Journal of
Linear and Topological Algebra
Vol. 07*, No.* 04*,* 2018*,* 307- 316

On the solving matrix equations by using the spectral representation

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Received 3 June 2018; Revised 13 November 2018, Accepted 10 December 2018.

Communicated by Mohammad Sadegh Asgari

Abstract. The purpose of this paper is to solve two types of Lyapunov equations and quadratic matrix equations by using the spectral representation. We focus on solving Lyapunov equations $AX+XA^* = C$ and $AX+XA^T = -bb^T$ for $A, X \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^{n\times s}$ with $s < n$, which X is unknown matrix. Also, we suggest the new method for solving quadratic matrix equations $AX^2 + BX + C = 0$, where $A, B, C, X \in \mathbb{C}^{n\times n}$ and X is unknown matrix with similar method.

Keywords: Lyapanov equation, square matrix equations, spectral representation.

2010 AMS Subject Classification: 15A24, 15A18, 15A42.

1. Introduction

Consider two square matrices $A, C \in \mathbb{C}^{n\times n}$. The problem is to find a square matrix $X \in \mathbb{C}^{n\times n}$ in

$$AX + XA^* = C, \tag{1}$$

which is called the Lyapunov equation. Many different algorithms are suitable for different situations depending on the properties of matrices A and C. For dense A, the Bartels-Stewart algorithm is the most widely used algorithm. It employs Schur decomposition and then builds simple linear equations which can be solved sequentially [2]. For sparse and large-scale A, if C is low-rank, Krylov-type methods may be more efficient [9]. This Lyapunov matrix equation has form $AX + XA^T = -bb^T$, where $A, X \in \mathbb{C}^{n\times n}$

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

and $b \in \mathbb{C}^{n\times s}$ with $s < n$. There is no need to have rank of matrix in this paper which shows our methodology.

In Lyapunov equation (1), suppose that the two matrices A and $-A^\star$ have not any common eigenvalues. It is well known that the Lyapunov equation (1) has an unique solution $X \in \mathbb{C}^{n\times n}$ if and only if $\lambda_i \neq \lambda_j$ for all $i, j = 1, \cdots, n$. In particular, if A is strictly stable; that is, $\lambda_i < 1$ for all $i = 1, \cdots, n$, then (1) has an unique solution, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of matrix A. Lyapunov matrix equation has many applications in control and system theory especially in controllability, and control filtering with singular measurement noise [3] and optimal control theory [4], model reduction of linear time-invariant systems [1, 6]. Another application is communicating system theory and power systems. This equation has an unique solution if and only if $\lambda_i + \lambda_j \neq 0$ for all i and j, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A [5, 10].

In the second part of this paper, we present a new method for solving quadratic matrix equations. Nazari et al. [7] solved the square root of matrix triangular equations for order $n = 3$ and Sambasiva Rao et al. [8] presented an extension of Nazari et al's method. Here, we solve a general quadratic matrix equations by spectral representation. Some different examples are presented.

2. Spectral representation of a matrix

At the first, we explain the spectral representation of a matrix and some of the applications. Let $A \in \mathbb{C}^{n\times n}$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be its eigenvalues. Assume that vectors $\mu_1, \mu_2, \cdots, \mu_n$ be its corresponding eigenvectors, respectively. Consequently, A is a diagonalizable matrix and is full rank. It is clear that $A\mu_i = \lambda_i\mu_i$ for every $i = 1, 2, \cdots, n$. Now, we multiply from the right by $\mu_i^\star$ that gives equation $A\mu_i\mu_i^\star = \lambda_i\mu_i\mu_i^\star$ for $i = 1, 2, \cdots, n$. Then by adding all n obtained above equations, we achieve

$$A\left(\mu_1\mu_1^\star + \mu_2\mu_2^\star + \cdots + \mu_n\mu_n^\star\right) = \lambda_1\mu_1\mu_1^\star + \cdots + \lambda_n\mu_n\mu_n^\star.$$

Assume that $S = \left(\mu_1\mu_1^\star + \cdots + \mu_n\mu_n^\star\right)^{-1}$ and $W_i = \mu_i\mu_i^\star S$ for $i = 1, 2, \cdots, n$, where W_i are rank-one matrices and S is positive definite matrix. It is easy to show that the matrix A can be written as linear combination of its eigenvalues and W_i is called spectral representation of A; that is, $A = \lambda_1W_1 + \lambda_2W_2 + \cdots + \lambda_nW_n$.

