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# Approximate solutions of homomorphisms and derivations of the generalized Cauchy-Jensen functional equation in $C^*$-ternary algebras

P. Kaskasem[a,*], C. Klin-eam[a,b]

[a] *Department of Mathematics, Faculty of Science Naresuan University, Phitsanulok, 65000, Thailand.*
[b] *Research center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok, Thailand.*

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**Abstract.** In this paper, we prove Hyers-Ulam-Rassias stability of $C^*$-ternary algebra homomorphism for the following generalized Cauchy-Jensen equation

$$\eta\mu f\left(\frac{x+y}{\eta}+z\right)=f(\mu x)+f(\mu y)+\eta f(\mu z)$$

for all $\mu \in \mathbb{S} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and for any fixed positive integer $\eta \geqslant 2$ on $C^*$-ternary algebras by using fixed poind alternative theorem. Moreover, we investigate Hyers-Ulam-Rassias stability of generalized $C^*$-ternary derivation for such function on $C^*$-algebras by the same method.



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**Keywords:** Cauchy-Jensen functional equations, Hyers-Ulam-Rassias stability, $C^*$-ternary algebras, alternative fixed point theorem.

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## 1. Introduction and Preliminaries

The initial concept of the stability theory of functional equations was introduced by Pólya and Szegő [24] which is stated as follows: For every real sequence $\{x_n\}_{n\in\mathbb{N}}$ with

$$\sup_{m,n\in\mathbb{N}} |x_{m+n} - x_m - x_n| \leqslant 1$$

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*Corresponding author.
E-mail address: prondanaik@nu.ac.th (P. Kaskasem); chakkridk@nu.ac.th (C. Klin-eam).

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there is a real number $x$ such that $\sup_{n\in\mathbb{N}} |x_n - nx| \leqslant 1$, where $x = \lim_{n\to\infty} \frac{x_n}{n}$. The motivation for studying of stability theory of functional equations was initiated by Ulam. In 1940, Ulam [27] proposed some unsolved problems that one of them is stability problem of functional equation concerning the stability of group homomorphisms as follows

"Let $(G_1, \cdot\,)$ be a group and $(G_2, \star, d)$ be a metric group with the metric $d(\cdot\,,\cdot\,)$. Given a real number $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x\cdot y), h(x) \star h(y)) < \delta$ for all $x, y, \in G_1$, then there exists a homomorphism $g : G_1 \to G_2$ with $d(h(x), g(x)) < \varepsilon$ for all $x \in G_1$?"

These questions form is the object of the stability theory. If the answers is affirmative, we say that the functional equation for homomorphism is stable. In 1941, Hyers [11] provided a first affirmative partial answer to Ulam's problem for the case of approximately additive mapping in which $X$ and $Y$ are Banach spaces. This result is stated as follows:

**Theorem 1.1** [11] Let $X$ and $Y$ be Banach spaces and let $f : X \to Y$ satisfy

$$\|f(x + y) - f(x) - f(y)\|_Y \leqslant \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon > 0$. Then, there exists a unique additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\|_Y \leqslant \varepsilon$ for all $x \in X$ where $g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

The method, which generates the additive mapping $g$ is called a direct method. This method is the most important for studying the stability of various functional equations.

In 1978, Rassias [26] provided a generalization of Hyers' theorem for linear mapping by considering an unbounded Cauchy difference $f(x + y) - f(x) - f(y)$ as follows:

**Theorem 1.2** [26] Let $X$ and $Y$ be Banach spaces. Let $f : X \to Y$ satisfy the inequality

$$\|f(x + y) - f(x) - f(y)\|_Y \leqslant \varepsilon(\|x\|_X^p + \|y\|_X^p)$$

for all $x, y \in X$ where $\varepsilon > 0$ and $0 \leqslant p < 1$. Then there exists a unique additive mapping $g : X \to Y$ defined by $g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ such that $\|f(x) - g(x)\|_Y \leqslant \frac{2\varepsilon}{2-2^p}\|x\|_X^p$ for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in X$, then $g$ is linear.

Next, a generalization of Rassias' results was developed by Găvruţa [23] in 1994 by replacing the unbounded Cauchy difference by a general control function.

**Theorem 1.3** [23] Let $G$ be commutative group and $X$ be Banach space. Let $\phi : G^2 \to [0, \infty)$ be a function satisfying $\Phi(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}\phi(2^k x, 2^k y) < \infty$ for all $x, y \in G$. If a mapping $f : G \to X$ satisfies the inequality $\|f(x + y) - f(x) - f(y)\| \leqslant \phi(x, y)$ for all $x, y \in G$, then there exists a unique additive function such that $\|f(x) - g(x)\| \leqslant \Phi(x, x)$ for all $x \in G$.

For more information on that subject and further references we refer to a survey paper [5] and to a recent monograph on Ulam stability [6].

In 2006, Baak [3] investigated the Cauchy-Rasssis stability of the following Cauchy-Jensen functional equations:

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) = f(x) + 2f(z), \tag{1}$$

$$f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x-y}{2} + z\right) = f(y), \tag{2}$$

$$2f\left(\frac{x+y}{2} + z\right) = f(x) + f(y) + 2f(z) \tag{3}$$

for all $x, y, z \in X$, in Banach spaces by using direct method. Later in the same year, Park [21] proved the Hyers-Ulam-Rassias stability of homomorphisms and derivation in $C^*$-ternary algebras for functional equations (1), (2) and (3) via direct method.

The fixed point method was applied to studying the stability of functional equations by Baker in 1991 [4] by using the Banach contraction principle. Next, Radu [25] proved a stability of functional equation by the alternative of fixed point, which was introduced by Diaz and Margolis [8]. The fixed point method has provided a lot of influence in the development of stability.

