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# Irreducibility of some tensor product of Albeverio's representations of the Braid groups $B_3$ and $B_4$

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**Abstract.** We consider Albeverio's linear representations of the braid groups $B_3$ and $B_4$. We specialize the indeterminates used in defining these representations to non zero complex numbers. We then consider the tensor products of the representations of $B_3$ and the tensor products of those of $B_4$. We then determine necessary and sufficient conditions that guarantee the irreducibility of the tensor products of the representations of $B_3$. As for the tensor products of the representations of $B_4$, we only find sufficient conditions for the irreducibility of the tensor product.



**Keywords:** Braid group, irreducible.

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## 1. Introduction

Let $B_n$ be the braid group on $n$ strings. It has many kinds of linear representations. The earliest was Artin's representation, which is an embedding $B_n \longrightarrow Aut(F_n)$, where $F_n$ is a free group with n generators. Applying the free differential calculus to elements of $Aut(F_n)$ sometimes gives rise to linear representations of $B_n$ and its normal subgroup, the pure braid group denoted by $P_n$. For more details, see [3,4]. The Lawrence- Krammer representation arises this way. Krammer's representation is a representation of the braid group $B_n$ in $GL(m, \mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$, where $m = \frac{n(n-1)}{2}$ ([6,7]). It was shown by Bigelow using topological methods, and independently by Krammer using algebraic methods to be

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faithful, thus proving the long open problem that the braid groups are linear. To prove linearity, Bigelow [2] used the Lawrence-Krammer representation. Other representations of braid groups were obtained by Albeverio in every dimension (see [1]). Using Burau unitarizable representation, Albeverio presented a class of non trivial unitary representations for the braid groups $B_3$ and $B_4$. These are unitary representations of the braid group on a small number of strands and they exist in every dimension depending on n parameters. In section 2, we write explicitly Albeverio's representations of the braid groups $B_3$ and $B_4$. In section 3, we write the main theorems of our work. In sections 4 and 5, we determine the tensor product of the representations of $B_3$ and $B_4$ respectively. In sections 6 and 7, we prove Theorem 3.1 and Theorem 3.2 concerning the irreducibility of the representations obtained by tensoring Albeverio's representations of $B_3$ and $B_4$ respectively. Theorem 3.1 gives necessary and sufficient conditions for the irreducibility of the tensor product of the representations of $B_3$. As for the tensor product of the representations of $B_4$, we fall short of finding necessary conditions of irreducibility. Theorem 3.2 gives only sufficient conditions of irreducibility of the representations of $B_4$. A similar study related to reducibility or irreducibility of braid groups representations exists for the Lawrence-Krammer representation. It was shown that the representation is generically irreducible, but when its two parameters are specified to some complex numbers, it becomes reducible. A complete criterion of irreducibility for the representation is provided in [8]. The latter paper provides a necessary and sufficient condition on the parameters so that the representation is reducible.

## 2. Sergio Albeverio representations of the Braid groups $B_3$ and $B_4$

Albeverio representations of the Braid Group $B_3$: Consider the braid group $B_3$ and the product of the generators $J = \sigma_1\sigma_2$ and $S = \sigma_1 J$. This means that $B_3$ will be generated by $J$ and $S$, and has only one relation $S^2 = J^3$. Denote the representation of $B_3$ by $\pi_3$, where $\pi_3(S) = U$ and $\pi_3(J) = V$. Here $U$ and $V$ are $2n + m \times 2n + m$ block matrices given by

$$U = 2\begin{pmatrix} A - I_n/2 & B & C \\ B^* & B^*A^{-1}B - I_n/2 & B^*A^{-1}C \\ C^* & C^*A^{-1}B & C^*A^{-1}C - I_m/2 \end{pmatrix}$$

and $V = diag(I_n, \beta I_n, \beta^2 I_m)$. We have $\beta = \sqrt[3]{1}$ is a primitive root, $1 \leqslant m \leqslant n$, $A$ and $B$ are $n \times n$ matrices and $C$ is an $n \times m$ matrix. We also have $V^3 = I_{2n+m}$. If $A = A^*$and $BB^* + CC^* = A - A^2$, we get $U = U^*$ and $U^2 = I_{2n+m}$. For more details, see [1].

**Proposition 2.1** $A$ and $B$ are invertible, $rank(C) = m$, $B^*B$ is a diagonal matrix with simple spectrum and every entry of $A$ is non-zero then the Albeverio representation is irreducible.

Albeverio representations of the Braid Group $B_4$: Consider the braid group $B_4$ generated by $\sigma_1$, $\sigma_2$ and $\sigma_3$. Denote the representation of $B_4$ by $\pi_4$. The representation $\pi_4$ is constructed using the reduced Burau representation (see [5]) written in the base where every matrix $\pi_4(\sigma_i)$ is unitary, $\pi_4(\sigma_1) = diag(u, 1, 1)$,

$$\pi_4(\sigma_2) = \begin{pmatrix} (u-1)\alpha_1 + 1 & (u-1)\sqrt{\alpha_1 - \alpha_1^2} & 0 \\ (u-1)\sqrt{\alpha_1 - \alpha_1^2} & (1-u)\alpha_1 + u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\pi_4(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (u-1)\alpha_2+1 & (u-1)\sqrt{\alpha_2-\alpha_2^2} \\ 0 & (u-1)\sqrt{\alpha_2-\alpha_2^2} & (1-u)\alpha_2+u \end{pmatrix},$$

where $u\overline{u} = 1$, $\alpha_1 = -u/(u-1)^2$, $\alpha_2 = \alpha_1/(1-\alpha_1)$. We assume further that $u \neq 1$ and $u \neq \frac{1\pm i\sqrt{3}}{2}$. We observe that $\alpha_1$ and $\alpha_2$ are real numbers. Let us specialize $u$ to a non zero complex number and write $\pi_4(u)$ instead of $\pi_4$. Since the representation $\pi_4(u)$ is unitary then the orthogonal complement of a proper invariant subspace is again a proper invariant subspace. To study the irreducibility of $\pi_4(u)$, it suffices to study the existence of a one-dimensional invariant subspace. The possible one-dimensional invariant subspaces are $\langle e_1\rangle$ and $\langle ae_2 + be_3\rangle$, where $a$ and $b$ are scalars, and $e_1$, $e_2$ and $e_3$ standard unit vectors. Easy calculations give the following proposition.