Theorem 2.1 The matrices $W_1, ..., W_n$ satisfy in the following conditions: (I) S is symmetric positive definite matrix,
(II) $\operatorname{rank}(W_i) = 1$ for $i = 1, 2, \cdots, n$,
(III) $W_1 + W_2 + \cdots + W_n = I_n$, I_n denotes the $n \times n$ identity matrix,
(IV) $W_iW_j = 0_n$ for $i, j = 1, 2, \cdots, n$ with $i \neq j$, 0_n denotes the $n \times n$ zero matrix,
(V) $W_i^r = W_i$ for $i = 1, 2, \cdots, n$ with $r > 0$,
(VI) $P(A)W_i = P(\lambda_i)W_i$ for every polynomial,
(VII) $W_iA = AW_i = \lambda_iW_i$ for $i = 1, 2, \cdots, n$,
(VIII) $1 \leqslant \|W_i\|_2 \leqslant \|S\|_2$ for $i = 1, 2, \cdots, n$,
(IX) If $\sigma(A)$ denotes the spectrum of the matrix A, then $\sigma(W_i) = \{1, 0, \cdots, 0\}$,
(X) For any complex numbers α, β it hold that

$$\sigma(\alpha W_i + \beta W_j) = \alpha\sigma(W_i) + \beta\sigma(W_j) = \begin{cases} \{\alpha, \beta, 0, \cdots, 0\}, & i \neq j \\ \\ \{\alpha + \beta, 0, \cdots, 0\}, & i = j \end{cases}$$

(XI) Consider a square matrix $B (\neq A)$. There exist n diagonal matrix $R_1, R_2, \cdots, R_n$ can be computed such that $B = W_1R_1 + \cdots + W_nR_n$.

Proof. We just only prove the last part and the other parts are trivial. The significant property $W_iW_j = 0_n$ for $i, j = 1, 2, \cdots, n$ with $i \neq j$ concludes $W_iB = W_iR_i$ for $i = 1, 2, \cdots, n$. If $W^c_{i,j}$ denotes the j-th column of W_i for every $i, j = 1, 2, \cdots, n$ and

$$R_i = \operatorname{diag}(r^i_{1,1}, r^i_{2,2}, \cdots, r^i_{n,n}), \tag{2}$$

then

$$W_iB = W_iR_i = [W^c_{1,i}W^c_{2,i} \cdots W^c_{n,i}][\operatorname{diag}(r^i_{1,1}, r^i_{2,2}, \cdots, r^i_{n,n})] = [r^i_{1,1}W_{1,i}r^i_{2,2}W_{2,i} \cdots r^i_{n,n}W_{n,i}]$$

and consequently,

$$r^i_{j,j} = \frac{(W_iB)_{1,j}}{(W_{j,i})_1} \tag{3}$$

for $i, j = 1, 2, \cdots, n$, where $(W_iB)_{1,j}$ is the element of the matrix W_iB in row (arbitrary chosen) 1 and column j. Moreover, $(W_{j,i})_1$ is the first (due to first row) element of the columns $W_{j,i}$ by (3). ■

We show that by helping spectral representation how we can compute $A^{-1}_{n\times n}$ and $A^r_{n\times n}$. Let A be an invertible matrix. Thus, $\lambda_1, \cdots, \lambda_n$ are non-zero and it is easy to see that

$$A^{-1} = \frac{1}{\lambda_1}W_1 + \frac{1}{\lambda_2}W_2 + \cdots + \frac{1}{\lambda_n}W_n.$$

Moreover, for every $r \in \mathbb{R}$, the matrix A^r satisfies $A^r = \lambda^r_1W_1 + \lambda^r_2W_2 + \cdots + \lambda^r_nW_n$.

In the first part of this paper, we explain a method for solving two cases of Lyapunov equation $AX + XA^\star = C$ and $AX + XA^T = -bb^T$ that discussed above.