In 2008, Park and An [22] proved the Hyers-Ulam-Rassias stability of $C^*$-algebra homomorphisms and generalized derivations on $C^*$-algebras by using alternative of fixed point theorem for the Cauchy-Jensen functional equation $2f\left(\frac{x+y}{2}+z\right) = f(x) + f(y) + 2f(z)$, which was introduced and investigated by Baak [3]. After that Gao et al. [10] introduced generalized Cauchy-Jensen equation. Let $G$ be an $n$-divisible abelian group where $n \in \mathbb{N}$ and $X$ be a normed space with norm $\|\cdot\|_X$. For a mapping $f : G \to X$, the equation

$$nf\left(\frac{x+y}{n}+z\right) = f(x) + f(y) + nf(z) \tag{4}$$

for all $x, y, z \in G$ and $n \in \mathbb{N}\backslash\{0\}$ is said to be a generalized Cauchy-Jensen equation, shortly GCJE. In particular, when $n = 2$, it is called a Cauchy-Jensen equation. Moreover, they gave the following useful properties as follow:

**Proposition 1.4** [10] Let $G$ be an $n$-divisible abelian group for some positive integer $n$ and $X$ be a normed space with norm $\|\cdot\|_X$. Then a mapping $f : G \to X$ is additive if and only if it satisfies $\|f(x) + f(y) + nf(z)\|_X \leqslant \|nf(\frac{x+y}{n} + z)\|_X$ for all $x, y, z \in G$.

The following corollary is an immediate consequence of Proposition 1.4.

**Corollary 1.5** [10] For a mapping $f : G \to X$, the following statements are equivalent.

(a) $f$ is additive.
(b) $f(x) + f(y) + nf(z) = nf(\frac{x+y}{n} + z)$, for all $x, y, z \in G$.
(c) $\|f(x) + f(y) + nf(z)\|_X \leqslant \|nf(\frac{x+y}{n} + z)\|_X$, for all $x, y, z \in G$.

Clearly, a vector space is $n$-divisible abelian group, so Corollary 1.5 is right when $G$ is a vector space. We refer stability results of the functional equation (4) to [9, 13–16]. Next, we recall the concept of $C^*$-ternary algebras.

A $C^*$-ternary algebras is a complex Banach space $\mathcal{A}$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $\mathcal{A}^3$ into $\mathcal{A}$, which is $\mathbb{C}$-linear in the outer variables and conjugate $\mathbb{C}$-linear in the middle variable:

(i) $[\lambda x + y, v, w] = \lambda[x, v, w] + [y, v, w]$,
(ii) $[v, w, \lambda x + y] = \lambda[v, w, x] + [v, w, y]$,
(iii) $[v, \lambda x + y, w] = \bar{\lambda}[v, x, w] + [v, y, w]$

and associative in the sense that $[[v, w, x], y, z] = [v, [y, x, w], z] = [v, w, [x, y, z]]$ and, satisfies $\|[x, y, z]\| \leqslant \|x\|\|y\|\|z\|$ and $\|[x, x, x]\| = \|x\|^3$ for all $v, w, x, y, z \in \mathcal{A}$.

If a $C^*$-ternary algebras $(\mathcal{A}, [\cdot\,, \cdot\,, \cdot\,])$ has an identity, i.e. an element $e \in \mathcal{A}$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in \mathcal{A}$, then it is routine to verify that $\mathcal{A}$, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is unital $C^*$-algebra. Conversely, if $(\mathcal{A}, \circ)$ is a unital $C^*$-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes $\mathcal{A}$ into $C^*$-ternary algebra.

A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a $C^*$-ternary algebra homomorphism if $H([x, y, z]) = [H(x), H(y), H(z)]$ for all $x, y, z \in \mathcal{A}$. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a $C^*$-ternary derivation if $\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)]$ for all $x,y,z \in \mathcal{A}$.

A $C^*$-ternary algebras have many applications in fractional quantum Hall effect, the nonstandard statistics, hypothetical, supersymmetric theory, and Yang-Baxter equation (see [1, 17, 28]).

Throughout this paper, assume that $\mathcal{A}$ is a unital $C^*$-ternary algebra with norm $\|\cdot\|_{\mathcal{A}}$ and unit $e$, and that $\mathcal{B}$ is a $C^*$-ternary algebra with norm $\|\cdot\|_{\mathcal{B}}$ and unit $e'$. The purpose of the present paper is to investigate the stability of homomorphisms and derivations of the functional equation (4) in $C^*$-ternary algebras by using the alternative fixed point theorem. Next, we recall a fundamental results in fixed point theory. The following is the definition of generalized metric space which was introduced by Luxemburg in 1958 [19].

**Definition 1.6** [19] Let $X$ be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions

(1) $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$, for all $x,y \in X$,
(3) $d(x,z) \leqslant d(x,y) + d(y,z)$, for all $x,y,z \in X$.

The following fixed point theorems will play important roles in proving our main results.

**Theorem 1.7** ([7] alternative of fixed point) Let $(X,d)$ be a complete metric space and let $\Lambda : X \to X$ be strictly contractive, that is, $d(\Lambda x,\Lambda y) \leqslant \gamma d(x,y)$ for all $x,y \in X$ and for some Lipshitz $\gamma < 1$. Then, the following conditions hold.

(1) The mapping $\Lambda$ has a unique fixed point $x^* = \Lambda x^*$.
(2) The fixed point $x^*$ is globally attractive, that is,

$$\lim_{n\to\infty} \Lambda^n x = x^* \tag{5}$$

for any starting point $x \in X$.
(3) One has the following estimation inequalities:

$$d(\Lambda^n x, x^*) \leqslant \gamma^n d(x,x^*),\ d(\Lambda^n x, x^*) \leqslant \frac{1}{1-\gamma} d(\Lambda^n x, \Lambda^{n+1}x),\ d(x,x^*) \leqslant \frac{1}{1-\gamma} d(x,\Lambda x)$$

for all nonnegative integers $n$ and all $x \in X$.