**Proposition 2.2** $\pi_4(u)$ is irreducible if and only if $u \neq \pm i$.

## 3. Main theorems of the paper

We would take the 3-dimensional 1-parameter based representations of $B_3$, namely, $\pi_3 C)$, with $B$ and $C$ non-zero reals and $A$ specialized to $\frac{1}{2}$, $\beta$ specialized to $e^{\frac{2\pi i}{3}}$, tensor with the 3-dimensional 1-parameter based representation of $B_3$, namely $\pi_3(C')$ with $B$ and $C$' non-zero reals and $A$ specialized to $\frac{1}{2}$, $\beta$ specialized to $e^{\frac{4\pi i}{3}}$.

**Theorem 3.1** For non zero real numbers $C, C' \in ]-\frac{1}{2}, \frac{1}{2}[$, the tensor product of the real specializations of Albeverio's representations $\rho_3 = \pi_3(C) \times \pi_3(C') : B_3 \longrightarrow GL(9,\mathbb{C})$ is irreducible if and only if $C^2 \neq C'^2$.

On the other hand, we take the 3-dimensional 1-parameter based representation of $B_4$, $\pi_4(u)$ as defined formerly tensor with $\pi_4(u')$.

**Theorem 3.2** For non zero complex numbers $u, u' \notin \left\{1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\right\}$, the tensor product of the complex specializations of Albeverio's representations $\rho_4 = \pi_4(u)\otimes\pi_4(u') : B_4 \longrightarrow GL(9,\mathbb{C})$ is irreducible if $u \neq u'$ and $uu' \neq 1$.

## 4. Tensor product of Albeverio's representations of $B_3$

Consider the braid group $B_3$ generated by $S$ and $V$. Take $n = m = 1$ with $B$ and $C$ non-zero real numbers and $A$ is specialized to the value $\frac{1}{2}$. This implies that $B = B^*$, $C = C^*$ and $B^2+C^2 = A-A^2$. For $A = \frac{1}{2}$, we have $B^2 = \frac{1}{4}-C^2$. We require $-\frac{1}{2} < C < \frac{1}{2}$. We substitute the value of $B$ in $U$. So, we get

$$\pi_3(S) = U = 2\begin{pmatrix} 0 & \sqrt{\frac{1}{4}-C^2} & C \\ \sqrt{\frac{1}{4}-C^2} & -2C^2 & 2C\sqrt{\frac{1}{4}-C^2} \\ C & 2C\sqrt{\frac{1}{4}-C^2} & 2C^2-\frac{1}{2} \end{pmatrix} \text{ and } \pi_3(J) = V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}.$$

Here, $\beta$ is a 3rd root of unity. That is, $\beta^3 = 1$. The representation $\pi_3$ is now a one parameter representation. The matrices $U$ and $V$ are given in terms of the real number $C$. Let $\beta = e^{\frac{2\pi i}{3}}$ and we denote $\pi_3$ by $\pi_3(C)$. On the other hand, we change the real number

$C$ to $C'$ and $\beta$ to $e^{\frac{4\pi i}{3}}$, which we will denote by $\pi_3(C')$. We also require $-\frac{1}{2} < C' < \frac{1}{2}$. We denote $\pi_3(C) \times \pi_3(C')$ by $\rho_3$ given by

$$\rho_3(J) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{4\pi i}{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{2\pi i}{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{4\pi i}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{4\pi i}{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_3(S) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & a & b & 0 & d & e \\ 0 & 0 & 0 & a & f & g & d & h & i \\ 0 & 0 & 0 & b & g & j & e & i & k \\ \hline 0 & a & b & 0 & u & l & 0 & m & n \\ a & f & g & u & o & p & m & q & s \\ b & g & j & l & p & r & n & s & t \\ \hline 0 & d & e & 0 & m & n & 0 & w & x \\ d & h & i & m & q & s & w & y & z \\ e & i & k & n & s & t & x & z & v \end{array}\right),$$

