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$(F, \varphi, \alpha)_s$-contractions in b-metric spaces and applications

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Abstract. In this paper, we introduce more general contractions called φ-fixed point point for $(F, \varphi, \alpha)_s$ and $(F, \varphi, \alpha)_s$-weak contractions. We prove the existence and uniqueness of φ-fixed point point for $(F, \varphi, \alpha)_s$ and $(F, \varphi, \alpha)_s$-weak contractions in complete b-metric spaces. Some examples are supplied to support the usability of our results. As applications, necessary conditions to ensure the existence of a unique solution for a nonlinear inequality problem are also discussed. Also, some new fixed point results in partial metric spaces are proved.

Keywords: b-metric space, partial b-metric space, $(F, \varphi, \alpha)_s$-contraction, φ-fixed point.

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1. Introduction

The Banach contraction principle is one of the most important subjects in mathematical analysis, it guarantees the existence and uniqueness of a fixed point [4]. By using this principle, most authors have proved several fixed point theorems for various mappings in several metric spaces (see [1, 2, 5–7, 11–16, 20, 21, 25, 27]. For example, Mathews [17] introduced the concept of partial metric space and showed that the Banach contraction principle can be generalized in partial metric space. Bakhtin [3] and Czerwik [8] introduced b-metric spaces as a generalization of metric spaces and proved the contraction mapping principle in b-metric spaces that is an extension of the Banach contraction principle in metric spaces. Since then, a number of authors have investigated fixed point theorems in b-metric spaces (see [9, 10, 18, 22]). Later, Shukla [24] generalized the concept of both b-metric and partial metric space by presenting the partial b-metric space.

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On the other hand, some authors proved fixed point results by offering its various variations of the Banach contraction mapping principle. For example, Jleli et al. [25] introduced the concept of φ-fixed point and established some existence results of φ-fixed points for various classes of operators in metric spaces. Also, Samet et al. [23] introduced the notion of α-admissible mapping in metric spaces. Later, Sintunavarat [26] introduced the concepts of α-admissible mapping type S, as some generalizations of α-admissible mapping and then he proved some fixed point theorems by using his new types of α-admissibility mapping in b-metric spaces.

In this paper, we establish the existence and uniqueness of φ-fixed points for $(F, \varphi, \alpha)_s$-contraction and $(F, \varphi, \alpha)_s$-weak contraction in complete b-metric space. The presented theorems extend and generalize the φ-fixed point results. Some examples are supplied in order to support the useability of our results. As applications of the obtained results, we presented the existence of a unique solution for nonlinear Volterra integral equations. Also, some fixed point theorems in partial b-metric spaces are derived from our main theorems.

2. Preliminaries

Definition 2.1 [8] Let X be a nonempty set and $s \geqslant 1$ a real number. A mapping $d_b : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $d_b(x, y) = 0$ if and only if $x = y$,
(ii) $d_b(x, y) = d_b(y, x)$,
(iii) $d_b(x, z) \leqslant s[d_b(x, y) + d_b(y, z)]$.
In this case, (X, d) is called a b-metric space.

Definition 2.2 [9] A sequence $\{x_n\}$ in a b-metric space (X, d_b) is said to be:
(i) b-convergent to a point $x \in X$ if $\lim_{n\to\infty} d_b(x_n, x) = 0$.
(ii) A sequence $\{x_n\}$ in a b-metric space (X, d_b) is called a Cauchy sequence if $\lim_{n,m\to\infty} d_b(x_n, x_m) = 0$.
(iii) A b-metric space (X, d_b) is called complete if every Cauchy sequence $\{x_n\}$ in X b-converges to a point $x \in$X.
(iv) A function $f : X \to Y$ is b-continuous at a point $x \in X$ if $\{x_n\} \subset X$ b-converges to x, then $\{fx_n\} \subset Y$ b-converges to fx, where (Y, ρ) is a b-metric space.

Definition 2.3 [26] Let X be a nonempty set and $s \geqslant 1$ a given real number. Let $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. We say T is an $\alpha-$admissible mapping type S if for all $x, y \in X$, $\alpha(x, y) \geqslant s$ leads to $\alpha(Tx, Ty) \geqslant s$. In particular, T is called $\alpha-$admissible mapping if $s = 1$.

