r)$ for each $r \in D(T)$.

Proposition 2.7 [27] Every partial p-metric p_p on a non-empty set G with a super-additive function Ω, defines a p-metric d_{p_p}, where

$$d_{p_p}(g_1, g_2) = 2p_p(g_1, g_2) - p_p(g_1, g_1) - p_p(g_2, g_2), \quad \text{for each } g_1, g_2 \in G.$$

Lemma 2.8 [27] Let (G, p_p) be a partial p-metric space. Then the following conditions are satisfied:

(i) if $p_p(g_1, g_2) = 0$, then $g_1 = g_2$;
(ii) if $g_1 \neq g_2$, then $p_p(g_1, g_2) > 0$.

Lemma 2.9 [27] Let (G, p_p) be a partial p-metric space with super-additive function Ω.

(i) A sequence $\{g_n\}$ is a p_p-Cauchy in (G, p_p) iff it is a p-Cauchy in the p-metric space $\left(G, d_{p_p}\right)$.
(ii) The space (G, p_p) is p_p-complete iff the p-metric space $\left(G, d_{p_p}\right)$ is p-complete. Moreover,

$$\lim_{n\to\infty} d_{p_p}(g, g_n) = 0 \Leftrightarrow \lim_{n\to\infty} p_p(g, g_n) = \lim_{n,m\to\infty} p_p(g_n, g_m) = p_p(g, g).$$

Lemma 2.10 [27] Let (G, p_p) be a partial p-metric space. Suppose that $\{g_n\}$ and $\{g_n^*\}$ are convergent to g and g^*, respectively. Then

$$\begin{aligned}
&\Omega^{-1}(\Omega^{-1}[p_p(g, g^*) - p_p(g, g)] - 2p_p(g, g)) - p_p(g^*, g^*) \\
&\quad \leqslant \liminf_{n\to\infty}\ p_p(g_n, g_n^*) \leq \limsup_{n\to\infty}\ p_p(g_n, g_n^*) \\
&\quad \leqslant (2p_p(g, g) + \Omega[p_p(g, g^*) + p_p(g^*, g^*)]) + p_p(g, g).
\end{aligned}$$

In particular, if $p_p(g, g^*) = 0$, then $\lim_{n\to\infty} p_p(g_n, g_n^*) = 0$.

In addition, for all $g_1 \in G$,

$$\begin{aligned}
&\Omega^{-1}[p_p(g, g_1) - p_p(g, g)] - p_p(g, g) \\
&\quad \leqslant \liminf_{n\to\infty}\ p_p(g_n, g_1) \leqslant \limsup_{n\to\infty}\ p_p(g_n, g_1) \\
&\quad \leqslant \Omega[p_p(g, g) + p_p(g, g_1)] + p_p(g, g).
\end{aligned}$$

In particular, if $p_p(g, g_1) = 0$, then $\lim_{n\to\infty} p_p(g_n, g_1) = 0$.

Definition 2.11 [31] A mapping $\psi : [0, +\infty) \to [0, +\infty)$ is called a generalized altering distance function if

(i) ψ is non-decreasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Set
$\Psi = \{\psi\colon\ \psi : [0, +\infty) \to [0, +\infty)$ is a generalized altering distance function$\}$
and
$\Phi = \{\varphi\colon\ \varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right upper semi-continuous function such that $\psi(t) > \varphi(t)$ for all $t > 0$ where $\psi \in \Psi\}$.

3. Main results

A triplet $(G, \preceq, p_p)$ will be called an ordered partial p-metric space (ordered PPMS, for short) if $(G, \preceq)$ is a partially ordered set and p_p is a partial p-metric on G.

For arbitrary points $g_1,\ g_2 \in G$, it is said to be

(i) g_1 and g_2 are comparable point, if either $g_1 \preceq g_2$ or $g_2 \preceq g_1$;
(ii) T is nondecreasing function, if $Tg_1 \preceq Tg_2$ whenever $g_1 \preceq g_2$;
(iii) $(G, \preceq, p_p)$ is regular, if for each nondecreasing sequence $\{g_n\} \in G$ convergent to some point $g \in G$, then $g_n \preceq g,$ for all $n \in \mathbb{N}$.