3. Solving Lyapunov equation

In this section, we assume that A is invertible and we present a new method of solving

$$AX + XA^\star = C. \tag{4}$$

We have

$$A = \lambda_1W_1 + \lambda_2W_2 + \cdots + \lambda_nW_n, \tag{5}$$

$$C = W_1R_1 + W_2R_2 + \cdots + W_nR_n, \tag{6}$$

$$X = W_1D_1 + W_2D_2 + \cdots + W_nD_n, \tag{7}$$

where R_i for $i = 1, 2, \cdots, n$ is denoted in (2) and (3) and $D_1, D_2, \cdots, D_n$ that are unknown matrices that must be determined for equation $AX+XA^\star = C$. By substitution (5)-(7) in Lyapanov equation (4), we have
$(\lambda_1W_1+\cdots+\lambda_nW_n)(W_1D_1+\cdots+W_nD_n)+(W_1D_1+\cdots+W_nD_n)A^\star = W_1R_1+\cdots+W_nR_n$
such that
$(W_1\lambda_1D_1 + \cdots + W_n\lambda_nD_n) + (W_1D_1A^\star + \cdots + W_nD_nA^\star) = W_1R_1 + W_2R_2 + \cdots + W_nR_n$.
By classifying above equation, we have

$$W_1(\lambda_1D_1 + D_1A^\star) + \cdots + W_n(\lambda_nD_n + D_nA^\star) = W_1R_1 + \cdots + W_nR_n.$$

Thus, $\lambda_iD_i + D_iA^\star = R_i$ for $i = 1, 2, \cdots, n$ is achieved. Therefore, the matrices $D_1, D_2, \cdots, D_n$ are computed as following $D_i = R_i(\lambda_iI_n + A^\star)^{-1}$ for $i = 1, 2, \cdots, n$.

***Example* 3.1** For solving Lyapunov equation $AX + XA^T = -bb^T$, consider the 5×5 full rank following matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 2 \\ -2 & 1 & -3 & 7 & -5 \\ 3 & -3 & 1 & 4 & -5 \\ 0 & 1 & -2 & 3 & -4 \\ -2 & 3 & -1 & 5 & 1 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -5 \\ 4 \end{bmatrix}.$$

We use software Matlab and compute the solution of Lyapanove equation $AX + XA^T = -bb^T$. At the first, we obtain the eigenvectors of matrix A, as following matrix

$$\mu_i = \begin{bmatrix} 0.3066 & -0.2764 + 0.6004i & -0.2764 - 0.6004i & -0.3730 + 0.1079i & -0.3730 - 0.1079i \\ 0.8028 & -0.1127 - 0.1043i & -0.1127 + 0.1043i & 0.2723 - 0.3409i & 0.2723 + 0.3409i \\ 0.4373 & 0.6647 & 0.6647 & 0.5791 & 0.5791 \\ -0.1658 & 0.0188 + 0.0573i & 0.0188 - 0.0573i & 0.2322 - 0.0981i & 0.2322 + 0.0981i \\ -0.2069 & -0.3042 - 0.0392i & -0.3042 + 0.0392i & -0.2727 - 0.4308i & -0.2727 + 0.4308i \end{bmatrix}$$

and

$$\sigma(A) = \{\lambda_1 = -1.5558, \lambda_2 = 2.6624 + 3.8198i, \lambda_3 = 2.6624 - 3.8199i, \lambda_4 = 1.6155 + 5.3682i, \lambda_5 = 1.6155 - 5.3682i\},$$

that associated to the eigenvectors of A. So,

$$S = (\mu\mu^\star)^{-1} = \begin{bmatrix} 1.1425 & -0.2970 & 0.7204 & 0.4961 & 0.5598 \\ -0.2970 & 1.6762 & -1.5053 & 1.0178 & -1.6513 \\ 0.7204 & -1.5053 & 3.1215 & -2.5528 & 3.2401 \\ 0.4961 & 1.0178 & -2.5528 & 9.5772 & -2.6400 \\ 0.5598 & -1.6513 & 3.2401 & -2.6400 & 4.7439 \end{bmatrix}.$$

Thus, matrices R_i and D_i for $i = 1, 2, \cdots, 5$ as they are shown above, are presented in the following:

$$\begin{aligned}
R_1 &= \mathrm{diag}\left([-17.0684\ 10.1633\ -89.6434\ -16.4277\ 55.7326]\right), \\
R_2 &= \mathrm{diag}\Big(\Big[1.1302 - 16.4245i\ \ -33.3898 - 17.8968i\ \ -30.8001 - 13.7871i \\
&\qquad -38.3588 + 26.2608i\ \ -62.5822 - 23.5840i\Big]\Big), \\
R_3 &= \mathrm{diag}\Big(\Big[1.1302 + 16.4245i\ \ -33.3898 + 17.8968i\ \ -30.8001 + 13.7871i \\
&\qquad -38.3588 - 26.2608i\ \ -62.5822 + 23.5840i\Big]\Big), \\
R_4 &= \mathrm{diag}\Big(\Big[-96.5124 - 3.0525i\ \ -65.9130 + 29.5523i\ \ -53.3405 - 34.0764i \\
&\qquad -38.7462 + 21.3551i\ \ -33.9249 - 22.8532i\Big]\Big), \\
R_5 &= \mathrm{diag}\Big(\Big[-96.5124 + 3.0525i\ \ -65.9130 - 29.5523i\ \ -53.3405 + 34.0764i \\
&\qquad -38.7462 - 21.3551i\ \ -33.9249 + 22.8532i\Big]\Big)
\end{aligned}$$