**Theorem 1.8** [8] Let $(X,d)$ be a complete generalized metric space and $\Lambda : X \to X$ be a strictly contractive mapping with Lipschitz constant $\gamma < 1$. Then for each given element $x \in X$, either

$$d(\Lambda^n x, \Lambda^{n+1}x) = \infty \tag{6}$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

(1) $d(\Lambda^n x, \Lambda^{n+1}x) < \infty$ for all $n \geqslant n_0$,
(2) the sequence $\{\Lambda^n x\}$ converges to a fixed point $y^*$ of $\Lambda$,
(3) $y^*$ is the unique fixed point of $\Lambda$ in the set $Y = \{y \in X : d(\Lambda^{n_0}x, y) < \infty\}$,
(4) $d(y,y^*) \leqslant \left(\frac{1}{1-\gamma}\right) d(y,\Lambda y)$, for all $y \in Y$.

The following theorem shows that each element $S$ of $\{x \in \mathcal{A} : \|x\| = 1\}$ is mean of a finite number of unitary elements of $\mathcal{A}$.

**Theorem 1.9** [12] If the element $S$ of a $C^*$-algebra $\mathcal{A}$ has the property that $\|S\|_{\mathcal{A}} < 1-\frac{2}{n}$ for some integer $n$ greater than 2, then there are $n$ elements $S_1, S_2, \ldots, S_n$ in $\mathcal{A}$ with $\|S_i\| = 1$ for all $i = 1, 2, \ldots, n$ such that $S = \frac{1}{n}(S_1 + S_2 + \cdots + S_n)$.

The following lemmas is useful results for proving our main results.

**Lemma 1.10** [20] Let $f : \mathcal{A} \to \mathcal{B}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in \mathcal{A}$ and all $\mu \in \mathbb{S} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then the mapping $f$ is $\mathbb{C}$-linear.

The following lemma shows that $[\cdot,\cdot,\cdot] : \mathcal{A}^3 \to \mathcal{A}$ is continuous.

**Lemma 1.11** [18] Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be convergent sequence in $\mathcal{A}$. Then the sequence $\{[x_n, y_n, z_n]\}$ is convergent in $\mathcal{A}$.

## 2. Stability of homomorphisms in $C^*$-ternary algebras

For a given mapping $f : \mathcal{A} \to \mathcal{B}$, we define

$$E_\mu f(x, y, z) := \eta\mu f\left(\frac{x+y}{\eta} + z\right) - f(\mu x) - f(\mu y) - \eta f(\mu z), \tag{7}$$

for all $\mu \in \mathbb{S} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $x, y, z \in \mathcal{A}$.

We prove the Hyers-Ulam-Rassias stability of $C^*$-ternary algebra homomorphism for the functional equation $E_\mu f(x, y, z) = 0$.

**Theorem 2.1** Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there exists a function $\psi : \mathcal{A}^3 \to [0, \infty)$ such that

$$\lim_{j\to\infty} \left(\frac{\eta}{2+\eta}\right)^j \cdot \psi\left(\left(\frac{2+\eta}{\eta}\right)^j x, \left(\frac{2+\eta}{\eta}\right)^j y, \left(\frac{2+\eta}{\eta}\right)^j z\right) = 0, \tag{8}$$

$$\|E_\mu f(x, y, z)\|_{\mathcal{B}} \leqslant \psi(x, y, z), \tag{9}$$

$$\|f[x, y, z] - [f(x), f(y), f(z)]\|_{\mathcal{B}} \leqslant \psi(x, y, z) \tag{10}$$

for all $\mu \in \mathbb{S}$ and all $x, y, z \in \mathcal{A}$. If there exists a $\gamma < 1$ such that

$$\psi(x, x, x) \leqslant \frac{2+\eta}{\eta}\gamma\psi\left(\frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x\right) \tag{11}$$

for all $x \in \mathcal{A}$, then there exists a unique $C^*$-ternary algebra homomorphism $\Gamma : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \Gamma(x)\|_{\mathcal{B}} \leqslant \frac{1}{(1-\gamma)(2+\eta)}\psi(x, x, x) \tag{12}$$

for all $x \in A$.

**Proof.** Consider the set $X := \{g : \mathcal{A} \to \mathcal{B}\}$ and introduce the generalized metric on $X$ as follows:

$$d(g, h) = \inf\{M \in (0, \infty) : \|g(x) - h(x)\|_{\mathcal{B}} \leqslant M\psi(x, x, x), \forall x \in \mathcal{A}\}. \tag{13}$$

It is easy to show that $(X, d)$ is complete. Now, we consider the linear mapping $\Lambda : X \to X$ such that $\Lambda g(x) := \frac{\eta}{2+\eta} g\left(\frac{2+\eta}{\eta}x\right)$ for all $x \in \mathcal{A}$. Next, we will show that $\Lambda$ is a strictly contractive self-mapping of X with the Lipschitz constant $\gamma$. For any $g, h \in X$, let $d(g, h) = K$ for some $K \in \mathcal{R}_+$. Then, we have $\|g(x) - h(x)\|_{\mathcal{B}} \leqslant K\psi(x, x, x)$ for all $x \in \mathcal{A}$, which implies that

$$\left\|g\left(\frac{2+\eta}{\eta}x\right) - h\left(\frac{2+\eta}{\eta}x\right)\right\|_{\mathcal{B}} \leqslant K\psi\left(\frac{2+\eta}{\eta}x, \frac{2+\eta}{\eta}x, \frac{2+\eta}{\eta}x\right).$$

Thus,

$$\left\|\frac{\eta}{2+\eta}g\left(\frac{2+\eta}{\eta}x\right) - \frac{\eta}{2+\eta}h\left(\frac{2+\eta}{\eta}x\right)\right\|_{\mathcal{B}} \leqslant \frac{\eta}{2+\eta}K\psi\left(\frac{2+\eta}{\eta}x, \frac{2+\eta}{\eta}x, \frac{2+\eta}{\eta}x\right).$$