Definition 2.4 [15] Let (X, d) be a metric space, $\varphi : X \longrightarrow [0, \infty)$ be a given function and $T : X \to X$ be an operator. We denote by $T^0 = 1_X$, $T^1 = T$ and $T^{n+1} = T \circ T^n$ for $n \in \mathbb{N}$, the iterate operators of T. The set of all fixed points of the operator T will be denoted by $F_T = \{x \in X : Tx = x\}$ and the set all zeros of the function φ will be denoted by $Z_\varphi = \{x \in X : \varphi(x) = 0\}$.
(D-1) An element $z \in X$ is said to be a $\varphi-$fixed point of the operator T if and only if $z \in F_T \cap Z_\varphi$.
(D-2) T is a $\varphi-$Picard operator if and only if
(i) $F_T \cap Z_\varphi = \{z\}$,
(ii) $T^n x \to z$ as $n \to \infty$, for each $x \in X$.

(D-3) T is a weakly $\varphi-$Picard operator if and only if
(i) $F_T \cap Z_\varphi \neq \varnothing$,
(ii) the sequence $\{T^n x\}$ converges for each $x \in X$ and the limit is a $\varphi-$fixed point of the operator T.

Let $\mathcal{F}$ be the set of functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:
(F1) $\max\{a, b\} \leqslant F(a, b, c)$ for all $a, b, c \in [0, \infty)$,
(F2) $F(0, 0, 0) = 0$,
(F3) F is continuous.
The following functions are given as examples:
(i) $F(a, b, c) = a + b + c$,
(ii) $F(a, b, c) = \max\{a, b\} + c$,
(iii) $F(a, b, c) = a + a^2 + b + c$.

Definition 2.5 [15] Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. The operator $T : X \to X$ is an $(F, \varphi)-$contraction if and only if

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leqslant kF(d(x, y), \varphi(x), \varphi(y)), \quad x, y \in X$$

for some constant $k \in (0, 1)$.

Definition 2.6 [15] Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. The operator $T : X \to X$ is an $(F, \varphi)-$weak contraction if and only if for $x, y \in X$,

$$\begin{aligned} F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) &\leqslant kF(d(x, y), \varphi(x), \varphi(y)) \\ &\quad + L(F(d(y, Tx), \varphi(y), \varphi(Tx) - F(0, \varphi(y), \varphi(Tx))) \end{aligned}$$

for some constant $k \in (0, 1)$ and $L \geqslant 0$.

3. Main Results

Definition 3.1 Let (X, d_b) be a b-metric space with coefficient $s \geqslant 1$, $\alpha : X \times X \to [0, \infty)$

be a mapping, $\varphi : X \to [0, \infty)$ be lower semi continuous function, $F \in \mathcal{F}$, $\lambda \in (0, 1)$ and $\varepsilon > 1$ be a constant. A mapping $T : X \to X$ is said to be an $(F, \varphi, \alpha)_s-$contraction mapping if

$$\begin{aligned} &x, y \in X \text{ with } \alpha(x, y) \geqslant s \Longrightarrow \\ &s^\varepsilon F(d_b(Tx, Ty), \varphi(Tx), \varphi(Ty) \leqslant \lambda F(d_b(x, y), \varphi(x), \varphi(y)). \end{aligned} \tag{1}$$

Theorem 3.2 Let (X, d_b) be a complete b-metric space with coefficient $s \geqslant 1$ and $T : X \to X$ be $\alpha-$admissible mapping type S.
Suppose that the following conditions hold:
(1) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant s$,
(2) T is an $(F, \varphi, \alpha)_s-$contraction mapping,
(3) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant s$ and $x_n \to x$ then $\alpha(x_n, x) \geqslant s$ for all $n \in \mathbb{N}$.
Then
(i) $F_T \subseteq Z_\varphi$,

(ii) T is φ- Picard operator. Moreover, if $\alpha(x,y) \geqslant s$ for all $x,y \in F_T$, then T has a unique φ-fixed point.