and

$$D_1 = \begin{bmatrix} -2.2144 & 0.1681 & 4.2221 & -0.3877 & -2.2090 \\ -2.6138 & -2.2132 & -0.9100 & 0.0932 & -0.0646 \\ -19.6751 & -1.1401 & -8.3363 & -8.7737 & 0.7154 \\ -1.7248 & -3.5827 & 2.7825 & 2.3113 & 2.6545 \\ 2.8059 & 13.3522 & 16.5956 & 7.3191 & -2.3172 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -1.1071 - 2.2871i & -0.9303 - 0.1774i & 0.4354 + 0.7158i & 0.0506 - 0.0892i & -0.2935 - 0.4802i \\ 1.1943 + 1.1459i & -4.9332 + 3.4206i & -1.0338 + 2.3994i & 0.8207 - 1.3558i & 1.7427 - 1.4840i \\ -1.2733 + 1.2896i & -1.6759 + 1.0771i & -6.1326 + 0.8245i & -1.4314 + 0.5095i & 0.1345 - 0.7887i \\ -1.7745 - 1.6297i & -6.8371 - 11.1354i & -5.5727 - 9.1883i & -2.6287 + 3.2181i & 3.8724 - 2.7097i \\ 3.6023 - 2.3514i & -6.7103 - 0.7777i & -9.1271 + 3.8297i & -5.2301 + 3.2357i & -7.6078 - 2.5234i \end{bmatrix},$$

$$D_3 = \begin{bmatrix} -1.1071 + 2.2871i & -0.9303 + 0.1774i & 0.4354 - 0.7158i & 0.0506 + 0.0892i & -0.2935 + 0.4802i \\ 1.1943 - 1.1459i & -4.9333 - 3.4206i & -1.0338 - 2.3994i & 0.8207 + 1.3558i & 1.7427 + 1.4840i \\ -1.2733 - 1.2896i & -1.6759 - 1.077i & -6.1326 - 0.8245i & -1.4314 - 0.5095i & 0.1345 + 0.7887i \\ -1.7745 + 1.6297i & -6.8371 + 11.1354i & -5.5727 + 9.1883i & -2.6287 - 3.2181i & 3.8724 + 2.7097i \\ 3.6023 + 2.3514i & -6.7103 + 0.7777i & -9.1271 - 3.8297i & -5.2301 - 3.2357i & -7.6078 + 2.5234i \end{bmatrix},$$

$$D_4 = \begin{bmatrix} -11.8361 + 10.4423i & 1.5012 + 4.9073i & 2.3249 - 4.9892i & -0.4762 + 0.3532i & -1.8107 + 3.4898i \\ 2.2610 - 2.6180i & 0.0674 + 8.4109i & 2.5443 + 2.1430i & -2.3504 - 1.8132i & -1.5340 - 4.2153i \\ -0.9908 + 2.6725i & -3.2591 + 3.0570i & -12.0836 - 4.3282i & -2.6638 + 2.4017i & -1.0698 - 2.2036i \\ -0.2100 - 0.4351i & -9.7579 - 7.0743i & -9.6808 - 4.8815i & -4.1294 + 3.5173i & 2.6023 - 6.1498i \\ 2.4845 - 2.6157i & -5.5814 + 1.5571i & -5.5854 + 5.6476i & -3.1861 + 3.2901i & -6.6945 - 2.9140i \end{bmatrix},$$

$$D_5 = \begin{bmatrix} -11.8361 - 10.4423i & 1.5012 - 4.9073i & 2.3249 + 4.9892i & -0.4762 - 0.3523i & -1.8107 - 3.4898i \\ 2.2610 + 2.6180i & 0.0674 - 8.4109i & 2.5443 - 2.1430i & -2.3504 + 1.8132i & -1.5340 + 4.2153i \\ -0.9908 - 2.6725i & -3.2591 - 3.0570i & -12.0836 - 4.3282i & -2.6638 - 2.4017i & -1.0698 + 2.2036i \\ -0.2100 + 0.4351i & -9.7579 + 7.0743i & -9.6808 - 4.8815i & -4.1294 - 3.5173i & 2.6023 + 6.1498i \\ 2.4845 + 2.6157i & -5.5814 - 1.5571i & -5.5854 - 5.6476i & -3.1861 - 3.2901i & -6.6945 + 2.9140i \end{bmatrix}.$$