Definition 3.1 Let $(G, \preceq, p_p)$ be an ordered partial p-metric space with function Ω. An operator $T : G \to G$ is called Su-type contractive mapping, if there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned}\psi(\Omega^2(2p_p(Tg_1, Tg_2))) \leqslant \varphi(\max\{&p_p(g_1, g_2), p_p(g_1, Tg_1) + p_p(g_2, Tg_2),\\ &p_p(g_1, Tg_2) - p_p(g_1, g_1), p_p(g_2, Tg_1)\}),\end{aligned} \tag{1}$$

for each comparable $g_1, g_2 \in G$.

Theorem 3.2 Let $(G, \preceq, p_p)$ be an ordered p_p-complete PPMS with super-additive function Ω. Let $T : G \to G$ be a non-decreasing continuous Su-type contractive mapping. If there exists $g_0 \in G$ such that $g_0 \preceq Tg_0$, then T has a fixed point.

Proof. Let $g_0 \in G$ be an arbitrary point such that $g_0 \preceq Tg_0$. Let $\{g_n\}$ be the Picard sequence with initial point g_0, that is, $g_n = T^n g_0 = Tg_{n-1}$ and $u_n = p_p(g_n, g_{n+1}) = p_p(T^n g_0, T^{n+1} g_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Since $g_0 \preceq Tg_0 = g_1$ and T is non-decreasing, $g_1 = Tg_0 \preceq g_2 = Tg_1$. By induction, we get

$$g_0 \preceq g_1 \preceq g_2 \preceq \cdots \preceq g_n \preceq g_{n+1} \preceq \cdots .$$

If $g_{n_0} = g_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $g_{n_0} = Tg_{n_0}$ and so g_{n_0} is a fixed point of T. Therefore, assume that $g_n \neq g_{n+1}$ for all $n \in \mathbb{N}$. By contractive condition (1) with $g_1 = g_{n-1}$ and $g_2 = g_n$, we have

$$\begin{aligned}\psi(\Omega(2u_n)) \leq \psi(\Omega^2(2u_n)) &= \psi(\Omega^2(2p_p(Tg_{n-1}, Tg_n)))\\ &\leq \varphi(\max\{p_p(g_{n-1}, g_n), p_p(g_{n-1}, Tg_{n-1}) + p_p(g_n, Tg_n),\\ &\quad p_p(g_{n-1}, Tg_n) - p_p(g_{n-1}, g_{n-1}), p_p(g_n, Tg_{n-1})\})\\ &= \varphi(\max\{u_{n-1}, u_{n-1} + u_n, p_p(g_{n-1}, g_{n+1}) - p_p(g_{n-1}, g_{n-1}),\\ &\quad p_p(g_n, g_n)\})\\ &\leqslant \varphi(\max\{u_{n-1} + u_n, \Omega(u_{n-1} + u_n), p_p(g_n, g_n)\})\\ &= \varphi(\Omega(u_{n-1} + u_n))\\ &< \psi(\Omega(u_{n-1} + u_n)).\end{aligned} \tag{2}$$

By the properties of ψ and Ω, it follows that $2u_n < u_{n-1} + u_n$; that is,

$$u_n < u_{n-1}, \qquad \text{for all } n \in \mathbb{N}. \tag{3}$$

From (3), the sequence $\{u_n\}$ is decreasing and hence converges to a real number $r \geqslant 0$. We now show that $r = 0$. Assume on the contrary that $r > 0$. Then, letting $n \to \infty$ in (2), we have

$$\begin{aligned}\psi(\Omega(2r)) &\leqslant \lim_{n\to\infty} \psi\left(\Omega(2u_n)\right)\\ &\leqslant \lim_{n\to\infty} \varphi\left(\Omega(u_{n-1}+u_n)\right) \leqslant \varphi\left(\Omega(2r)\right),\end{aligned}$$

a contradiction and hence $r = 0$, that is, $\lim_{n\to\infty} u_n = 0$, and thus

$$\lim_{n\to\infty} p_p(g_n, g_{n+1}) = \lim_{n\to\infty} p_p(g_n, g_n) = 0. \tag{4}$$