where

| | | |
|---|---|---|
| $a = 4\sqrt{\left(\frac{1}{4} - C^2\right)\left(\frac{1}{4} - C'^2\right)}$ | $b = 4C'\sqrt{\left(\frac{1}{4} - C^2\right)}$ | $d = 4C\sqrt{\left(\frac{1}{4} - C'^2\right)}$ |
| $e = 4CC'$ | $f = -8C'^2\sqrt{\left(\frac{1}{4} - C^2\right)}$ | $g = 8C'\sqrt{\left(\frac{1}{4} - C^2\right)\left(\frac{1}{4} - C'^2\right)}$ |
| $h = -8CC'^2$ | $i = 8CC'\sqrt{\left(\frac{1}{4} - C'^2\right)}$ | $j = (8C'^2 - 2)\sqrt{\left(\frac{1}{4} - C^2\right)}$ |
| $k = 2C(4C'^2 - 1)$ | $l = -8C'C^2$ | $m = 8C\sqrt{\left(\frac{1}{4} - C^2\right)\left(\frac{1}{4} - C'^2\right)}$ |
| $n = 8CC'\sqrt{\left(\frac{1}{4} - C^2\right)}$ | $o = 16C^2C'^2$ | $p = -16C'C^2\sqrt{\left(\frac{1}{4} - C'^2\right)}$ |
| $q = -16CC'^2\sqrt{\left(\frac{1}{4} - C^2\right)}$ | $r = -4C^2\left(4C'^2 - 1\right)$ | $s = 16CC'\sqrt{\left(\frac{1}{4} - C^2\right)\left(\frac{1}{4} - C'^2\right)}$ |
| $t = 4C\left(4C'^2 - 1\right)\sqrt{\left(\frac{1}{4} - C^2\right)}$ | $u = -8C^2\sqrt{\left(\frac{1}{4} - C'^2\right)}$ | $v = \left(4C^2 - 1\right)\left(4C'^2 - 1\right)$ |
| $w = \left(8C^2 - 2\right)\sqrt{\left(\frac{1}{4} - C'^2\right)}$ | $x = 2C'(4C^2 - 1)$ | $y = -4C'^2(4C^2 - 1)$ |
| $z = 4C'(4C^2 - 1)\sqrt{\left(\frac{1}{4} - C'^2\right)}$ | | |

## 5. Tensor Product of Albeverio's Representations of $B_4$

Consider the braid group $B_4$, where $B_4$ is the braid group generated by the standard generators $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\pi_4$ is a one parameter representation of $B_4$. The images of the generators $\pi_4(\sigma_1)$,$\pi_4(\sigma_2)$ and $\pi_4(\sigma_3)$ are given in terms of u only, and so we get the representation $\pi_4(u)$. On the other hand, we change $u$ to $u'$ and denote it by $\pi_4(u')$. We require $u \neq 1$, $u \neq \frac{1 \pm i\sqrt{3}}{2}$, $u' \neq 1$ and $u' \neq \frac{1 \pm i\sqrt{3}}{2}$. The representation $\pi_4(u)$ is given by

$$\sigma_1 = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} \frac{-1}{u-1} & \frac{\sqrt{-u^3+u^2-u}}{u-1} & 0 \\ \frac{\sqrt{-u^3+u^2-u}}{u-1} & \frac{u^2}{u-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{u^2-u+1} & \frac{(u-1)\sqrt{-u(u^2+1)}}{|u^2-u+1|} \\ 0 & \frac{(u-1)\sqrt{-u(u^2+1)}}{|u^2-u+1|} & \frac{u^3}{u^2-u+1} \end{pmatrix}.$$

Then, we determine the tensor product $\rho_4 = \pi_4(u) \times \pi_4(u')$.

$$\rho_4(\sigma_1) = \begin{pmatrix} uu' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \rho_4(\sigma_2) = \left(\begin{array}{ccc|ccc|ccc} xx' & r & 0 & r' & vv' & 0 & 0 & 0 & 0 \\ r & y' & 0 & vv' & z' & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & v & 0 & 0 & 0 \\ \hline r' & vv' & 0 & y & z & 0 & 0 & 0 & 0 \\ vv' & z' & 0 & z & ww' & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & w & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & x' & v' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v' & w' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

where

| | |
|---|---|
| $r = \frac{\sqrt{-u'^3+u'^2-u'}}{(1-u)(u'-1)}$ | $r' = \frac{\sqrt{-u^3+u^2-u}}{(1-u')(u-1)}$ |
| $v = \frac{\sqrt{(-u^3+u^2-u)}}{(u-1)}$ | $v' = \frac{\sqrt{(-u'^3+u'^2-u')}}{(u'-1)}$ |
| $w = \frac{u^2}{(u-1)}$ | $w' = \frac{u'^2}{(u'-1)}$ |
| $x = \frac{-1}{u-1}$ | $x' = \frac{-1}{u'-1}$ |
| $y = \frac{-u^2}{(u-1)(u'-1)}$ | $y' = \frac{-u'^2}{(u'-1)(u-1)}$ |
| $z = \frac{u^2\sqrt{(-u'^3+u'^2-u')}}{(u-1)(u'-1)}$ | $z' = \frac{u'^2\sqrt{-u^3+u^2-u}}{(u-1)(u'-1)}$ |

and

$$\rho_4(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 & f & g & 0 & h & i \\ 0 & 0 & 0 & 0 & g & j & 0 & i & k \\ 0 & 0 & 0 & e & 0 & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & h & i & 0 & m & n \\ 0 & 0 & 0 & 0 & i & k & 0 & n & o \end{pmatrix},$$

where

| | | |
|---|---|---|
| $a = \frac{1}{u'^2-u'+1}$ | $b = \frac{(u'-1)\sqrt{-u'(u'^2+1)}}{\lvert u'^2-u'+1\rvert}$ | $c = \frac{u'^3}{u'^2-u'+1}$ |
| $d = \frac{1}{u^2-u+1}$ | $e = \frac{(u-1)\sqrt{-u(u^2+1)}}{\lvert u^2-u+1\rvert}$ | $f = \frac{1}{(u^2-u+1)(u'^2-u'+1)}$ |
| $g = \frac{(u'-1)\sqrt{-u'(u'^2+1)}}{(u^2-u+1)\lvert u'^2-u'+1\rvert}$ | $h = \frac{(u-1)\sqrt{-u(u^2+1)}}{(u'^2-u'+1)\lvert\lvert u^2-u+1\rvert}$ | $i = \frac{(u-1)(u'-1)\sqrt{uu'(u^2+1)(u'^2+1)}}{\lvert u^2-u+1\rvert\lvert u'^2-u'+1\rvert}$ |
| $j = \frac{u'^3}{(u'^2-u'+1)(u^2-u+1)}$ | $k = \frac{u'^3(u-1)\sqrt{-u(u^2+1)}}{(u'^2-u'+1)\lvert u^2-u+1\rvert}$ | $l = \frac{u^3}{u^2-u+1}$ |
| $m = \frac{u^3}{(u^2-u+1)(u'^2-u'+1)}$ | $n = \frac{u^3(u'-1)\sqrt{-u'(u'^2+1)}}{(u^2-u+1)\lvert u'^2-u'+1\rvert}$ | $o = \frac{u^3u'^3}{(u^2-u+1)(u'^2-u'+1)}$ |