Proof. (i) Assume that $\xi \in X$ is a fixed point of T such that $\alpha(\xi,\xi) \geqslant s$. Applying (1) with $x = y = \xi$, we obtain

$$F(0,\varphi(\xi),\varphi(\xi)) \leqslant s^{\varepsilon} F(0,\varphi(\xi),\varphi(\xi)) \leqslant \lambda F(0,\varphi(\xi),\varphi(\xi)) \quad (2)$$

then we get $F(0,\varphi(\xi),\varphi(\xi) \leqslant \lambda F(0,\varphi(\xi),\varphi(\xi))$, which is implies that

$$F(0,\varphi(\xi),\varphi(\xi)) = 0. \quad (3)$$

On the other hand, from (F1), we have

$$\varphi(\xi) \leqslant F(0,\varphi(\xi),\varphi(\xi)). \quad (4)$$

From (3) and (4), we obtain $\varphi(\xi) = 0$, which proves (i).
(ii) Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geqslant s$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. By condition (1), we get $\alpha(x_0,x_1) = \alpha(x_0,Tx_0) \geqslant s$ and we deduce that $\alpha(x_1,x_2) = \alpha(Tx_0,Tx_1) \geqslant s$. Continuing this process, we get $\alpha(x_n,x_{n+1}) \geqslant s$ for all $n \in \mathbb{N}\cup\{0\}$. If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = Tx_n$. Thus, x_n is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Using condition (1) as $\alpha(x_{n-1},x_n) \geqslant s$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} F(d_b(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) &\leqslant s^{\varepsilon} F(d_b(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)) \\ &\leqslant \lambda F(d_b(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)) \\ &\quad ... \\ &\leqslant \lambda^n F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)). \end{aligned} \quad (5)$$

Then, from (F1), we have

$$\max\{d_b(x_n,x_{n+1}),\varphi(x_n)\} \leqslant \lambda^n F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)), \quad (6)$$

which implies

$$d_b(x_n,x_{n+1}) \leqslant \lambda^n F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)). \quad (7)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $k \in \mathbb{N}$ such that $k > 0$. By using the triangle inequality, we get

$$\begin{aligned} d_b(x_n,x_{n+k}) &\leqslant s d_b(x_n,x_{n+1}) + s^2 d_b(x_{n+1},x_{n+2}) + ... + s^k d_b(x_{n+k-1},x_{n+k}) \\ &\leqslant s\lambda^n F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)) + s^2\lambda^{n+1} F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)) \\ &\quad + .. + s^k \lambda^{n+k-1} F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)) \\ &= \frac{1}{s^{n-1}}[s^n\lambda^n F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)) + s^{n+1}\lambda^{n+1} F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1)) \\ &\quad + .. + s^{n+k-1}\lambda^{n+k-1} F(d_b(x_0,x_1),\varphi(x_0),\varphi(x_1))]. \end{aligned}$$

Since $\lambda \in (0,1)$, the passing to limit in above the inequality, we obtain $d_b(x_n, x_{n+k}) \to 0$. This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d_b) is complete, then the sequence $\{x_n\}$ converges some $z \in X$ and

$$\lim_{n\to\infty} d_b(x_n, z) = 0. \tag{8}$$

Now, we shall prove that z is a $\varphi-$fixed point of T. Observe that from (6), we have

$$\lim_{n\to\infty} \varphi(x_n) = 0. \tag{9}$$

Since φ is lower semi continuous, from (8) and (9) we obtain

$$\varphi(z) = 0. \tag{10}$$

Using condition (2), we have

$$s^{\varepsilon} F(d_b(x_{n+1}, Tz), \varphi(x_{n+1}), \varphi(Tz)) \leqslant \lambda F(d_b(x_n, z), \varphi(x_n), \varphi(z)). \tag{11}$$

Letting $n \to \infty$ in (11), using (8), (9), (10), (F2) and the continuity of F, we have

$$s^{\epsilon} F(\lim_{n\to\infty} d_b(x_{n+1}, Tz), 0, \varphi(Tz)) \leqslant \lambda F(0,0,0) = 0,$$

which implies from condition (F1) that

$$\lim_{n\to\infty} d_b(x_{n+1}, Tz) = 0. \tag{12}$$

On the other hand, from the condition (iii) of definition b-metric space, we have

$$d_b(z, Tz) \leqslant s[d_b(z, x_{n+1}) + d_b(x_{n+1}, Tz)].$$

Taking the limit as $n \to \infty$ in above the inequality, using (8) and (12), we get $d_b(z, Tz) = 0$, that is $Tz = z$. Hence, z is a $\varphi-$fixed point of T. Now we show that z is the unique $\varphi-$fixed point of T. Assume that $w \in X$ is another $\varphi-$fixed point of T. From (1), we have

$$s^{\varepsilon} F(d_b(z, w), \varphi(z), \varphi(w)) \leqslant \lambda F(d_b(z, w), \varphi(z), \varphi(w))$$

and thus, $s^{\varepsilon} F(d_b(z, w), 0, 0) \leqslant \lambda F(d_b(z, w), 0, 0)$, which implies $d_b(z, w) = 0$, that is $z = w$. ■