To show that $\{g_n\}$ is a p_p-Cauchy sequence in G, we have to show that $\{g_n\}$ is a p-Cauchy sequence in (G, d_{p_p}) (see Lemma 2.9). Suppose the contrary, that is, $\{g_n\}$ is not a p-Cauchy sequence. Then, there exists an $\varepsilon > 0$ for which we can find two subsequences $\{g_{m_k}\}$ and $\{g_{n_k}\}$ of $\{g_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and

$$d_{p_p}(g_{m_k}, g_{n_k}) \geqslant \varepsilon \quad \text{ and } \quad d_{p_p}(g_{m_k}, g_{n_k-1}) < \varepsilon. \tag{5}$$

Using the triangular inequality and (5), we get

$$\begin{aligned}\varepsilon \leq d_{p_p}(g_{m_k}, g_{n_k}) &\leqslant \Omega(d_{p_p}(g_{m_k}, g_{n_k-1}) + d_{p_p}(g_{n_k-1}, g_{n_k}))\\ &< \Omega(\varepsilon + d_{p_p}(g_{n_k-1}, g_{n_k})).\end{aligned}$$

Taking the upper limit as $k \to \infty$, and using (4) and (5), we get

$$\Omega^{-1}(\varepsilon) \leqslant \limsup_{k\to\infty} d_{p_p}(g_{m_k}, g_{n_k}) \leqslant \varepsilon,$$

and so

$$\varepsilon \leqslant \liminf_{k\to\infty} d_{p_p}(g_{m_k}, g_{n_k}) \leq \limsup_{k\to\infty} d_{p_p}(g_{m_k}, g_{n_k}) \leqslant \Omega(\varepsilon). \tag{6}$$

Since

$$\begin{aligned}\varepsilon &\leqslant d_{p_p}(g_{m_k}, g_{n_k}) \leq \Omega(d_{p_p}(g_{m_k}, g_{m_k+1}) + d_{p_p}(g_{m_k+1}, g_{n_k}))\\ &\leq \Omega(d_{p_p}(g_{m_k}, g_{m_k+1}) + \Omega(d_{p_p}(g_{m_k+1}, g_{n_k+1}) + d_{p_p}(g_{n_k+1}, g_{n_k}))),\end{aligned}$$

by using (4), we obtain

$$\varepsilon \leqslant \Omega(\varepsilon) \leqslant \liminf_{k\to\infty} \Omega^2(d_{p_p}(g_{m_k+1}, g_{n_k+1})). \tag{7}$$

On the other hand, by the definition of d_{p_p} and (4),

$$\liminf_{k\to\infty} d_{p_p}(g_{m_k}, g_{n_k}) = 2 \liminf_{k\to\infty} p_p(g_{m_k}, g_{n_k}). \tag{8}$$

From (1), we get

$$\begin{aligned}
\psi(\Omega^2(2p_p(g_{m_k+1}, g_{n_k+1}))) &= \psi(\Omega^2(2p_p(Tg_{m_k}, Tg_{n_k}))) \\
&\le \varphi(\max\{p_p(g_{m_k}, g_{n_k}), p_p(g_{m_k}, g_{m_k+1}) + p_p(g_{n_k}, g_{n_k+1}), \\
&\qquad p_p(g_{m_k}, g_{n_k+1}) - p_p(g_{m_k}, g_{m_k}), p_p(g_{n_k}, g_{m_k+1})\}) \\
&\le \varphi(\max\{p_p(g_{m_k}, g_{n_k}), p_p(g_{m_k}, g_{m_k+1}) + p_p(g_{n_k}, g_{n_k+1}), \\
&\qquad \Omega(p_p(g_{m_k}, g_{n_k}) + p_p(g_{n_k}, g_{n_k+1})), \Omega(p_p(g_{n_k}, g_{m_k}) + p_p(g_{m_k}, g_{m_k+1}))\}).
\end{aligned}$$