## 6. On the Irreducibility of the Tensor Product of the Real Specializations of Albeverio's Representations of $B_3$

We specialize the indeterminates involved in defining the tensor product of Albeverio's representations of $B_3$ to non zero real numbers. We investigate whether or not there are invariant subspaces under the tensor product of the representations. Now, we prove Theorem 3.1.

*Sufficient conditions of irreducibility:* For a unitary representation, the orthogonal complement of a proper invariant subspace is again a proper invariant subspace. Thus, to show irreducibility, it suffices only to show that there are no proper invariant subspaces of dimensions 1, 2, 3 and 4 if $C^2$ is distinct from $C'^2$.

**Invariant subspaces of dimension one.** If x is a generator of a one-dimensional invariant subspace, then since $\rho_3(J)(x) = \lambda x$, some scalar $\lambda$, we must have x belongs to span$\langle e_1, e_5, e_9\rangle$ or x belongs to span $\langle e_2, e_6, e_7\rangle$ or x belongs to span$\langle e_3, e_4, e_8\rangle$. From the shape of the matrix $\rho_3(S)$ and the allowed specializations for C and C', the third possibility is to rule out. And so is the second one. It remains to study the first possibility, where x belongs to span$\langle e_1, e_5, e_9\rangle$. Let $A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9\rangle$. Then, $\rho_3(S)\, e_1 = \lambda \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & \alpha_9 \end{pmatrix}^T$ for some scalar $\lambda$. Here we notice that $fk - gi = 0$ but $fn - ui \neq 0$ if $C^2$ is distinct from $C'^2$. This implies $\alpha_5 = \alpha_9 = 0$. Then from the eight row, we derive $\alpha_1 = 0$. Hence if $C^2$ is distinct from $C'^2$, there is no one-dimensional invariant subspace.

**Invariant subspaces of dimension two.** From the diagonal shape of $\rho_3(J)$, the possible 2-dimensional invariant subspaces are $\langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8\rangle$, $\langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_2 e_2 + \alpha_6 e_6 + \alpha_7 e_7\rangle$, $\langle \alpha_2 e_2 + \alpha_6 e_6 + \alpha_7 e_7, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8\rangle$, $\langle e_1, e_i\rangle$ for $i = 5, 9$, $\langle e_2, e_i\rangle$ for $i = 6, 7$ and $\langle e_3, e_i\rangle$ for $i = 4, 8$. Here, $\alpha_i' s$ are scalars. We observe that for any vector belonging to either one of these spaces, we have at least two of its components are zeros. From the shape of $\rho_3(S)$ and the specializations, we rule out the subspaces of the form $\langle e_i, e_j\rangle$. Since the argument is quite similar in handling all the other subspaces, we take $A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8\rangle$ as an example to show that the subspace is not invariant if $C^2 \neq C'^2$.

If $A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8\rangle$, then $\rho_3(S)\,(\alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9) \in A$. We show that the 2nd entry and the 7th entry cannot be both zeros. If 2nd entry $f\alpha_5 + i\alpha_9 = 0$, we get

$$-8C'^2\sqrt{\left(\frac{1}{4} - C^2\right)}\alpha_5 + 8CC'\sqrt{\left(\frac{1}{4} - C'^2\right)}\alpha_5 = 0. \tag{1}$$

If the 7-th entry $m\alpha_5 + x\alpha_9 = 0$, we get

$$8C\sqrt{\left(\frac{1}{4} - C^2\right)\left(\frac{1}{4} - C^2\right)}\alpha_5 + 2C'(4C^2 - 1)\alpha_9 = 0. \tag{2}$$

By solving equations (1) and (2), we get $C'^2 = C^2$, which is a contradiction. So, the 2nd and 7th entries cannot be zeros at the same time. Thus, $\rho_3(U)\,(\alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9) \notin A$. Therefore, the subspace $A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8\rangle$ is not invariant.

**Invariant subspaces of dimension three.** As in dimension 2, we only consider the three dimensional invariant subspace which contains vectors whose none of their components are zeros. Take $A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8, \alpha_2 e_2 + \alpha_6 e_6 + \alpha_7 e_7\rangle$. Let $V_1 = \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9$, $V_2 = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8$ and $V_3 = \alpha_2 e_2 + \alpha_6 e_6 + \alpha_7 e_7$.

- $\rho_3(S)(V_1) \in A$.