By (11), we obtain that

$$\|\Lambda g(x) - \Lambda h(x)\|_{\mathcal{B}} \leqslant \frac{\eta}{2+\eta}K\frac{2+\eta}{\eta}\gamma\psi\left(\frac{\eta}{2+\eta}\cdot\frac{2+\eta}{\eta}x, \frac{\eta}{2+\eta}\cdot\frac{2+\eta}{\eta}x, \frac{\eta}{2+\eta}\cdot\frac{2+\eta}{\eta}x\right)$$

which implies that $\|\Lambda g(x) - \Lambda h(x)\|_{\mathcal{B}} \leqslant K\gamma\psi(x, x, x)$ for all $x \in \mathcal{A}$. Thus, $d(\Lambda g, \Lambda h) \leqslant K\gamma$. Hence, we obtain $d(\Lambda g, \Lambda h) \leqslant \gamma d(g, h)$. Letting $\mu = 1$ and $x = y = z$ in (7), we get

$$\begin{aligned} E_\mu f(x, x, x) &= \eta f\left(\frac{x+x}{\eta} + x\right) - f(x) - f(x) - \eta f(x) \\ &= \eta f\left(\frac{2+\eta}{\eta}x\right) - (2+\eta)f(x) \end{aligned}$$

for all $x \in \mathcal{A}$. By (9), we have

$$\|E_\mu f(x, x, x)\|_{\mathcal{B}} = \left\|\eta f\left(\tfrac{2+\eta}{\eta}x\right) - (2+\eta)f(x)\right\|_{\mathcal{B}} \leqslant \psi(x, x, x)$$

which implies that

$$\left\|f(x) - \tfrac{\eta}{(2+\eta)}f\left(\tfrac{(2+\eta)}{\eta}x\right)\right\|_{\mathcal{B}} \leqslant \tfrac{1}{(2+\eta)}\cdot\psi(x, x, x)$$

for all $x \in \mathcal{A}$, that is, $\|f(x) - \Lambda f(x)\|_{\mathcal{B}} \leqslant \frac{1}{2+\eta}\cdot\psi(x, x, x)$ for all $x \in \mathcal{A}$. It follows from (13) that we have $d(f, \Lambda f) \leqslant \frac{1}{2+\eta}$. By Theorem 1.8, there exists a mapping $\Gamma : \mathcal{A} \to \mathcal{B}$ such that the following conditions hold.

(1) $\Gamma$ is a fixed point of $\Lambda$, that is, $\Lambda\Gamma(x) = \Gamma(x)$ for all $x \in \mathcal{A}$. Then we have

$$\Gamma(x) = \Lambda\Gamma(x) = \frac{\eta}{2+\eta}\Gamma\left(\frac{2+\eta}{\eta}x\right) \Rightarrow \Gamma\left(\frac{2+\eta}{\eta}x\right) = \frac{2+\eta}{\eta}\Gamma(x)$$

for all $x \in \mathcal{A}$. The mapping $\Gamma$ is a unique fixed point of $\Lambda$ in the set $Y = \{g \in X : d(f, g) < \infty\}$, that is, $d(f, \Gamma) < \infty$. From (13), there exists $C \in (0, \infty)$ satisfying $\|f(x) - \Gamma(x)\|_{\mathcal{B}} \leqslant C\psi(x, x, x)$ for all $x \in \mathcal{A}$.

(2) The sequence $\{\Lambda^n f\}$ converges to $\Gamma$. This implies that the equality

$$\Gamma(x) = \lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right) \tag{14}$$

for all $x \in \mathcal{A}$.

(3) We obtain that $d(f, \Gamma) \leqslant \left(\frac{1}{1-\gamma}\right) d(f, \Lambda f)$, which implies that

$$d(f,\Gamma) \leqslant \left(\frac{1}{1-\gamma}\right) d(f,\Lambda f) \leqslant \frac{1}{(1-\gamma)(2+\eta)}.$$

Therefore, the inequality (12) holds.
It follow from (8), (9) and (14) that

$$\begin{aligned}
&\left\|\eta\Gamma\left(\frac{x+y}{\eta}+z\right)-\Gamma(x)-\Gamma(y)-\eta\Gamma(z)\right\|_{\mathcal{B}}\\
=&\left\|\eta\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n f\left(\left(\frac{2+\eta}{\eta}\right)^n\left(\frac{x+y}{\eta}+z\right)\right)-\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right.\\
&\left.-\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n f\left(\left(\frac{2+\eta}{\eta}\right)^n y\right)-\eta\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n f\left(\left(\frac{2+\eta}{\eta}\right)^n z\right)\right\|_{\mathcal{B}}\\
&\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n\left\|\eta f\left(\frac{\left(\frac{2+\eta}{\eta}\right)^n x+\left(\frac{2+\eta}{\eta}\right)^n y}{\eta}+\left(\frac{2+\eta}{\eta}\right)^n z\right)-f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right.\\
&\left.-f\left(\left(\frac{2+\eta}{\eta}\right)^n y\right)-\eta f\left(\left(\frac{2+\eta}{\eta}\right)^n z\right)\right\|_{\mathcal{B}}\\
\leqslant&\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^n\psi\left(\left(\frac{2+\eta}{\eta}\right)^n x,\left(\frac{2+\eta}{\eta}\right)^n y,\left(\frac{2+\eta}{\eta}\right)^n z\right)\\
&=0
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Hence, we have

$$\eta\Gamma\left(\frac{x+y}{\eta}+z\right)=\Gamma(x)+\Gamma(y)+\eta\Gamma(z) \tag{15}$$

for all $x, y, z \in \mathcal{A}$. From Corollary 1.5, we get that $H$ is additive, that is,

$$\Gamma(x+y)=\Gamma(x)+\Gamma(y) \tag{16}$$

for all $x, y \in \mathcal{A}$. Next, we can show that $\Gamma : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear. Firstly, we will show that for any $x \in \mathcal{A}$, $\Gamma(\mu x) = \mu\Gamma(x)$ for all $\mu \in \mathbb{S}$. For each $\mu \in \mathbb{S}$, substituting $x, y, z$ in (7) by $\left(\frac{2+\eta}{\eta}\right)^n x$, we obtain that

$$\begin{aligned}
&E_\mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x,\left(\frac{2+\eta}{\eta}\right)^n x,\left(\frac{2+\eta}{\eta}\right)^n x\right)\\
&=\eta\mu f\left(\frac{\left(\frac{2+\eta}{\eta}\right)^n x+\left(\frac{2+\eta}{\eta}\right)^n x}{\eta}+\left(\frac{2+\eta}{\eta}\right)^n x\right)-f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right)\\
&-f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right)-\eta f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right)\\
&=\eta\mu f\left(\frac{(2+\eta)}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right)-(2+\eta)f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right)
\end{aligned}$$