Letting $k \to \infty$ in the previous inequality, by the properties of ψ and φ, we deduce

$$\begin{aligned}
\psi(\Omega(\varepsilon)) &\le \lim_{k\to\infty} \psi(\Omega^2(2p_p(g_{m_k+1}, g_{n_k+1}))) \\
&\le \lim_{k\to\infty} \varphi(\max\{p_p(g_{m_k}, g_{n_k}), p_p(g_{m_k}, g_{m_k+1}) + p_p(g_{n_k}, g_{n_k+1}), \\
&\qquad \Omega(p_p(g_{m_k}, g_{n_k}) + p_p(g_{n_k}, g_{n_k+1})), \Omega(p_p(g_{n_k}, g_{m_k}) + p_p(g_{m_k}, g_{m_k+1}))\}) \\
&\le \varphi(\max\{\varepsilon, 0, \Omega(\varepsilon), \Omega(\varepsilon)\}) = \varphi(\Omega(\varepsilon)),
\end{aligned}$$

which is a contradiction. Therefore $\{g_n\}$ is a p-Cauchy sequence in (G, d_{p_p}). Since (G, p_p) is p_p-complete, by Lemma 2.9, $\left(G, d_{p_p}\right)$ is a p-complete p-metric space. Hence, there exists $r \in G$ such that $\lim_{n\to\infty} d_{p_p}(g_n, r) = 0$ and

$$\lim_{n\to\infty} p_p(r, g_n) = \lim_{n\to\infty} p_p(g_n, g_n) = p_p(r, r).$$

Also, from (4), we have

$$\lim_{n\to\infty} p_p(r, g_n) = \lim_{n\to\infty} p_p(g_n, g_n) = p_p(r, r) = 0.$$

Applying triangular inequality, we obtain

$$p_p(r, Tr) - p_p(r, r) \leqslant \Omega\left(p_p(r, Tg_n) + p_p(Tg_n, Tr)\right).$$

Taking $n \to \infty$ and using the continuity of T and Ω, and $p_p(r, r) = 0$, we get

$$p_p(r, Tr) \leqslant \Omega\left(\lim_{n\to\infty} p_p(r, g_{n+1}) + \lim_{n\to\infty} p_p(Tg_n, Tr)\right) = \Omega(p_p(Tr, Tr)). \tag{9}$$

From (1), we deduce

$$\begin{aligned}
\psi(\Omega(2p_p(Tr, Tr))) &\le \psi(\Omega^2(2p_p(Tr, Tr))) \\
&\le \varphi(\max\{p_p(r, r), p_p(r, Tr) + p_p(r, Tr), \\
&\qquad p_p(r, Tr) - p_p(r, r), p_p(r, Tr)\}) \\
&= \varphi(2p_p(r, Tr)) \\
&< \psi(2p_p(r, Tr)).
\end{aligned} \tag{10}$$

Suppose that $p_p(r,Tr) > 0$. Since ψ is non-decreasing and Ω is super-additive, we deduce

$$2\Omega(p_p(Tr,Tr)) \leq \Omega(2p_p(Tr,Tr)) < 2p_p(r,Tr).$$

So (9) yields that

$$2\Omega(p_p(Tr,Tr)) < 2\Omega(p_p(Tr,Tr)),$$

which is a contradiction. Thus, $p_p(r,Tr) = 0$ and hence $r = Tr$. ■

Notice that the continuity of T in Theorem 3.2 is not necessary and can be dropped.

Theorem 3.3 Under the same hypotheses of Theorem 3.2 and without assuming the continuity of T, suppose that $(G,\preceq,p_p)$ is regular. Then T has a fixed point in G.

Proof. Following similar arguments to those given in Theorem 3.2, we construct a non-decreasing sequence g_n in G such that $g_n \to g$ for some $g \in G$. Using the regularity of G, we have $g_n \preceq g$ for all $n \in \mathbb{N}$. Now, we have to show that $Tg = g$. By contractive condition (1), we have

$$\psi(\Omega^2(2p_p(Tg_n,Tg))) \leqslant \varphi(H(g_n,g)), \tag{11}$$

where

$$\begin{aligned} &H(g_n,g)\\ &= \max\{p_p(g_n,g), p_p(g_n,Tg_n) + p_p(g,Tg), p_p(g_n,Tg) - p_p(g_n,g_n), p_p(g,Tg_n)\}\\ &= \max\{p_p(g_n,g), p_p(g_n,g_{n+1}) + p_p(g,Tg), p_p(g_n,Tg) - p_p(g_n,g_n), p_p(g,g_{n+1})\}. \end{aligned} \tag{12}$$

