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A new type of Hyers-Ulam-Rassias stability for Drygas functional equation

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Abstract. In this paper, we prove the generalized Hyers-Ulam-Rassias stability for the Drygas functional equation

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$$

in Banach spaces by using the Brzdęk's fixed point theorem. Moreover, we give a general result on the hyperstability of this equation. Our results are improvements and generalizations of the main result of M. Piszczek and J. Szczawińska [21].

Keywords: Stability, hyperstability, Drygas functional equation, fixed point method, Banach space.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

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We say a functional equation is hyperstable if any function f satisfying this equation approximately is a true solution of it. The first hyperstability result was published in [5] and concerned the ring homomorphisms. However, the term hyperstability has been used for the first time in [19]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. The hyperstability results of the several functional equation in the literature have been studied by many authors, see for example [4, 6, 7, 9, 11, 16, 19, 20].

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N} = \{1, 2, 3, \cdots\}$, the set of nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ and by $\mathbb{N}_m$ the set of all natural numbers greater than or equal to the natural number m. Let $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative real numbers. We write B^A to mean the family of all functions mapping from a nonempty set A into a nonempty set B, and we denote A^n the n-ary Cartesian power of A.

Before proceeding to the main results, we state the following definition and theorem which are useful for our purpose.

Definition 1.1 Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_+^{X^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n) \tag{1}$$

for $x_1, \ldots, x_n \in X$ is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d\left(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\right) \leqslant \varepsilon(x_1, \ldots, x_n)$$

fulfills the equation (1).

Theorem 1.2 ([10, Theorem 1]) Let X be a nonempty set, (Y, d) a complete metric space, $f_1, \ldots, f_s \colon X \to X$ and $L_1, \ldots, L_s \colon X \to \mathbb{R}_+$ be given mappings. Let $\Lambda \colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{s} L_i(x)\delta(f_i(x)) \tag{2}$$

for $\delta \in \mathbb{R}_+^X$ and $x \in X$. If $\mathcal{T} \colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \leqslant \sum_{i=1}^{s} L_i(x) d\big(\xi(f_i(x)), \mu(f_i(x))\big), \qquad (\xi, \mu \in Y^X, \quad x \in X),$$

and a function $\varepsilon \colon X \to \mathbb{R}_+$ and a mapping $\varphi \colon X \to Y$ satisfy $d\left(\mathcal{T}\varphi(x), \varphi(x)\right) \leqslant \varepsilon(x)$ and $\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k\varepsilon(x) < \infty$ for all $x \in X$, then for every $x \in X$, the limit $\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$ exists and the function $\psi \in Y^X$ so defined is the unique fixed point of $\mathcal{T}$ with $d\left(\varphi(x), \psi(x)\right) \leqslant \varepsilon^*(x)$ for all $x \in X$.

Characterizing quasi-inner product spaces, Drygas considers in [13] the functional equation

$$f(x) + f(y) = f(x - y) + \left\{f(\tfrac{x+y}{2}) - f(\tfrac{x-y}{2})\right\}$$

for all $x, y \in \mathbb{R}$, which can be reduced to the following equation [23, Remark 9.2, p. 131]

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \qquad (x, y \in \mathbb{R}). \tag{3}$$

This equation is known in the literature as Drygas equation and is a generalization of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \qquad (x,y\in\mathbb{R}).$$

The general solution of Drygas equation was given by Ebanks, Kannappan and Sahoo in [14]. It has the form $f(x)=A(x)+Q(y)$ for all $x\in\mathbb{R}$, where $A\colon\mathbb{R}\to\mathbb{R}$ is an additive function and $Q\colon\mathbb{R}\to\mathbb{R}$ is a quadratic function (see also [18]). A set-valued version of Drygas equation was considered by Smajdor in [25]. Recently, the hyperstability of the Drygas functional equation has been studied in [21] and [24].

In this paper, we discuss the generalized Hyers-Ulam-Rassias stability problem for the Drygas functional equation (3) in Banach spaces by using Theorem 1.2. We also introduce some hyperstability results for this equation. This approach to Ulam stability has been patterned on considerations in [1, 2, 8]. There are also recent results in [12].

2. Main results

In the sequel, for a nonempty set X we write $X_0 := X\setminus\{0\}$, and we denote by $Aut(X)$ for the family of all automorphisms of X. The identity function on X will be denoted by id_X, and for each $u\in X^X$ we write $ux := u(x)$ for $x\in X$ and we define $-u$ by $-ux := -u(x)$, $2ux := ux+ux$ and u' by $u'x := (id_X-u)x = x-ux$ for $x\in X$.

The following theorem is the main result concerning the stability of the functional equation (3).