for all $x \in \mathcal{A}$. By (9), we have

$$\begin{aligned} &\left\|E_\mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &= \left\|\eta\mu f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right) - (2+\eta) f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &\leqslant \psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x\right) \end{aligned} \tag{17}$$

for all $x \in \mathcal{A}$. From (17), we have

$$\begin{aligned} &\left\|\eta f\left(\frac{(2+\eta)}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right) - (2+\eta) f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &\leqslant \psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x\right) \end{aligned} \tag{18}$$

for all $x \in \mathcal{A}$. It follow from (17), (18) and (9) that

$$\begin{aligned} &\left\|(2+\eta) f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right) - (2+\eta)\mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &= \left\|(2+\eta) f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right) - \eta\mu f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right)\right. \\ &\left. +\eta\mu f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right) - (2+\eta)\mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &\leqslant \left\|(2+\eta) f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right) - \eta\mu f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &+ \left\|\mu f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right) - (2+\eta)\mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &\leqslant \left\|(2+\eta) f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right) - \eta\mu f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &+ |\mu| \left\|\eta f\left(\frac{2+\eta}{\eta}\cdot\left(\frac{2+\eta}{\eta}\right)^n x\right) - (2+\eta) f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &\leqslant 2\psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x\right) \end{aligned}$$

for all $x \in \mathcal{A}$. This implies that

$$\begin{aligned} &\left\|\left(\frac{\eta}{2+\eta}\right)^n f\left(\mu\left(\frac{2+\eta}{\eta}\right)^n x\right) - \left(\frac{\eta}{2+\eta}\right)^n \mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)\right\|_{\mathcal{B}} \\ &\leqslant \frac{2}{2+\eta}\left(\frac{\eta}{2+\eta}\right)^n \psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x\right) \\ &\leqslant \left(\frac{\eta}{2+\eta}\right)^n \psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n x\right) \end{aligned}$$

for all $x \in \mathcal{A}$. By (8), we have

$$\lim_{n\to\infty} \left\| \left(\frac{\eta}{2+\eta}\right)^n f\left(\mu \left(\frac{2+\eta}{\eta}\right)^n x\right) - \left(\frac{\eta}{2+\eta}\right)^n \mu f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right) \right\|_{\mathcal{B}} \to 0$$

which implies that

$$\Gamma(\mu x) = \mu\Gamma(x) \tag{19}$$

for all $x \in \mathcal{A}$. Next, we show that for any $x \in \mathcal{A}$, $\Gamma(\lambda x) = \lambda\Gamma(x)$ for all $\lambda \in \mathbb{C}$.

Let $\lambda \in \mathbb{C}$ and $M$ be an integer greater than $4|\lambda|$, $(M > 4|\lambda|)$. Then, we have $\frac{|\lambda|}{M} < \frac{1}{4} < \frac{1}{3} = 1 - \frac{2}{3}$. By Theorem 1.9, there exist $\lambda_1, \lambda_2, \lambda_3$ with $|\lambda_i| = 1$ for all $i = 1, 2, 3$ such that $\frac{\lambda}{M} = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$, that is, $3\frac{\lambda}{M} = (\lambda_1 + \lambda_2 + \lambda_3)$. Hence, we have $\Gamma(x) = \Gamma(\frac{1}{3}x + \frac{1}{3}x + \frac{1}{3}x) = \Gamma(\frac{1}{3}x) + \Gamma(\frac{1}{3}x) + \Gamma(\frac{1}{3}x) = 3\Gamma(\frac{1}{3}x)$, that is,

$$\frac{1}{3}\Gamma(x) = \Gamma(\frac{1}{3}x). \tag{20}$$

From (16), (19) and (20), we obtain that

$$\begin{aligned}
\Gamma(\lambda x) = \Gamma\left(M\cdot \frac{\lambda}{M}x\right) &= \Gamma\left(\overbrace{\frac{\lambda}{M}x + \frac{\lambda}{M}x + \cdots + \frac{\lambda}{M}x}^{M-\text{terms}}\right) \\
&= \overbrace{\Gamma\left(\frac{\lambda}{M}x\right) + \Gamma\left(\frac{\lambda}{M}x\right) + \cdots + \Gamma\left(\frac{\lambda}{M}x\right)}^{M-\text{terms}} \\
&= M\cdot\Gamma\left(\frac{\lambda}{M}x\right) = M\cdot\Gamma\left(\frac{1}{3}\cdot 3\frac{\lambda}{M}x\right) \\
&= \frac{M}{3}\Gamma\left(3\frac{\lambda}{M}x\right) \\
&= \frac{M}{3}\Gamma((\lambda_1 + \lambda_2 + \lambda_3)x) \\
&= \frac{M}{3}\Gamma(\lambda_1 x + \lambda_2 x + \lambda_3 x) \\
&= \frac{M}{3}\left(\Gamma(\lambda_1 x) + \Gamma(\lambda_2 x) + \Gamma(\lambda_3 x)\right) \\
&= \frac{M}{3}\left(\lambda_1\Gamma(x) + \lambda_2\Gamma(x) + \lambda_3\Gamma(x)\right) \\
&= \frac{M}{3}(\lambda_1 + \lambda_2 + \lambda_3)\Gamma(x) \\
&= \frac{M}{3}\left(3\cdot\frac{\lambda}{M}\right)\Gamma(x) \\
&= \lambda\Gamma(x)
\end{aligned}$$

for all $x \in \mathcal{A}$. This implies that $\Gamma(\zeta x+\eta y)=\Gamma(\zeta x)+\Gamma(\eta y)=\zeta\Gamma(x)+\eta\Gamma(y)$ for all $\zeta,\eta \in \mathbb{C}\backslash\{0\}$ and for all $x, y \in \mathcal{A}$ and so $\Gamma(0)=\Gamma(0\cdot x)=0\cdot\Gamma(x)=0$ for all $x \in \mathcal{A}$. Next, we will show that $\Gamma$ is a $C^*$-ternary algebra homomorphism.
It follows from (10) and Lemma 1.11 that we have