So the unknown matrix X is achieved from the following form:

$$X = W_1D_1 + W_2D_2 + W_3D_3 + W_4D_4 + W_5D_5 = \begin{bmatrix} -5.0309 & -1.7875 & -1.3281 & 0.8757 & 0.3085 \\ -1.7840 & -16.8314 & -25.4596 & -12.0316 & -3.8204 \\ -1.3271 & -25.4591 & -38.4578 & -12.9268 & -2.6543 \\ 0.8752 & -12.0329 & -12.9285 & -4.7302 & 3.0333 \\ 0.3090 & -3.8229 & -2.6544 & 3.0329 & -13.7378 \end{bmatrix}.$$

***Example* 3.2** For solving Lyapunov equation $AX + XA^{\star} = C$, consider the following matrixs 3×3, where $A, A^{*}, C \in \mathbb{C}^{n \times n}$ and A is a full rank matrix.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -5 & 2 \\ 3 & -4 & -1 \end{bmatrix}, \quad A^{\star} = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -5 & -4 \\ 3 & 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 27 & 22 & 32 \\ 21 & 52 & 50 \\ 22 & 78 & 31 \end{bmatrix},$$

At the first, we compute the eigenvalues and associated eigenvectors of matrix A as following:

$$\mu_i = \begin{bmatrix} 0.3341 & -0.7061 & -0.7061 \\ -0.4131 & -0.6307+0.0739i & -0.6307-0.0739i \\ -0.8472 & -0.0831-0.3023i & -0.0831+0.3023i \end{bmatrix}$$

is the matrix eigenvectors of A and the associated eigenvalues are:

$$\sigma(A) = \{-4.1337, -0.4331+1.4938i, -0.4331-1.4938i\}.$$

So,

$$S = (\mu\mu^{\star})^{-1} = \begin{bmatrix} 4.6175 & -4.8144 & 2.9964 \\ -4.8144 & 6.2806 & -3.6898 \\ 2.9964 & -3.6898 & 3.2919 \end{bmatrix}.$$

Thus, matrices R_i and D_i for $i = 1, 2, 3$ as they are shown above, are presented in the following:

$$R_1 = \mathrm{diag}\left([-1.6089\ 50.7883\ 114.8284]\right),$$
$$R_2 = \mathrm{diag}\left([25.0852+6.0470i\ -24.6205-22.0309i\ 1.6998+23.4335i]\right),$$
$$R_3 = \mathrm{diag}\left([25.0852-6.0470i\ -24.6205+22.0309i\ 1.6998-23.4335i]\right)$$

and

$$D_1 = \begin{bmatrix} 0.4627 & 0.2237 & 0.0961 \\ 5.9250 & -1.8859 & 4.9318 \\ -14.0779 & -10.9895 & -22.0317 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} -12.4816-31.4939i & -5.2289-25.7143i & -8.4475-2.9609i \\ 20.7342-32.6695i & 23.6428-19.9112i & -4.4235-17.4249i \\ 21.9898-8.9857i & 19.4954-6.1768i & 4.3673-13.3693i \end{bmatrix},$$

$$D_3 = \begin{bmatrix} -12.4816+31.4939i & -5.2289+25.7143i & -8.4475+2.9609i \\ 20.7342+32.6695i & 23.6428+19.9112i & -4.4235+17.4249i \\ 21.9898+8.9857i & 19.4954+6.1768i & 4.3673+13.3693i \end{bmatrix}.$$

So the unknown matrix X is achieved from the following form:

$$X = W_1D_1 + W_2D_2 + W_3D_3 = \begin{bmatrix} 0.9949 & -4.0042 & 2.9999 \\ 1.9971 & -7.0032 & 1.0015 \\ 3.9985 & -6.0016 & 5.005 \end{bmatrix}.$$