Definition 3.3 Let (X, d_b) be a b-metric space with coefficient $s \geqslant 1$, $\alpha : X \times X \to [0, \infty)$

be a mapping, $\varphi : X \to [0, \infty)$ be lower semi continuous function, $F \in \mathcal{F}$, $\lambda \in (0,1)$ and $\varepsilon > 1$ be a constant. A mapping $T : X \to X$ is said to be an $(F, \varphi, \alpha)_s-$weak contraction mapping if,

$$\begin{aligned} x, y \in X \text{ with } \alpha(x,y) \geqslant s \Longrightarrow s^{\varepsilon} F(d_b(Tx, Ty), \varphi(Tx), \varphi(Ty)) &\leqslant \lambda F(d_b(x, y), \varphi(x), \varphi(y)) \\ &+ L(F(d_b(y, Tx), \varphi(y), \varphi(Tx)) - F(0, \varphi(y), \varphi(Tx))). \end{aligned} \tag{13}$$

Theorem 3.4 Let (X, d_b) be a complete b-metric space with coefficient $s \geqslant 1$ and $T : X \to X$ be $\alpha-$admissible mapping type S. Suppose that the following conditions

hold:
(1) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant s$,
(2) T is an $(F, \varphi, \alpha)_s-$weak contraction mapping,
(3) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant s$ and $x_n \to x$ then $\alpha(x_n, x) \geqslant s$ for all $n \in \mathbb{N}$.
Then
(i) $F_T \subseteq Z_\varphi$,
(ii) T is φ- weakly Picard operator. Moreover, if $\alpha(x, y) \geqslant s$ for all $x, y \in F_T$, then T has a unique φ-fixed point.

Proof. (i) Assume that $\xi \in X$ is a fixed point of T such that $\alpha(\xi, \xi) \geqslant s$. Applying (13) with $x = y = \xi$, we obtain

$$\begin{aligned} F(0, \varphi(\xi), \varphi(\xi)) &\leqslant s^\varepsilon F(0, \varphi(\xi), \varphi(\xi)) \\ &\leqslant kF(0, \varphi(\xi), \varphi(\xi)) + L(F(0, \varphi(\xi), \varphi(\xi)) - F(0, \varphi(\xi), \varphi(\xi))) \\ &= kF(0, \varphi(\xi), \varphi(\xi)). \end{aligned} \tag{14}$$

Then we have $F(0, \varphi(\xi), \varphi(\xi) \leqslant kF(0, \varphi(\xi), \varphi(\xi))$, which is implies that

$$F(0, \varphi(\xi), \varphi(\xi)) = 0. \tag{15}$$

On the other hand, from (F1), we have

$$\varphi(\xi) \leqslant F(0, \varphi(\xi), \varphi(\xi)). \tag{16}$$

From (15) and (16), we obtain $\varphi(\xi) = 0$, which proves (i).
(ii) Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geqslant s$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. By condition (1), we get $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geqslant s$ and we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geqslant s$. Continuing this process, we get $\alpha(x_n, x_{n+1}) \geqslant s$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = Tx_n$. Thus, x_n is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Using condition (1) as $\alpha(x_{n-1}, x_n) \geqslant s$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} &F(d_b(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) \\ &\leqslant s^\varepsilon F(d_b(Tx_{n-1}, Tx_n), \varphi(Tx_{n-1}), \varphi(Tx_n)) \\ &\leqslant \lambda F(d_b(Tx_{n-2}, Tx_{n-1}), \varphi(Tx_{n-2}), \varphi(Tx_{n-1})) \\ &\quad + L(F(0, \varphi(Tx_{n-1}), \varphi(Tx_{n-1})) - F(0, \varphi(Tx_{n-1}), \varphi(Tx_{n-1}))) \\ &= \lambda F(d_b(Tx_{n-2}, Tx_{n-1}), \varphi(Tx_{n-2}), \varphi(Tx_{n-1})). \end{aligned} \tag{17}$$

By induction, we have

$$\begin{aligned} F(d_b(x_n, x_{n+1}), \varphi(x_n), \varphi(x_{n+1})) &\leqslant s^\varepsilon F(d_b(Tx_{n-1}, Tx_n), \varphi(Tx_{n-1}), \varphi(Tx_n)) \\ &\leqslant \lambda^n F(d_b(x, Tx), \varphi(x), \varphi(Tx)). \end{aligned}$$