So, we get $(a\alpha_5 + e\alpha_9)e_1 + (f\alpha_5 + i\alpha_9)e_2 + (g\alpha_5 + k\alpha_9)e_3 + (u\alpha_5 + n\alpha_9)e_4 + (a\alpha_1 + o\alpha_5 + s\alpha_9)e_5 + (b\alpha_1 + p\alpha_5 + t\alpha_9)e_6 + (m\alpha_5 + x\alpha_9)e_7 + (d\alpha_1 + q\alpha_5 + z\alpha_9)e_8 + (e\alpha_+ s\alpha_5 + v\alpha_9)e_9 \in A$. Thus, $\rho_3(S)(V_1) = k_1 V_1 + k_2 V_2 + k_3 V_3$, where $a\alpha_5 + e\alpha_9 = K_1\alpha_1$, $a\alpha_1 + o\alpha_5 + s\alpha_9 = K_1\alpha_5$, $e\alpha_+ s\alpha_5 + v\alpha_9 = K_1\alpha_9$, $g\alpha_5 + k\alpha_9 = K_2\alpha_3$, $u\alpha_5 + n\alpha_9 = K_2\alpha_4$, $d\alpha_1 + q\alpha_5 + z\alpha_9 = K_2\alpha_8$, $f\alpha_5 + i\alpha_9 = K_3\alpha_2$, $b\alpha_1 + p\alpha_5 + t\alpha_9 = K_3\alpha_6$ and $m\alpha_5 + x\alpha_9 = K_3\alpha_7$.

- $\rho_3(S)(V_2) \in A$

So we get $(d\alpha_8)e_1 + (a\alpha_4 + h\alpha_8)e_2 + (b\alpha_4 + i\alpha_8)e_3 + (b\alpha_3 + m\alpha_8)e_4 + (g\alpha_3 + u\alpha_4 + q\alpha_8)e_5 + (j\alpha_3 + l\alpha_4 + s\alpha_8)e_6 + (e\alpha_3 + w\alpha_8)e_7 + (i\alpha_3 + m\alpha_4 + y\alpha_8)e_8 + (k\alpha_3 + n\alpha_4 + z\alpha_8)e_9 \in A$. Thus, $\rho_3(S)(V_2) = d_1V_1 + d_2V_2 + d_3V_3$, where $d\alpha_8 = d_1\alpha_1$, $g\alpha_3 + u\alpha_4 + q\alpha_8 = d_1\alpha_5$, $k\alpha_3 + n\alpha_4 + z\alpha_8 = d_1\alpha_9$, $b\alpha_4 + i\alpha_8 = d_2\alpha_3$, $b\alpha_3 + m\alpha_8 = d_2\alpha_4$, $i\alpha_3 + m\alpha_4 + y\alpha_8 = d_2\alpha_8$, $a\alpha_4 + h\alpha_8 = d_3\alpha_2$, $j\alpha_3 + l\alpha_4 + s\alpha_8 = d_3\alpha_6$ and $e\alpha_3 + w\alpha_8 = d_3\alpha_7$

- $\rho_3(S)(V_3) \in A$

So, we get $(b\alpha_6)e_1 + (g\alpha_6 + d\alpha_7)e_2 + (j\alpha_6 + e\alpha_7)e_3 + (a\alpha_2 + l\alpha_6)e_4 + (f\alpha_2 + p\alpha_6 + m\alpha_7)e_5 + (g\alpha_2 + r\alpha_6 + n\alpha_7)e_6 + (d\alpha_2 + n\alpha_6)e_7 + (h\alpha_2 + s\alpha_6 + w\alpha_7)e_8 + (i\alpha_2 + t\alpha_6 + x\alpha_7)e_9 \in A$. Thus, $\rho_3(S)(V_3) = m_1V_1 + m_2V_2 + m_3V_3$, where $b\alpha_6 = m_1\alpha_1$, $f\alpha_2 + p\alpha_6 + m\alpha_7 = m_1\alpha_5$, $i\alpha_2 + t\alpha_6 + x\alpha_7 = m_1\alpha_9$, $j\alpha_6 + e\alpha_7 = m_2\alpha_3$, $a\alpha_2 + l\alpha_6 = m_2\alpha_4$, $h\alpha_2 + s\alpha_6 + w\alpha_7 = m_2\alpha_8$, $g\alpha_6 + d\alpha_7 = m_3\alpha_2$, $g\alpha_2 + r\alpha_6 + n\alpha_7 = m_3\alpha_6$ and $d\alpha_2 + n\alpha_6 = m_3\alpha_7$.

Without loss of generality, we assume $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Solving the system above, we get the following equations.

$$k\alpha_5 + n\alpha_4\alpha_5 + z\alpha_5\alpha_8 - g\alpha_9 - l\alpha_4\alpha_9 - s\alpha_8\alpha_9 = 0 \tag{3}$$
$$k + n\alpha_4 + z\alpha_8 - d\alpha_8\alpha_9 = 0 \tag{4}$$
$$g - l\alpha_4 - s\alpha_8 - d\alpha_5\alpha_8 = 0 \tag{5}$$
$$u\alpha_5 + i\alpha_9 - g\alpha_5\alpha_4 - k\alpha_9\alpha_4 = 0 \tag{6}$$
$$b + p\alpha_5 + t\alpha_9 - f\alpha_5\alpha_6 - i\alpha_9\alpha_6 = 0 \tag{7}$$
$$m\alpha_5 + x\alpha_9 - f\alpha_5\alpha_7 - i\alpha_9\alpha_7 = 0 \tag{8}$$
$$k + n\alpha_4 + z\alpha_8 - d\alpha_4\alpha_8\alpha_9 = 0 \tag{9}$$
$$g + l\alpha_4 + s\alpha_8 - d\alpha_5\alpha_8 = 0 \tag{10}$$
$$a + o\alpha_5 + s\alpha_9 - a\alpha_5^2 - e\alpha_5\alpha_9 = 0 \tag{11}$$
$$g\alpha_6\alpha_7 + d\alpha_7^2 - n\alpha_7 = 0 \tag{12}$$