4. Solving quadratic matrix equations

A manual approach in calculating the root of square matrix is studied in [10]. At first, consider $X^2 = A$ for $A, X \in \mathbb{C}^{n\times n}$, which X is unknown matrix. We compute square root of $n \times n$ as a prescribed matrix. $\sqrt{A}$ can be obtained as $\sqrt{\lambda_1}W_1 + \cdots + \sqrt{\lambda_n}W_n$. Therefore, $X = \sqrt{A} = \sqrt{\lambda_1}W_1 + \cdots + \sqrt{\lambda_n}W_n$. So, we can compute $(X + A)^2 = C$. As we know $C = \lambda_1W_1 + \cdots + \lambda_nW_n$, $(X + A) = \sqrt{\lambda_1}W_1 + \cdots + \sqrt{\lambda_n}W_n$, $X = \left(\sqrt{\lambda_1}W_1 + \cdots + \sqrt{\lambda_n}W_n\right) - A$.

Now, consider $X^2 + AX = C$ for $A, X \in \mathbb{C}^{n\times n}$, which X is unknown matrix. So, we can compute square root of matrix A. Consequently

$$\left(X + \frac{A}{2}\right)^2 = C + \frac{A^2}{4}, \qquad Y = X + \frac{A}{2}, \qquad H = C + \frac{A^2}{4}.$$

We have $H = \lambda_1 W_1 + \lambda_2 W_2 + \cdots + \lambda_n W_n$, where scalars $\lambda_1, \cdots, \lambda_n$ are eigenvalues of matrix H and $y = \sqrt{\lambda_1}W_1 + \cdots + \sqrt{\lambda_n}W_n$, as a result $X = Y - \frac{A}{2}$. Finally, for solving $AX^2 + BX = C$, where $A, B, C, X \in \mathbb{C}^{n\times n}$ and X is unknown matrix, we have $AX^2 + BX = C$. By multiplying it by A^{-1} from the left, we conclude $A^{-1}AX^2 + A^{-1}BX = A^{-1}C$ and $X^2 + A^{-1}BX = A^{-1}C$, that A^{-1} can be obtained by using spectral representation as the following

$$A^{-1} = \frac{1}{\lambda_1}W_1 + \frac{1}{\lambda_2}W_2 + \cdots + \frac{1}{\lambda_n}W_n.$$

Let $A^{-1}B = D$, $A^{-1}C = H$. Then

$$X^2 + DX = H, \left(X + \frac{D}{2}\right)^2 = H + \frac{D^2}{4}, Y = X + \frac{D}{2}, Z = H + \frac{D^2}{4}, Y^2 = Z. \tag{8}$$

Thus, we have $Z = \lambda_1 W_1 + \cdots + \lambda_n W_n$ that scalars $\lambda_1, \cdots, \lambda_n$ are eigenvalues of matrix Z. Also, $Y = \sqrt{\lambda_1}W_1 + \sqrt{\lambda_2}W_2 + \cdots + \sqrt{\lambda_n}W_n$, as a result $X + \frac{D}{2} = Y$, $X = Y - \frac{D}{2}$.

***Example* 4.1** For solving $X^2 = A$, consider the following 4×4 matrix

$$A = \begin{bmatrix} 0 & 10 & 10 & 0 \\ -20 & 54 & -14 & -16 \\ -4 & 28 & 0 & -59 \\ -2 & 5 & 9 & 29 \end{bmatrix}.$$

At first, we obtain the eigenvectors of matrix A as following matrix

$$\mu_i = \begin{bmatrix} -0.9169 & 0.2315 & 0.2292 - 0.2839i & 0.2292 + 0.2839i \\ -0.3074 & 0.8981 & 0.4187 - 0.0019i & 0.4187 + 0.0019i \\ 0.2394 & 0.0590 & 0.7672 & 0.7672 \\ -0.0867 & 0.3692 & -0.0828 - 0.3101i & -0.0828 + 0.3101i \end{bmatrix},$$

and its associated eigenvalues are
$\sigma(A) = \Big\{\lambda_1 = 0.7417, \lambda_2 = 41.3485, \lambda_3 = 20.4549 + 25.2592i, \lambda_4 = 20.4549 - 25.2592i\Big\}$.
So,

$$S = \begin{bmatrix} 1.4099 & -0.6843 & 0.1154 & -0.6126 \\ -0.6843 & 2.0900 & -1.1267 & -1.5458 \\ 0.1154 & -1.1267 & 1.5002 & 1.3746 \\ -0.6126 & -1.5458 & 1.3746 & 5.1663 \end{bmatrix}.$$