$$
\begin{aligned}
&\|\Gamma([x,y,z])-[\Gamma(x),\Gamma(y),\Gamma(z)]\|_{\mathcal{B}}\\
=&\Bigg\|\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{3n} f\left(\left[\left(\frac{2+\eta}{\eta}\right)^{n}x,\left(\frac{2+\eta}{\eta}\right)^{n}y,\left(\frac{2+\eta}{\eta}\right)^{n}z\right]\right)\\
&-\Bigg[\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{n} f\left(\left(\frac{2+\eta}{\eta}\right)^{n}x\right),\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{n} f\left(\left(\frac{2+\eta}{\eta}\right)^{n}y\right),\\
&\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{n} f\left(\left(\frac{2+\eta}{\eta}\right)^{n}z\right)\Bigg]\Bigg\|_{\mathcal{B}}\\
=&\Bigg\|\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{3n} f\left(\left[\left(\frac{2+\eta}{\eta}\right)^{n}x,\left(\frac{2+\eta}{\eta}\right)^{n}y,\left(\frac{2+\eta}{\eta}\right)^{n}z\right]\right)\\
&-\lim_{n\to\infty}\Bigg[\left(\frac{\eta}{2+\eta}\right)^{n} f\left(\left(\frac{2+\eta}{\eta}\right)^{n}x\right),\left(\frac{\eta}{2+\eta}\right)^{n} f\left(\left(\frac{2+\eta}{\eta}\right)^{n}y\right),\\
&\left(\frac{\eta}{2+\eta}\right)^{n} f\left(\left(\frac{2+\eta}{\eta}\right)^{n}z\right)\Bigg]\Bigg\|_{\mathcal{B}}\\
=&\Bigg\|\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{3n} f\left(\left[\left(\frac{2+\eta}{\eta}\right)^{n}x,\left(\frac{2+\eta}{\eta}\right)^{n}y,\left(\frac{2+\eta}{\eta}\right)^{n}z\right]\right)\\
&-\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{3n}\left[f\left(\left(\frac{2+\eta}{\eta}\right)^{n}x\right),f\left(\left(\frac{2+\eta}{\eta}\right)^{n}y\right),f\left(\left(\frac{2+\eta}{\eta}\right)^{n}z\right)\right]\Bigg\|_{\mathcal{B}}\\
=&\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{3n}\Bigg\|f\left(\left[\left(\frac{2+\eta}{\eta}\right)^{n}x,\left(\frac{2+\eta}{\eta}\right)^{n}y,\left(\frac{2+\eta}{\eta}\right)^{n}z\right]\right)\\
&-\left[f\left(\left(\frac{2+\eta}{\eta}\right)^{n}x\right),f\left(\left(\frac{2+\eta}{\eta}\right)^{n}y\right),f\left(\left(\frac{2+\eta}{\eta}\right)^{n}z\right)\right]\Bigg\|_{\mathcal{B}}\\
\leqslant&\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{3n}\psi\left(\left(\frac{2+\eta}{\eta}\right)^{n}x,\left(\frac{2+\eta}{\eta}\right)^{n}y,\left(\frac{2+\eta}{\eta}\right)^{n}z\right)\\
\leqslant&\lim_{n\to\infty}\left(\frac{\eta}{2+\eta}\right)^{n}\psi\left(\left(\frac{2+\eta}{\eta}\right)^{n}x,\left(\frac{2+\eta}{\eta}\right)^{n}y,\left(\frac{2+\eta}{\eta}\right)^{n}z\right)=0
\end{aligned}
$$

for all $x, y, z \in \mathcal{A}$. Thus $\Gamma([x,y,z])=[\Gamma(x),\Gamma(y),\Gamma(z)]$ for all $x, y, z \in \mathcal{A}$. Therefore, the mapping $\Gamma$ is a $C^*$-ternary algebra homomorphism. ■

**Corollary 2.2** Let $p \in [0,1)$, $\varepsilon \in [0,\infty)$ and let $f$ be a mapping of $\mathcal{A}$ into $\mathcal{B}$ such that

$$\|E_\mu f(x,y,z)\|_{\mathcal{B}} \leqslant \varepsilon\left(\|x\|_{\mathcal{A}}^p+\|y\|_{\mathcal{A}}^p+\|z\|_{\mathcal{A}}^p\right), \tag{21}$$

$$\|f[x,y,z]-[f(x),f(y),f(z)]\|_{\mathcal{B}} \leqslant \varepsilon(\|x\|_{\mathcal{A}}^p+\|y\|_{\mathcal{A}}^p) \tag{22}$$

for all $\mu \in \mathbb{S}$ and for all $x, y, z \in \mathcal{A}$. Then, there exists a unique $C^*$-ternary algebra homomorphism $\Gamma : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x)-\Gamma(x)\|_{\mathcal{B}} \leqslant \frac{3\varepsilon}{\left(1-\left(\frac{2+\eta}{\eta}\right)^{p-1}\right)(2+\eta)}\|x\|_{\mathcal{A}}^p.$$

**Proof.** From Theorem 2.1, we take $\psi(x,y,z) = \varepsilon(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}})$ for all $x,y,z \in \mathcal{A}$. Then, $\gamma = \left(\frac{2+\eta}{\eta}\right)^{p-1}$ and we get the desired results. ■

**Theorem 2.3** Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping for which there exists a function $\psi : \mathcal{A}^3 \to [0,\infty)$ satisfying (9) and (10) such that

$$\lim_{j\to\infty} \left(\frac{2+\eta}{\eta}\right)^{2j} \cdot \psi\left(\left(\frac{\eta}{2+\eta}\right)^j x, \left(\frac{\eta}{2+\eta}\right)^j y, \left(\frac{\eta}{2+\eta}\right)^j z\right) = 0 \tag{23}$$

for all $x,y,z \in \mathcal{A}$. If there exists an $\gamma < 1$ such that

$$\psi(x,x,x) \leqslant \frac{\eta}{2+\eta}\gamma\psi\left(\frac{2+\eta}{\eta}x, \frac{2+\eta}{\eta}x, \frac{2+\eta}{\eta}x\right), \tag{24}$$

then there exists a unique $C^*$-ternary algebra homomorphism $\Gamma : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \Gamma(x)\|_{\mathcal{B}} \leqslant \frac{\gamma}{(1-\gamma)(2+\eta)}\psi(x,x,x) \tag{25}$$

for all $x \in \mathcal{A}$.