The aim is to write the values of $\alpha_i$ in terms of $\alpha_5$ and $\alpha_9$ and then solve for $\alpha_5$ in terms of $\alpha_9$ $(i = 4, 6, 7, 8, 9)$. From equation (6), we get

$$\alpha_4 = \frac{8CC'\sqrt{(\frac{1}{4} - C^2)}\alpha_9 - 8C^2\sqrt{(\frac{1}{4} - C^2)}\alpha_5}{8C'\sqrt{(\frac{1}{4} - C^2)(\frac{1}{4} - C'^2)}\alpha_5 - 2C(4C'^2 - 1)\alpha_9}.$$

From equation (7), we get

$$\alpha_6 = \frac{16C^2C'\sqrt{(\frac{1}{4} - C'^2)}\alpha_5 - 4C(4C'^2 - 1)\sqrt{(\frac{1}{4} - C'^2)}\alpha_9 - 4C'\sqrt{(\frac{1}{4} - C'^2)}}{8C'^2\sqrt{\frac{1}{4} - C^2}\alpha_5 - 8CC'\sqrt{\frac{1}{4} - C'^2}\alpha_9}.$$

From equation (8), we get

$$\alpha_7 = \frac{8C\sqrt{(\frac{1}{4}-C^2)(\frac{1}{4}-C'^2)}\alpha_5 + 2C'(4C^2-1)\alpha_9}{8CC'\sqrt{\frac{1}{4}-C'^2}\alpha_9 - 8C^2\sqrt{\frac{1}{4}-C^2}\alpha_5}.$$

By solving equations (9) and (10), we get

$$\alpha_8 = \frac{(2c^2(4c'^2-1) + 8c'(\frac{1}{4}-c'^2)\sqrt{(\frac{1}{4}-c'^2)}}{4c^2\sqrt{(\frac{1}{4}-c'^2)}\alpha_9 + 4c\sqrt{\left(\frac{1}{4}-c^2\right)\left(\frac{1}{4}-c'^2\right)}\alpha_5}.$$

At last, we replace the values of $\alpha_6$ and $\alpha_7$ in equation (12) and then we get $\alpha_9$ in terms of $\alpha_5$. By solving equations (3)-(5), we get

$$\alpha_9 = \frac{(-4C^3C'^3 + 2C'C^3 + 2C'^3C)\alpha_5}{32C^2C'^2\alpha_5 - (2C^2+2C'^2-1)\sqrt{\left(\frac{1}{4}-C^2\right)\left(\frac{1}{4}-C'^2\right)}}.$$

Comparing both equations, we get a contradiction. Thus, the subspace

$$A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8, \alpha_2 e_2 + \alpha_6 e_6 + \alpha_7 e_7 \rangle$$

is not invariant.

**Invariant Subspaces of Dimension Four.** As in the previous cases, we will exclude these subspaces that are ruled out by just allowing the specializations for C and C'. We consider a possible invariant subspace when the zero argument cannot be applied. That is, at least one vector of the subspace has no zero components.

Let $A = \langle \alpha_1 e_1 + \alpha_5 e_5 + \alpha_9 e_9, \alpha_3 e_3 + \alpha_4 e_4 + \alpha_8 e_8, \alpha_2 e_2 + \alpha_6 e_6 + \alpha_7 e_7, e_8 \rangle$. Along the same lines as in dimension 3 and performing several computations, we get $\alpha_4 = 1$ and $\alpha_7 = \frac{CC'}{\sqrt{\left(\frac{1}{4}-C^2\right)\left(\frac{1}{4}-C'^2\right)}}$. This implies that

$$\alpha_6 = \frac{(8C'^2-2)\sqrt{\frac{1}{4}-C^2} - 8CC'^2}{4\sqrt{\left(\frac{1}{4}-C^2\right)\left(\frac{1}{4}-C'^2\right)}} \tag{13}$$

$$\alpha_6 = \frac{4(\frac{1}{4}-C^2)(\frac{1}{4}-C'^2) + 4C^2C'^2}{((8C'^2-2)\sqrt{\frac{1}{4}-C^2} - 8CC'^2)\sqrt{\left(\frac{1}{4}-C^2\right)\left(\frac{1}{4}-C'^2\right)}} \tag{14}$$

$$\alpha_6 = \frac{(1-4C'^2-4C^2)\sqrt{\frac{1}{4}-C^2}}{C'^2 - 2C'\sqrt{\frac{1}{4}-C^2}} \tag{15}$$

$$CC' + (8C'^2-2)\sqrt{1-4C^2} + 8CC'^2 = 0. \tag{16}$$

By (13) and (14), it follows that

$$4C'(-4C^2\sqrt{1-4C^2} + C'(1-8C^2-16C^4) + 16C^2C'^2\sqrt{1-4C^2} + 4C'^3(4C^2-1) = 0 \tag{17}$$

By (15) and (16), it follows that

$$(1-4C^2-4C'^2)\sqrt{1-4C^2}-C'(2C'-\sqrt{1-4C^2})(\frac{1}{4}-C^2)(-8C^2C'^2(4C'^2-1)(\frac{1}{4}-C^2)=0 \tag{18}$$

Solving equations (17) and (18), we get $(C,C')=(\pm\frac{1}{2},0)$, $(C,C')=(0,\pm\frac{1}{2})$ and $(C,C')=(0.45-0.09i,-0.36-0.014i)$ which are all rejected by our hypothesis. Thus, the subspace $A=\langle\alpha_1e_1+\alpha_5e_5+\alpha_9e_9,\alpha_3e_3+\alpha_4e_4+\alpha_8e_8,\alpha_2e_2+\alpha_6e_6+\alpha_7e_7,e_8\rangle$ is not invariant.
*Necessary conditions of irreducibility:* We assume that $C^2=C'^2$. In the case $C=C'$, it is easy to see that $a+e=a+o+s=e+s+v=1$ and $f+i=g+k=u+n=b+p+t=m+x=d+q+z=0$. Hence, the one-dimensional subspace generated by $\langle e_1+e_5+e_9\rangle$ is invariant. In the case $C=-C'$, we also see that $-a+e=-1$, $-a-o+s=-1$, $-e-s+v=1$ and $-f+i=-g+k=-u+n=-b-p+t=-m+x=-d-q+z=0$. Hence, the one-dimensional subspace generated by $\langle -e_1-e_5+e_9\rangle$ is invariant. ∎