Suppose that $p_p(g,Tg) > 0$. Taking limit of (12) as $n \to \infty$ and using Lemma 2.10, we get

$$\begin{aligned} \Omega^{-1}[p_p(g,Tg)] &= \min\{p_p(g,Tg), \Omega^{-1}[p_p(g,Tg) - p_p(g,g)] - p_p(g,g)\}\\ &\leq \liminf_{n\to\infty} H(g_n,g) \leq \limsup_{n\to\infty} H(g_n,g)\\ &\leq \max\{p_p(g,Tg), \Omega[p_p(g,Tg) + p_p(g,g)] + p_p(g,g)\}\\ &= \Omega[p_p(g,Tg)]. \end{aligned} \tag{13}$$

Again, taking the upper limit as $n \to \infty$ in (11) and using Lemma 2.10 and (13), we deduce

$$\begin{aligned} \psi(\Omega^2[\Omega^{-1}[p_p(g,Tg)]]) &\leq \limsup_{n\to\infty} \psi(\Omega^2[p_p(g_{n+1},Tg)])\\ &\leq \limsup_{n\to\infty} \psi(\Omega^2[2p_p(g_{n+1},Tg)])\\ &\leqslant \limsup_{n\to\infty} \varphi(H(g_n,g))\\ &\leqslant \varphi(\Omega[p_p(g,Tg)]), \end{aligned}$$

which implies that $p_p(g,Tg) = 0$ and so $g = Tg$. ■

By choosing $\varphi(t) = \psi(t) - \phi(t)$ in Theorems 3.2 and 3.3, we obtain the following result.

Corollary 3.4 [27] Let $(G, \preceq, p_p)$ be an ordered p_p-complete PPMS with super-additive function Ω and $T : G \to G$ be a non-decreasing mapping. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\psi(\Omega^2(2p_p(Tg_1, Tg_2))) \leqslant \psi(H(g_1, g_2)) - \phi(H(g_1, g_2)),$$

for each comparable $g_1, g_2 \in G$, where

$$H(g_1, g_2) = \max\{p_p(g_1, g_2), p_p(g_1, Tg_1) + p_p(g_2, Tg_2), p_p(g_1, Tg_2) - p_p(g_1, g_1), p_p(g_2, Tg_1)\}.$$

Suppose that also the following conditions hold:

(i) There exists $g_0 \in G$ such that $g_0 \preceq Tg_0$;
(ii) T is continuous or $(G, \preceq, p_p)$ is regular.

Then T has a fixed point.

Corollary 3.5 Let $(G, \preceq, p_p)$ be an ordered p_p-complete PPMS with super-additive function Ω and $T : G \to G$ be a non-decreasing mapping. Assume that there exist $\alpha, \beta, \gamma, \delta \geqslant 0$ with $\alpha + \beta + \gamma + \delta \in (0, 1)$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned}\psi(\Omega^2(2p_p(Tg_1, Tg_2))) \leqslant \varphi(&\alpha p_p(g_1, g_2) + \beta[p_p(g_1, Tg_1) + p_p(g_2, Tg_2)] \\ &+\gamma[p_p(g_1, Tg_2) - p_p(g_1, g_1)] + \delta p_p(g_2, Tg_1)),\end{aligned}$$

for each comparable $g_1, g_2 \in G$. Suppose that also the following conditions hold:

(i) There exists $g_0 \in G$ such that $g_0 \preceq Tg_0$;
(ii) T is continuous or $(G, \preceq, p_p)$ is regular.

Then T has a fixed point.

If we take $\psi(t) = t$ and $\varphi(t) = kt$ in Theorems 3.2 and 3.3, we obtain the following result.

Corollary 3.6 Let $(G, \preceq, p_p)$ be an ordered p_p-complete PPMS with super-additive function Ω and $T : G \to G$ be a non-decreasing mapping. Assume that there exists $k \in [0, 1)$ such that

$$\begin{aligned}\Omega^2(2p_p(Tg_1, Tg_2)) \leqslant k \max\{&p_p(g_1, g_2), p_p(g_1, Tg_1) + p_p(g_2, Tg_2), \\ &p_p(g_1, Tg_2) - p_p(g_1, g_1), p_p(g_2, Tg_1)\},\end{aligned}$$

for each comparable $g_1, g_2 \in G$. Suppose that also the following conditions hold:

(i) There exists $g_0 \in G$ such that $g_0 \preceq Tg_0$;
(ii) T is continuous or $(G, \preceq, p_p)$ is regular.