**Proof.** We consider the linear mapping $\Lambda : X \to X$ such that $\Lambda g(x) := \frac{2+\eta}{\eta} g\left(\frac{\eta}{2+\eta}x\right)$ for all $x \in \mathcal{A}$. By similar proof of Theorem 2.1, $\Lambda$ is a strictly contractive self-mapping of X with the Lipschitz constant $\gamma$. Letting $\mu = 1$ and substituting $x,y,z$ in (9) by $\frac{\eta}{2+\eta}x$, we have

$$\begin{aligned} \left\|E_\mu f\left(\frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x\right)\right\|_{\mathcal{B}} &= \left\|\eta f(x) - (2+\eta) f\left(\frac{\eta}{2+\eta}x\right)\right\|_{\mathcal{B}} \\ &\leqslant \psi\left(\frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x\right) \end{aligned} \tag{26}$$

for all $x \in \mathcal{A}$. From this inequality we get

$$\begin{aligned} \left\|f(x) - \frac{2+\eta}{\eta} f\left(\frac{\eta}{2+\eta}x\right)\right\|_{\mathcal{B}} &\leqslant \frac{1}{\eta}\psi\left(\frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x, \frac{\eta}{2+\eta}x\right) \\ &\leqslant \frac{1}{\eta}\cdot\frac{\eta}{2+\eta}\gamma\psi\left(\frac{2+\eta}{\eta}\cdot\frac{\eta}{2+\eta}x, \frac{2+\eta}{\eta}\cdot\frac{\eta}{2+\eta}x, \frac{2+\eta}{\eta}\cdot\frac{\eta}{2+\eta}x\right) \\ &= \frac{\gamma}{2+\eta}\cdot\psi(x,x,x) \end{aligned}$$

for all $x \in \mathcal{A}$, that is, $\|\Lambda f(x) - f(x)\|_{\mathcal{B}} \leqslant \frac{\gamma}{2+\eta}\psi(x,x,x)$ for all $x \in \mathcal{A}$. Hence, we obtain that $d(f, \Lambda f) \leqslant \frac{\gamma}{2+\eta}$. By Theorem 1.8, there exists a mapping $\Gamma : \mathcal{A} \to \mathcal{B}$ such that the following conditions hold.

(1) $\Gamma$ is a fixed point of $\Lambda$, that is, $\Lambda\Gamma(x) = \Gamma(x)$ for all $x \in \mathcal{A}$. Then we have

$$\Gamma(x) = \Lambda\Gamma(x) = \frac{2+\eta}{\eta}\Gamma\left(\frac{\eta}{2+\eta}x\right) \Rightarrow \Gamma\left(\frac{\eta}{2+\eta}x\right) = \frac{\eta}{2+\eta}\Gamma(x)$$

for all $x \in \mathcal{A}$. The mapping $\Gamma$ is a unique fixed point of $\Lambda$ in the set

$$Y = \{g \in X : d(f, g) < \infty\},$$

that is, $d(f, \Gamma) < \infty$. From (13), there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - \Gamma(x)\|_{\mathcal{B}} \leqslant C\psi(x, x, x),$$

for all $x \in \mathcal{A}$.

(2) The sequence $\{\Lambda^n f\}$ converges to $\Gamma$. This implies that the equality

$$\Gamma(x) = \lim_{n\to\infty} \left(\frac{2+\eta}{\eta}\right)^n f\left(\left(\frac{\eta}{2+\eta}\right)^n x\right)$$

for all $x \in \mathcal{A}$.

(3) We obtain that $d(f, \Gamma) \leqslant \left(\frac{1}{1-\gamma}\right) d(f, \Lambda f)$, which implies that

$$d(f, \Gamma) \leqslant \left(\frac{1}{1-\gamma}\right) d(f, \Lambda f) \leqslant \frac{\gamma}{(1-\gamma)(2+\eta)}.$$

Therefore, the inequality (25) holds.

The rest of the proof is similar to the proof of Theorem 2.1. ■

**Corollary 2.4** Let $p \in (2, \infty)$, $\varepsilon \in [0, \infty)$ and $f$ be a mapping of $\mathcal{A}$ into $\mathcal{B}$ such that (21) and (22). Then, there exists a unique $C^*$-ternary algebra homomorphism $\Gamma : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \Gamma(x)\|_{\mathcal{B}} \leqslant \frac{3\varepsilon}{\left(\left(\frac{2+\eta}{\eta}\right)^{p-1} - 1\right)(2+\eta)} \|x\|^p_{\mathcal{A}}.$$

**Proof.** The proof follow from Theorem 2.3 and Corollary 2.2, when we take

$$\psi(x, y, z) = \varepsilon(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}})$$

for all $x, y, z \in \mathcal{A}$. Then, $\gamma = \left(\frac{2+\eta}{\eta}\right)^{1-p}$ and we get the desired results. ■

## 3. Stability of derivations in $C^*$-ternary algebras

For a given mapping $f : \mathcal{A} \to \mathcal{A}$, we define

$$E_\mu f(x, y, z) := \eta\mu f\left(\frac{x+y}{\eta} + z\right) - f(\mu x) - f(\mu y) - \eta(f\mu z)$$

for all $\mu \in \mathbb{S}$ and all $x, y, z \in \mathcal{A}$.

We recall definition of $C^*$-ternary derivation.

**Definition 3.1** [2] A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a $C^*$-ternary derivation if $\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$ for all $x, y, z \in \mathcal{A}$.