## 7. On the Irreducibility of the Tensor Product of the Complex Specializations of Albeverio's Representations of $B_4$

We specialize the indeterminates $u$ and $u'$ to non zero complex numbers. Our aim is to study the irreducibility of the tensor product of complex specializations of Albeverio's representations of $B_4$. The representations are $9\times 9$ matrices. We determine whether or not there are invariant subspaces under the tensor product of the representation. Now, we prove Theorem 3.2.

We show that there are no non trivial proper invariant subspaces. Assume that $u\neq u'$ and $uu'\neq 1$ in order to reduce the number of possible invariant subspaces where we need to study.

**Invariant Subspaces of Dimension one.** If there exists a one-dimensional invariant subspace spanned by $x=(\alpha_1,...,\alpha_9)^T$, assuming that $u\neq u'$ and $uu'\neq 1$, then $\rho_4(\sigma_1)(x)=\lambda x$ for some scalar $\lambda$ forces the following set of conditions on the $\alpha_i's$ :
1) $\alpha_1\neq 0$ implies $\alpha_i=0$ for all $i\neq 1$.
2) $\alpha_2\neq 0$ implies $\alpha_i=0$ for all $i\neq 2,3$.
3) $\alpha_4\neq 0$ implies $\alpha_i=0$ for all $i\neq 4,7$.
4) $\alpha_5\neq 0$ implies $\alpha_i=0$ for all $i\neq 5,6,8,9$.

Then, the one dimensional invariant subspaces candidates to study are:
1) $S=\langle e_1\rangle$.
2) $S=\langle\alpha_2e_2+\alpha_3e_3\rangle$.
3) $S=\langle\alpha_4e_4+\alpha_7e_7\rangle$.
4) $S=\langle\alpha_5e_5+\alpha_6e_6+\alpha_8e_8+\alpha_9e_9\rangle$.

**Case 1.** Consider $S=\langle e_1\rangle$. $\rho_4(\sigma_2)(e_1)=\left(xx'\ r\ 0\ r'\ vv'\ 0\ 0\ 0\ 0\right)^T$ $r\neq 0$ implies that $S$ is not invariant.

**Case 2.** Consider $S=\langle\alpha_2e_2+\alpha_3e_3\rangle$. Then

$$\rho_4(\sigma_2)(\alpha_2e_2+\alpha_3e_3)=\left(r\alpha_2\ y'\alpha_2\ x\alpha_3\ vv'\alpha_2\ z'\alpha_2\ v\alpha_3\ 0\ 0\ 0\right)^T.$$

If $S$ is invariant then $r\neq 0$ implies that $\alpha_2$ must be zero and $v\neq 0$ implies that $\alpha_3$ must be zero, a contradiction.

**Case 3.** Consider $S=\langle\alpha_4e_4+\alpha_7e_7\rangle$. Then

$$\rho_4(\sigma_2)(\alpha_4e_4+\alpha_7e_7)=\left(r'\alpha_4\ vv'\alpha_4\ 0\ y\alpha_4\ z\alpha_4\ 0\ x'\alpha_7\ v'\alpha_7\ 0\right)^T$$

The values of $r'$ and $v'$ are both non zero which force respectively that $\alpha_4=0$ and $\alpha_7=0$.

**Case 4.** Consider $S = \langle \alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9 \rangle$. Then $\rho_4(\sigma_2)(\alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9) = \left( vv'\alpha_5 \; z'\alpha_5 \; v\alpha_6 \; z\alpha_5 \; ww'\alpha_5 \; w\alpha_6 \; v'\alpha_8 \; w'\alpha_8 \; \alpha_9 \right)^T$. Same reasoning as in Case 3, using $v$ and $v'$ to get $\alpha_5 = 0$, $\alpha_6 = 0$, $\alpha_8 = 0$. Thus, we have $\rho_4(\sigma_3)(\alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9) = \left( 0\;0\;0\;0\; i\alpha_9 \; k\alpha_9 \; 0 \; n\alpha_9 \; o\alpha_9 \right)^T$. This implies that $\alpha_9 = 0$, a contradiction.

**Invariant Subspaces of Dimension Two.** As before, we could apply the zero argument and we take

$$S = \langle \alpha_4 e_4 + \alpha_7 e_7, \alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9 \rangle.$$

We have

$$\rho_4(\sigma_2)(\alpha_4 e_4 + \alpha_7 e_7) = \left( r'\alpha_4 \; vv'\alpha_4 \; 0 \; y\alpha_4 \; z\alpha_4 \; 0 \; x'\alpha_7 \; v'\alpha_7 \; 0 \right)^T$$

and

$$\rho_4(\sigma_3)(\alpha_4 e_4 + \alpha_7 e_7) = \left( 0\;0\;0\; d\alpha_4 + l\alpha_7 \; 0\;0\; e\alpha_4 + m\alpha_7 \; 0\;0 \right)^T.$$

Since $r'$ is non zero, it follows from $\rho_4(\sigma_2)(\alpha_4 e_4 + \alpha_7 e_7)$ that $\alpha_4 = 0$ and so $\alpha_7 = 0$ by considering the fourth component of $\rho_4(\sigma_3)(\alpha_4 e_4 + \alpha_7 e_7)$ is a contradiction.