The rest of the proof follows using similar arguments to the proof of Theorem 3.2. ■

***Example* 3.5** Let $X = [1, \infty)$ and $d_b : X \times X \to [0, \infty)$ be defined by $d_b(x, y) = |x - y|$ for all $x, y \in X$. Let $F : [0, \infty)^3 \to [0, \infty)$ and $\varphi : X \to [0, \infty)$ be defined by $F(a, b, c) =$

$a+b+c$ and $\varphi(x)=\ln x$. Then, it is obvious that $F\in\mathcal{F}$ and $Z_\varphi=\{1\}$. Define $T:X\to X$ by $Tx=\dfrac{x+1}{2}$ and $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & x,y\in X,\\ 0 & otherwise.\end{cases}$$

Now, we will show that T is an $(F,\varphi,\alpha)_s$- contraction. If we take $s=1$, then we have

$$s^\varepsilon F(d_b(Tx,Ty),\varphi(Tx),\varphi(Ty))\leqslant\lambda F(d_b(x,y),\varphi(x),\varphi(y)).$$

Therefore, by Theorem 3.2, we conclude that T has a unique φ-fixed point which is $F_T\cap Z_\varphi=\{1\}$.

4. Applications

Application to integral equations:

In this section, firstly we shall apply Theorem 3.2 to show the existence of solution of Volterra integral equation. Then, we give some fixed point results in partial b-metric spaces, using the main results in the previous section. Now, we investigate the existence and uniqueness of solution of Volterra integral equation:

$$x(t)=v(t)+\mu\int_a^t K(t,p)x(p)dp,$$

where for all $t,p\in[a,c]$, $v:[a,c]\to\mathbb{R}$, $K:[a,c]\times[a,c]\to\mathbb{R}$ and μ is a real number.

Theorem 4.1 Consider the Volterra integral equation. Suppose that the following conditions are satisfied:
(i) $K:[a,c]\times[a,c]\to\mathbb{R}$ is continuous,
(ii) for all $t,p\in[a,c]$, $\varepsilon>1$ and $\lambda\in(0,1)$, we have

$$\sup_{a\leqslant p\leqslant c}\int_a^c|K(t,p)|\,dt\leqslant\frac{\lambda}{3^\varepsilon|\mu|}.$$

Then the Volterra integral equation has a unique solution.

Proof. Let $X=C[a,c]$ and let $T:X\to X$ be defined by

$$Tx(t)=v(t)+\mu\int_a^c K(t,p)x(p)dp$$

for all $x\in X$. Let $d_b(x,y)=\|x(p)-y(p)\|_X=\int_a^c|x(p)-y(p)|\,dp$ for all $x,y\in X$. We define $F:[0,\infty)^3\to[0,\infty)$ and $\varphi:X\to[0,\infty)$ by

$$F(k,l,m)=k+l+m \text{ for all } k,l,m\in[0,\infty)$$

and $\varphi(x)=0$ for all $x\in X$. Now, we define $\alpha:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}3 & x(p)\leqslant y(p) \text{ for all } p\in[a,c],\\ 0 & otherwise.\end{cases}$$

Assume that $x, y \in X$ such that $\alpha(x,y) \geqslant s = 3$, that is $x(p) \leqslant y(p)$ for all $p \in [a,c]$. Here, we will show that T is an $(F,\varphi,\alpha)_s-$ contraction mapping. Suppose that $x, y \in X$ and $t, p \in [a,c]$. Then, we have

$$\begin{aligned} s^{\varepsilon} d_b(Tx, Ty) &= 3^{\varepsilon} \|Tx(p) - Ty(p)\| \\ &= 3^{\varepsilon} \int_a^c |Tx(p) - T(p)|\, dp \\ &= 3^{\varepsilon} \int_a^c \left| \mu \int_a^c K(t,p)x(p)dp - \mu \int_a^c K(t,p)y(p)dp \right| dt \\ &= 3^{\varepsilon} \int_a^c \left| \mu \int_a^c K(t,p)[x(p) - y(p)]dp \right| dt \\ &\leqslant 3^{\varepsilon} |\mu| \sup_{a \leqslant s \leqslant b} \int_a^c |K(t,p)|\, dt \int_a^c |x(p) - y(p)|\, dp \\ &\leqslant 3^{\varepsilon} |\mu| \frac{\lambda}{3^{\varepsilon} |\mu|} d_b(x,y) \\ &= \lambda d_b(x,y) \end{aligned}$$

for all $x, y \in X$. It follows that

$$s^{\varepsilon} F(d_b(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leqslant \lambda F(d_b(x,y), \varphi(x), \varphi(y))$$

for all $x, y \in X$. Thus, T is an $(F,\varphi,\alpha)_s-$ contraction mapping. Thus all the conditions of Theorem 3.2 are satisfied. Then, T has a unique $\varphi-$fixed point in X. This implies that there exists a unique solution of the Volterra integral equation. ■