**Theorem 3.2** Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there exists a function $\psi : \mathcal{A}^3 \to [0, \infty)$ satisfying (8) and

$$\|E_\mu f(x, y, z)\|_{\mathcal{A}} \leqslant \psi(x, y, z) \tag{27}$$

and

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_{\mathcal{A}} \leqslant \psi(x,y,z) \tag{28}$$

for all $\mu \in \mathbb{S}$ and all $x, y, z \in \mathcal{A}$. If there exists a $\gamma < 1$ satisfying (11) for all $x \in \mathcal{A}$, then there exists a unique $C^*$-ternary derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathcal{A}} \leqslant \frac{1}{(1-\gamma)(2+\eta)}\psi(x,x,x) \tag{29}$$

for all $x \in \mathcal{A}$.

**Proof.** Similarly to the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (29). The mapping $\delta : \mathcal{A} \to \mathcal{A}$ is given by

$$\delta(x) = \lim_{n\to\infty} \left(\frac{\eta}{2+\eta}\right)^n f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right)$$

for all $x \in \mathcal{A}$. It follows that (8) and (28) that

$$\begin{aligned}
&\|\delta([x,y,z]) - [\delta(x),y,z] - [x,\delta(y),z] - [x,y,\delta(z)]\|_{\mathcal{A}} \\
&= \lim_{n\to\infty} \left(\frac{\eta}{2+\eta}\right)^{3n} \left\| f\left(\left[\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n y, \left(\frac{2+\eta}{\eta}\right)^n z\right]\right)\right. \\
&\quad - \left[f\left(\left(\frac{2+\eta}{\eta}\right)^n x\right), \left(\frac{2+\eta}{\eta}\right)^n y, \left(\frac{2+\eta}{\eta}\right)^n z\right] \\
&\quad - \left[\left(\frac{2+\eta}{\eta}\right)^n x, f\left(\left(\frac{2+\eta}{\eta}\right)^n y\right), \left(\frac{2+\eta}{\eta}\right)^n z\right] \\
&\quad \left. - \left[\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n y, f\left(\left(\frac{2+\eta}{\eta}\right)^n z\right)\right]\right\|_{\mathcal{A}} \\
&\leqslant \lim_{n\to\infty} \left(\frac{\eta}{2+\eta}\right)^{3n} \psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n y, \left(\frac{2+\eta}{\eta}\right)^n z\right) \\
&\leqslant \lim_{n\to\infty} \left(\frac{\eta}{2+\eta}\right)^{n} \psi\left(\left(\frac{2+\eta}{\eta}\right)^n x, \left(\frac{2+\eta}{\eta}\right)^n y, \left(\frac{2+\eta}{\eta}\right)^n z\right) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Therefore, $\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)]$ for all $x, y, z \in \mathcal{A}$. Hence $\delta : \mathcal{A} \to \mathcal{A}$ is a generalized derivation satisfying (29). ■

**Corollary 3.3** Let $r < 1$ and $\varepsilon$ be nonnegative real number, and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping such that

$$\|E_\mu f(x,y,z)\|_{\mathcal{A}} \leqslant \varepsilon \cdot \|x\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|y\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|z\|_{\mathcal{A}}^{\frac{r}{3}} \tag{30}$$

and

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_{\mathcal{A}} \leqslant \varepsilon \cdot \|x\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|y\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|z\|_{\mathcal{A}}^{\frac{r}{3}} \tag{31}$$

for all $\mu \in \mathbb{S}$ and all $x, y, z \in \mathcal{A}$. Then there exists a unique $C^*$-ternary derivation

$\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathcal{A}} \leqslant \frac{\varepsilon}{(1 - \left(\frac{2+\eta}{\eta}\right)^{r-1})(2+\eta)} \|x\|_{\mathcal{A}}^{r}$$

for all $x \in X$.

**Proof.** The proof follows from Theorem 3.2 by taking $\psi(x,y,z) = \varepsilon \cdot \|x\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|y\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|z\|_{\mathcal{A}}^{\frac{r}{3}}$ for all $x, y, z \in \mathcal{A}$. Then, $\gamma = \left(\frac{2+\eta}{\eta}\right)^{r-1}$ and we get the desired results. ■

**Theorem 3.4** Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there exists a function $\psi : \mathcal{A}^3 \to [0, \infty)$ satisfying (27), (28) and

$$\lim_{j\to\infty} \left(\frac{2+\eta}{\eta}\right)^{3j} \cdot \psi\left(\left(\frac{\eta}{2+\eta}\right)^{j} x, \left(\frac{\eta}{2+\eta}\right)^{j} y, \left(\frac{\eta}{2+\eta}\right)^{j} z\right) = 0$$

for all $\mu \in \mathbb{S}$ and all $x, y, z \in \mathcal{A}$. If there exists a $\gamma < 1$ satisfying (24) for all $x \in \mathcal{A}$, then there exist a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathcal{A}} \leqslant \frac{\gamma}{(1-\gamma) \cdot (2+\eta)} \psi(x, x, x)$$

for all $x \in \mathcal{A}$. Moreover $\delta : \mathcal{A} \to \mathcal{A}$ is generalized derivation on $\mathcal{A}$.

**Proof.** The proof is similar to the proofs of Theorems 2.3 and Theorem 3.2. ■

**Corollary 3.5** Let $r > 3$ and $\varepsilon$ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying (30) and (31). Then, there exists a unique generalized derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\|_{\mathcal{A}} \leqslant \frac{\varepsilon}{\left(\left(\frac{\eta}{2+\eta}\right)^{1-r} - 1\right)(2+\eta)} \|x\|_{\mathcal{A}}^{r}$$

for all $x \in \mathcal{A}$.

**Proof.** The proof follows Theorem 3.4 by taking $\psi(x,y,z) := \varepsilon \cdot \|x\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|y\|_{\mathcal{A}}^{\frac{r}{3}} \cdot \|z\|_{\mathcal{A}}^{\frac{r}{3}}$ for all $x, y, z \in \mathcal{A}$. Then, $\gamma = \left(\frac{2+\eta}{\eta}\right)^{1-r}$ and we get the desired results. ■

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