**Invariant Subspaces of Dimension Three.** As in dimension 2, we can use the zero argument. To see this, take

$$S = \langle \alpha_2 e_2 + \alpha_3 e_3, \alpha_4 e_4 + \alpha_7 e_7, \alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9 \rangle.$$

We have

$$\rho_4(\sigma_2)(\alpha_2 e_2 + \alpha_3 e_3) = \left( r\alpha_2 \; y'\alpha_2 \; x'\alpha_3 \; vv'\alpha_2 \; z'\alpha_2 \; v\alpha_3 \; 0\;0\;0 \right)^T$$

and

$$\rho_4(\sigma_3)(\alpha_2 e_2 + \alpha_3 e_3) = \left( 0 \; a\alpha_2 + b\alpha_3 \; b\alpha_2 + c\alpha_3 \; 0\;0\;0\;0\;0\;0 \right)^T.$$

Since $r$ is non zero, it follows from $\rho_4(\sigma_2)(\alpha_2 e_2 + \alpha_3 e_3)$ that $\alpha_2 = 0$ and so $\alpha_3 = 0$ by considering the second component of $\rho_4(\sigma_3)(\alpha_2 e_2 + \alpha_3 e_3)$, a contradiction.

**Invariant Subspaces of Dimension Four.** We cannot always use zero argument as in dimension 3. For instance, we take

$$S = \langle \alpha_2 e_2 + \alpha_3 e_3, \alpha_4 e_4 + \alpha_7 e_7, \alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9, e_1 \rangle.$$

Let $V_1 = \alpha_2 e_2 + \alpha_3 e_3$, $V_2 = \alpha_4 e_4 + \alpha_7 e_7$ and $V_3 = \alpha_5 e_5 + \alpha_6 e_6 + \alpha_8 e_8 + \alpha_9 e_9$. Then

- $\rho_4(\sigma_2)(V_1) = K_1 V_1 + K_2 V_2 + K_3 V_3$, where

$\frac{-u'^2}{(u-1)(u'-1)}\alpha_2 = K_1\alpha_2$, $\frac{-1}{(u-1)}\alpha_3 = K_1\alpha_3$, $\frac{\sqrt{(u^3-u^2+u)(u'^3-u'^2+u)}}{(u-1)(u'-1)}\alpha_2 = K_2\alpha_4$, $\frac{u'^2\sqrt{(-u^3+u^2-u)}}{(u-1)(u'-1)}\alpha_2 = K_3\alpha_5$ and $\frac{\sqrt{(-u^3+u^2-u)}}{u-1}\alpha_3 = K_3\alpha_6$.

- $\rho_4(\sigma_2)(V_2) = m_1 V_1 + m_2 V_2 + m_3 V_3$, where

$\frac{\sqrt{(u^3-u^2+u)(u'^3-u'^2+u)}}{(u-1)(u'-1)}\alpha_4 = m_1\alpha_2$, $\frac{-u^2}{(u-1)(u'-1)}\alpha_4 = m_2\alpha_4$, $\frac{-1}{(u'-1)}\alpha_7 = m_2\alpha_7$, $\frac{u'^2\sqrt{(-u^3+u^2-u)}}{(u-1)(u'-1)}\alpha_4 = m_3\alpha_5$ and $\frac{\sqrt{(-u'^3+u'^2-u')}}{u'-1}\alpha_7 = m_3\alpha_8$.

- $\rho_4(\sigma_2)(V_3) = d_1 V_1 + d_2 V_2 + d_3 V_3$, where

$\frac{u'^2\sqrt{(-u^3+u^2-u)}}{(u-1)(u'-1)}\alpha_5 = d_1\alpha_2$, $\frac{\sqrt{(-u^3+u^2-u)}}{u-1}\alpha_6 = d_1\alpha_3$, $\frac{u^2\sqrt{(-u'^3+u'^2-u')}}{(u-1)(u'-1)}\alpha_5 = d_2\alpha_4$, $\frac{\sqrt{(-u'^3+u'^2-u')}}{u'-1}\alpha_8 = d_2\alpha_7$, $\frac{u^2u'^2}{(u-1)(u'-1)}\alpha_5 = d_3\alpha_5$, $\frac{u^2}{u-1}\alpha_6 = d_3\alpha_6$, $\frac{u'^2}{u'-1}\alpha_8 = d_3\alpha_8$ and

$\alpha_9 = d_3\alpha_9$. Solving all the above we get the following equations.

$$u'^2 - u' + 1 = 0 \quad (19)$$

$$u'^2\alpha_2\alpha_6 - (u' - 1)\alpha_2\alpha_6 = 0 \quad (20)$$

$$u^2 - u + 1 = 0 \quad (21)$$

$$u\alpha_5 + i\alpha_9 - g\alpha_5\alpha_4 - k\alpha_9\alpha_4 = 0 \quad (22)$$

$$u^2\alpha_4\alpha_8 - (u - 1)\alpha_5\alpha_7 = 0 \quad (23)$$

$$u'^2\alpha_3\alpha_5 - (u' - 1)\alpha_2\alpha_6 = 0 \quad (24)$$

$$u^2\alpha_5\alpha_7 - (u' - 1)\alpha_4\alpha_8 = 0 \quad (25)$$

$$u^2(u' - 1) - u'^2(u - 1) = 0 \quad (26)$$

$$u^2u' - (u - 1)(u' - 1) = 0 \quad (27)$$

Solving equations (20) and (27), we get $(u-1)(u') - (u-1)(u'-1) = 0$. So, $(u-1)(u' - u' + 1) = 0$. Then, we have $u = 1$, which is a contradiction. So, $S$ is not invariant. ∎

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