Application to partial b-metric spaces:

Now, let us recall some basic definitions on partial b-metric spaces.

Definition 4.2 [24] Let X be a nonempty set and and $s \geqslant 1$ be a given real number. A function $p_b : X \times X \to R^+$ is a partial b-metric if for all $x, y, z \in X$, the following conditions are satisfied:
(p_1) $x = y \Leftrightarrow p_b(x,x) = p_b(x,y) = p_b(y,y)$,
(p_2) $p_b(x,x) \leqslant p_b(x,y)$,
(p_3) $p_b(x,y) = p_b(y,x)$,
(p_4) $p_b(x,y) \leqslant s(p_b(x,z) + p_b(z,y) - p_b(z,z)) + (\frac{1-s}{2})(p_b(x,x) + p_b(y,y))$.

Definition 4.3 [19] A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is said to be:
(i) p_b-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x, x_n) = p_b(x,x)$.
(ii) A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is called a Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists and is finite.
(iii) A partial b-metric space (X, p_b) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in$X such that, $\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n,m\to\infty} p_b(x_n, x) = p_b(x,x)$.

Proposition 4.4 [19] Every partial b-metric p_b defines a b-metric d_{p_b}, where

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$$

for all $x, y \in X$.

Lemma 4.5 [19] Let (X, p_b)be a partial b-metric space. Then,
(i) A sequence $\{x_n\}_{n\in N}$ in a partial b-metric space (X, p_b) is a Cauchy sequence if and only if it is a Cauchy sequence in the b-metric space (X, d_b).
(ii) A partial b-metric space (X, p_b) is complete if and only if the b-metric space (X, d_b) is complete.
(iii) Given a sequence $\{x_n\}_{n\in N}$ in a partial b-metric space (X, p_b) and $x \in X$, we have

$$\lim_{n\to\infty} p_b(x, x_n) = 0 \Leftrightarrow p_b(x, x) = \lim_{n\to\infty} p_b(x, x_n) = 0 = \lim_{n,m\to\infty} p_b(x_n, x_m).$$

Now, we give our some results in partial b-metric spaces.

Corollary 4.6 Let (X, p_b) be a complete partial b-metric space with coefficient $s \geqslant 1$. Let $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be a given mapping such that $s^{\varepsilon} p_b(Tx, Ty) \leqslant \lambda p_b(x, y)$ for all $x, y \in X$ and for some constant $\lambda \in (0, 1)$ and $\varepsilon > 1$. Then T has a unique fixed point.

Proof. Consider the metric $d_{p_b} = 2p_b(x, y) - p_b(x, x) - p_b(y, y)$ on X and the function $\varphi : X \to [0, \infty)$ defined by $\varphi(x) = p_b(x, x)$. Applying Theorem 3.2 with $F(a, b, c) = a + b + c$, we obtain the desired result. ■

Corollary 4.7 Let (X, p_b) be a complete partial b-metric space with coefficient $s \geqslant 1$. Let $\alpha : X \times X \to [0, \infty)$, $T : X \to X$ be given mappings such that for all $x, y \in X$ and for some constant $\lambda \in (0, 1)$ and $\varepsilon > 1$,

$$s^{\varepsilon} p_b(Tx, Ty) \leqslant \lambda p_b(x, y) + L(p_b(Ty, Tx) - \frac{p_b(y, y) + p_b(Tx, Tx)}{2}).$$

Then T has a unique fixed point.

Proof. Consider the metric $d_{p_b} = 2p_b(x, y) - p_b(x, x) - p_b(y, y)$ on X and the function $\varphi : X \to [0, \infty)$ defined by $\varphi(x) = p_b(x, x)$. Applying Theorem 3.4 with $F(a, b, c) = a + b + c$, we obtain the desired result. ■

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