Then T has a fixed point.

If we take $p_p(g_1, g_2) = 1 + \sinh(d(g_1, g_2)^2)$ in Corollary 3.6, where $(G, \preceq, d)$ is a complete ordered metric space, we have the following result.

Corollary 3.7 Let $(G, \preceq, d)$ be a complete ordered metric space and $T : G \to G$ be a non-decreasing mapping. Assume that there exists $k \in [0,1)$ such that

$$\sinh[2\sinh[4+4\sinh(d(Tg_1,Tg_2)^2)]] \leqslant k[1+\sinh(d(g_1,g_2)^2)],$$

for each comparable $g_1, g_2 \in G$. Suppose that also the following conditions hold:

(*i*) There exists $g_0 \in G$ such that $g_0 \preceq Tg_0$;
(*ii*) T is continuous or $(G, \preceq, d)$ is regular.

Then T has a fixed point.

Remark 4 *In Theorems 3.2 and 3.3, it can be proved in a standard way that T has a unique fixed point provided that all fixed points of T are comparable.*

***Example* 3.8** Let $G = \{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ be equipped with the following partial order $\preceq$:

$$\preceq := \{(0,0), (\frac{1}{3},\frac{1}{3}), (\frac{2}{3},\frac{2}{3}), (1,1), (\frac{4}{3},\frac{4}{3}), (\frac{5}{3},\frac{2}{3}), (\frac{5}{3},\frac{5}{3}), (2,2)\}.$$

Define $p_p : G \times G \to [0,+\infty)$ by

$$p_p(g_1,g_2) = \begin{cases} 0, & \text{if } g_1 = g_2, \\ 1+\sinh[(g_1+g_2)^2], & \text{if } g_1 \neq g_2. \end{cases}$$

It is easy to see that (G, p_p) is a p_p-complete PPMS, with $\Omega(t) = \sinh 2t$ (which is super-additive).

Define $T : G \to G$ by

$$T = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ \frac{1}{3} & \frac{2}{3} & 1 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

Obviously, T is non-decreasing and continuous.

Define $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = t/2$ and $\varphi(t) = t/4$. In order to see that T is F_Ω-contractive mapping for nontrivial cases, we only need to check the case $g_1 = 2/3$ and $g_2 = 5/3$ or $g_1 = 5/3$ and $g_2 = 2/3$. Then

$$\begin{aligned}
&\psi(\Omega^2(2p_p(T\frac{2}{3}, T\frac{5}{3}))) = \psi(\Omega^2(0)) = 0 \\
&\quad \leqslant \varphi(\max\{p_p(\frac{2}{3},\frac{5}{3}), p_p(\frac{2}{3},T\frac{2}{3}) + p_p(\frac{5}{3},T\frac{5}{3}), p_p(\frac{2}{3},T\frac{5}{3}) - p_p(\frac{2}{3},\frac{2}{3}), p_p(\frac{5}{3},T\frac{2}{3})\}) \\
&\quad = \varphi(\max\{p_p(\frac{2}{3},\frac{5}{3}), p_p(\frac{2}{3},1) + p_p(\frac{5}{3},1), p_p(\frac{2}{3},1), p_p(\frac{5}{3},1)\}) \\
&\quad = \varphi(p_p(\frac{2}{3},1) + p_p(\frac{5}{3},1)) \\
&\quad = \varphi(1+\sinh(25/9) + 1 + \sinh(64/9)) \approx 155.69
\end{aligned}$$

Thus, all the hypotheses of Theorem 3.3 are satisfied and so T possesses a fixed point. In fact, 1 and 2 are two fixed points of T. Note that the set $(\{1,2\}, \preceq)$ is not well ordered (i.e., elements 1 and 2 are not comparable).