Consequently, matrices W_i for $i = 1, 2, 3, 4$ as they are shown above, are presented in the following form:

$$W_1 = \begin{bmatrix} 0.9182 & 0.1382 & -0.4406 & -0.8416 \\ 0.3079 & 0.0463 & -0.1477 & -0.2822 \\ -0.2398 & -0.0361 & 0.1150 & 0.2198 \\ 0.0869 & 0.0131 & -0.0417 & -0.0796 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.1175 & 0.2503 & -0.0901 & 0.1061 \\ -0.4559 & 0.9713 & -0.3495 & 0.4116 \\ -0.0300 & -0.0638 & -0.0230 & 0.0270 \\ -0.1874 & 0.03993 & -0.1437 & 0.1692 \end{bmatrix},$$

$$W_3 = \begin{bmatrix} 0.0996 - 0.0020i & -0.1943 - 0.1483i & 0.2653 - 0.0633i & 0.3677 + 0.3723i \\ 0.0740 + 0.0872i & -0.0088 - 0.2804i & 0.2486 + 0.1902i & -0.0647 + 0.5970i \\ 0.1349 + 0.1604i & -0.0139 - 0.5138i & 0.4540 + 0.3505i & -0.1234 + 1.0933i \\ 0.0503 - 0.0718i & -0.2062 + 0.0610i & 0.0927 - 0.2213i & 0.4552 - 0.0681i \end{bmatrix},$$

$$W_4 = \begin{bmatrix} 0.0996 + 0.0020i & -0.1943 + 0.1483i & 0.2653 + 0.0633i & 0.3677 - 0.3723i \\ 0.0740 - 0.0827i & -0.0088 + 0.2804i & 0.2486 - 0.1902i & -0.0647 - 0.5970i \\ 0.1349 - 0.1604i & -0.0139 + 0.5138i & 0.4540 - 0.3505i & -0.1234 - 1.0933i \\ 0.0503 + 0.0718i & -0.2062 - 0.0610i & 0.0927 + 0.2213i & 0.4552 + 0.0681i \end{bmatrix}.$$

Then the unknown matrix X computes as the following

$$X = \sqrt{A} = \sqrt{\lambda_1} W_1 + \sqrt{\lambda_2} W_2 + \sqrt{\lambda_3} W_3 + \sqrt{\lambda_4} W_4.$$

Hence,

$$X = \begin{bmatrix} 1.0705 & 0.4576 & 2.0826 & 1.9146 \\ -2.3328 & 7.5712 & -0.7497 & -1.1931 \\ 0.2017 & 2.7586 & 2.9028 & -6.2737 \\ -0.2604 & 0.1572 & 1.0804 & 6.0386 \end{bmatrix}.$$

***Example* 4.2** For solving $X^2 + AX = C$, $A_{4\times4}$ and $C_{4\times4}$ as they are given. Then we compute X as it is described above.

$$A = \begin{bmatrix} -4 & 3 & -4 & 0 \\ -1 & 3 & 8 & -2 \\ 2 & -5 & 0 & -1 \\ -4 & 3 & 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 21 & 0 & -27 & 7 \\ -9 & 10 & 16 & -6 \\ -26 & -2 & 31 & -13 \\ 14 & -15 & 1 & 47 \end{bmatrix}.$$

The matrix eigenvectors and its associated eigenvalues are computed in the following:

$$\mu_i = \begin{bmatrix} -0.3015 & -0.1921 + 0.1007i & -0.1921 - 0.1007i & -0.5095 \\ 0.2736 & 0.8457 & 0.8457 & 0.2655 \\ 0.3732 & 0.1570 + 0.1607i & 0.1570 - 0.1607i & 0.5201 \\ -0.8336 & 0.4310 - 0.0396i & 0.4310 + 0.0396i & 0.6320 \end{bmatrix}$$

and

$$\sigma(H) = \{\lambda_1 = 57.4109, \lambda_2 = 0.6322 + 4.2275i, \lambda_3 = 0.6322 - 4.2275i, \lambda_4 = 29.3247\}.$$

Thus,

$$S = (\mu\mu^\star)^{-1} = \begin{bmatrix} 9.9279 & 1.0381 & 6.7054 & 0.5379 \\ 1.0381 & 1.2468 & -0.3104 & -0.3654 \\ 6.7054 & -0.3106 & 7.4384 & 0.5461 \\ 0.5379 & -0.3654 & 0.5461 & 0.8843 \end{bmatrix}.$$