4. An application

In this section, we display to view of existence of solutions for a Fredholm type integral equations using our result. Consider the following Fredholm integral equation:

$$\chi(r) = \gamma(r) + \int_0^1 D(r, s, \chi(s))\, ds, \tag{14}$$

$r, s \in I = [0,1]$, where $\gamma : I \to \mathbb{R}$ and $D : I \times I \times \mathbb{R} \to \mathbb{R}$. Let the space $G = C(I, \mathbb{R})$ consist of all real-valued functions which are continuous on I equipped with the partial order

$$g_1 \preceq g_2 \iff g_1(r) \leqslant g_2(r),\ r \in I.$$

For $\chi \in G$, define

$$||\chi|| = \sup_{r \in I} |\chi(r)|.$$

Notice that $||\cdot||$ is a norm equivalent to the supremum norm and $(G, ||\cdot||)$ is a Banach space. The metric induced by this norm is given by

$$d(g_1, g_2) = ||g_1 - g_2|| = \sup_{r \in I} |g_1(r) - g_2(r)|,$$

for all $g_1, g_2 \in G$. Now, let $\xi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing continuous function with $r \leqslant \xi(r)$ and consider G endowed with the partial p-metric given by

$$\rho(g_1, g_2) = 1 + \xi(d(g_1, g_2)), \quad \text{for all } g_1, g_2 \in G.$$

It is easy to see that (G, ρ) is complete and $(G, \preceq, \rho)$ is regular.

Define $T : G \to G$ by

$$T(\chi(r)) = \gamma(r) + \int_0^1 D(r, s, \chi(s)) ds, \qquad \chi \in G, r \in I.$$

Clearly, a function $g \in G$ is a solution of (14) if and only if it is a fixed point of T. Now, we prove the following theorem to validate the existence of solution for the integral equation (14).

Theorem 4.1 Suppose that the following assertions are satisfied:

(*i*) $\gamma : I \to \mathbb{R}$ and $D : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous functions;

(*ii*) if $g_1 \preceq g_2$, then

$$D(r, s, g_1(s)) \leqslant D(r, s, g_2(s)), \quad \text{for all } r, s \in I;$$

(*iii*) for all $g_1, g_2 \in G$ with $g_1 \preceq g_2$ and $g_1 \neq g_2$, and for all $r \in I$,

$$\xi^2(2 + 2\xi(\int_0^1 |D(r, s, g_1(s)) - D(r, s, g_2(s))| ds)) \leqslant \theta(|g_1(s) - g_2(s)|),$$

where $\theta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right upper semi-continuous function with $\theta(0) = 0$ and $\theta(t) < t$ for all $t > 0$;

(iv) There exists a continuous function $g_0 : I \to \mathbb{R}$ such that

$$g_0(r) \leqslant \gamma(r) + \int_0^1 D(r, s, g_0(s))\, ds, \quad r \in I.$$

Proof. It follows from (ii) that the mapping T is non-decreasing w.r.t. $\preceq$.

Let $g_1, g_2 \in G$ with $g_1 \preceq g_2$ and $Tg_1 \neq Tg_2$. Then, for all $r \in I$,

$$\begin{aligned}
&\xi^2(2 + 2\xi(|Tg_1(r) - Tg_2(r)|)) \\
&\quad \leq \xi^2(2 + 2\xi(\int_0^1 |D(r, s, g_1(s)) - D(r, s, g_2(s))| ds)) \\
&\quad \leqslant \theta(|g_1(s) - g_2(s)|) \leqslant \theta(d(g_1, g_2)) \\
&\quad \leqslant \theta(H(g_1, g_2)),
\end{aligned}$$

where

$$H(g_1, g_2) = \max\{\rho(g_1, g_2), \rho(g_1, Tg_1) + \rho(g_2, Tg_2), \rho(g_1, Tg_2) - \rho(g_1, g_1), \rho(g_2, Tg_1)\}.$$

Putting $\psi(t) = t$, $\varphi(t) = \theta(t)$ and $\Omega = \xi$, we have

$$\psi(\Omega^2(2\rho(Tg_1, Tg_2))) \leqslant \varphi(H(g_1, g_2))$$

for each $g_1, g_2 \in G$ with $g_1 \preceq g_2$. Thus, all the conditions of Theorem 3.3 are satisfied and so the integral equation (14) possesses the required solution. ■

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