Consequently, matrices W_i for $i = 1, 2, 3, 4$ as they are shown above, are presented in the following form:

$$W_1 = \begin{bmatrix} 0.1974 & -0.0654 & -0.0647 & 0.2398 \\ -0.1792 & 0.0594 & 0.0587 & -0.2177 \\ -0.2444 & 0.0810 & 0.0801 & -0.2969 \\ 0.5459 & -0.1808 & -0.1788 & 0.6632 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.1579 + 0.4207i & -0.1177 + 0.0786i & 0.2144 + 0.3403i & 0.0003 + 0.0260i \\ 0.2164 - 1.7388i & 0.5487 - 0.0585i & -0.1243 - 1.5637i & 0.0462 - 0.0904i \\ 0.3706 - 0.2817i & 0.1130 + 0.0934i & 0.2741 - 0.3140i & 0.0258 - 0.0080i \\ 0.0289 - 0.8964i & 0.2769 - 0.0555i & -0.1365 - 0.7912i & 0.0193 - 0.0483i \end{bmatrix},$$

$$W_3 = \begin{bmatrix} 0.1579 - 0.4207i & -0.1177 - 0.0786i & 0.2144 - 0.3403i & 0.0003 - 0.0260i \\ 0.2164 + 1.7388i & 0.5487 + 0.0585i & -0.1243 + 1.5637i & 0.0462 + 0.0904i \\ 0.3706 + 0.2817i & 0.1130 - 0.0934i & 0.2741 + 0.3140i & 0.0258 + 0.0080i \\ 0.0289 + 0.8964i & 0.2769 + 0.0555i & -0.1365 + 0.7912i & 0.0193 + 0.0483i \end{bmatrix},$$

$$W_4 = \begin{bmatrix} 0.4867 & 0.3007 & -0.3642 & -0.2404 \\ -0.2537 & -0.1567 & 0.1898 & 0.1253 \\ -0.4968 & -0.3069 & 0.3718 & 0.2454 \\ -0.6038 & -0.3730 & 0.4518 & 0.2982 \end{bmatrix}.$$

Then the unknown matrix X is computed as the following:

$$X = \begin{bmatrix} 5.4909 & -0.9479 & -0.7092 & 0.4460 \\ 3.1398 & -0.0223 & 1.3038 & 0.4179 \\ -3.6208 & 1.5531 & 4.3259 & -0.3186 \\ 5.3766 & -3.8728 & 1.7995 & 8.3304 \end{bmatrix}.$$

***Example* 4.3** For solving $AX^2 + BX = C$ with the following matrix $A_{3\times3}, B_{3\times3}, C_{3\times3}$, we compute X.

$$A = \begin{bmatrix} -4 & 2 & -1 \\ 3 & 7 & -3 \\ 4 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 3 \\ 4 & -2 & 3 \\ 5 & -1 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} -38 & 8 & -18 \\ 120 & -54 & 164 \\ 58 & -60 & 206 \end{bmatrix},$$

Since the matrix A is invertible, we compute the matrix A^{-1} and multiply the above quadratic matrix equation from the left by A^{-1}. Thus,

$$A^{-1} = \begin{bmatrix} -0.2160 & 0.0560 & -0.0080 \\ 0.2400 & 0.1600 & 0.1200 \\ 0.3440 & 0.0960 & 0.2720 \end{bmatrix}.$$

Then, by relations (8), we have

$$H = \begin{bmatrix} 14.4639 & -4.2720 & 11.4239 \\ 17.0400 & -13.9200 & 46.6400 \\ 14.2240 & -18.7520 & 65.5840 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.4000 & -0.5360 & -0.5280 \\ 1.0000 & 0.0400 & 1.9200 \\ 1.4000 & 0.2240 & 2.9520 \end{bmatrix},$$

$$Z = \begin{bmatrix} 14.1851 & -4.3605 & 10.7242 \\ 17.8220 & -13.9461 & 47.9441 \\ 15.4532 & -18.7720 & 67.6853 \end{bmatrix}.$$

and the unknown matrix X is computed in the following:

$$X = \begin{bmatrix} 3.5705 & -0.4204 & 1.4301 \\ 2.5657 & -1.4077 & 4.9393 \\ 1.0828 & -2.4447 & 7.4329 \end{bmatrix}.$$

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