Theorem 2.1 Let X be a normed space with the norm $\|.\|_X$, Y be a Banach space with the norm $\|.\|_Y$, $\varepsilon\colon X_0^2\to\mathbb{R}_+$ and

$$l(X) := \Big\{u\in Aut(X) : -u, u', (id_X-2u)\in Aut(X), \quad \alpha_u<1\Big\} \tag{4}$$

be an infinite set, where

$$\begin{aligned}&\alpha_u := 2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(id_X-2u),\\ &\lambda(u) := \inf\Big\{t\in\mathbb{R}_+ : \varepsilon(ux,uy)\leqslant t\varepsilon(x,y), \quad \forall x,y\in X_0\Big\}\end{aligned}$$

for $u\in Aut(X)$. Assume that $f:X\longrightarrow Y$ satisfies the inequality

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\|_Y\leqslant\varepsilon(x,y) \tag{5}$$

for all $x,y\in X_0$ such that $x+y\neq0$ and $x-y\neq0$. Then, for each nonempty subset $\mathcal{U}\subset l(X)$ such that

$$u\circ v = v\circ u, \qquad (u,v\in\mathcal{U}), \tag{6}$$

there exists a unique function $D:X\longrightarrow Y$ satisfies the equation (3) and

$$\|f(x)-D(x)\|_Y\leqslant\tilde{\varepsilon}(x) \tag{7}$$

for $x\in X_0$, where $\tilde{\varepsilon}(x) := inf\Big\{\frac{\varepsilon(u'x,ux)}{1-\alpha_u} : u\in\mathcal{U}\Big\}$.

Proof. Let us fix arbitrarily $u \in \mathcal{U}$. Replacing x with $u'x$ and y with ux in (5), we get

$$\big\|f(x) + f\big((id_X - 2u)x\big) - 2f(u'x) - f(ux) - f(-ux)\big\|_Y \leqslant \varepsilon(u'x, ux) := \varepsilon_u(x) \tag{8}$$

for all $x \in X_0$. We define the operators $\mathcal{T}_u \colon Y^{X_0} \to Y^{X_0}$ and $\Lambda_u \colon \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ by

$$\begin{aligned} \mathcal{T}_u\xi(x) &:= 2\xi(u'x) + \xi(ux) + \xi(-ux) - \xi\big((id_X - 2u)x\big), \\ \Lambda_u\delta(x) &:= 2\delta(u'x) + \delta(ux) + \delta(-ux) + \delta\big((id_X - 2u)x\big) \end{aligned} \tag{9}$$

for all $x \in X_0$, $\xi \in Y^{X_0}$ and $\delta \in \mathbb{R}_+^{X_0}$. Then (8) becomes $\|f(x) - \mathcal{T}_u f(x)\|_Y \leqslant \varepsilon_u(x)$ for all $x \in X_0$.

The operator Λ_u has the form given by (2) with $s = 4$ and $f_n(x) = u'x$, $f_2(x) = ux$, $f_3(x) = -ux$, $f_4(x) = (id_X - 2u)x$, $L_1(x) = 2$, $L_2(x) = L_3(x) = L_4(x) = 1$ for all $x \in X_0$. Further,

$$\begin{aligned} \Big\|\mathcal{T}_u\xi(x) - \mathcal{T}_u\mu(x)\Big\|_Y &= \Big\|2\xi(u'x) + \xi(ux) + \xi(-ux) - \xi\big((id_X - 2u)x\big) \\ &\qquad - 2\mu(u'x) - \mu(ux) - \mu(-ux) + \mu\big((id_X - 2u)x\big)\Big\|_Y \\ &\leqslant 2\big\|\xi(u'x) - \mu(u'x)\big\|_Y + \big\|\xi(ux) - \mu(ux)\big\|_Y + \big\|\xi(-ux) - \mu(-ux)\big\|_Y \\ &\qquad + \big\|\xi\big((id_X - 2u)x\big) - \mu\big((id_X - 2u)x\big)\big\|_Y \end{aligned}$$

for all $x \in X_0$ and $\xi, \mu \in Y^{X_0}$.

Note that, in view of the definition of $\lambda(u)$, $\varepsilon(ux, uy) \leqslant \lambda(u)\varepsilon(x, y)$ for all $x, y \in X_0$. So it is easy to show by induction on s that $\Lambda_u^s\varepsilon_u(x) \leqslant \alpha_u^s\varepsilon(u'x, ux)$ for all $x \in X_0$, where

$$\alpha_u = 2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(id_X - 2u).$$

Hence,

$$\varepsilon^*(x) := \sum_{r=0}^{\infty} \Lambda_u^r\varepsilon_u(x) \leqslant \varepsilon(u'x, ux) \sum_{r=0}^{\infty} \alpha_u^r = \frac{\varepsilon(u'x, ux)}{1 - \alpha_u} < \infty$$

for all $x \in X_0$. By Theorem 1.2, there exists a unique solution $D_u : X \to Y$ of the equation

$$D_u(x) = 2D_u(u'x) + D_u(ux) + D_u(-ux) - D_u\big((id_X - 2u)x\big) \tag{10}$$

for all $x \in X_0$, which is a fixed point of $\mathcal{T}_u$ such that

$$\Big\|D_u(x) - f(x)\Big\|_Y \leqslant \frac{\varepsilon(u'x, ux)}{1 - \alpha_u} \tag{11}$$

for all $x \in X_0$. Moreover, $D_u(x) = \lim_{r\to\infty} \mathcal{T}_u^r f(x)$ for all $x \in X_0$.

To prove that D_u satisfies the functional equation (3) on X_0, just prove the following inequality

$$\Big\|\mathcal{T}_u^r f(x+y) + \mathcal{T}_u^r f(x-y) - 2\mathcal{T}_u^r f(x) - \mathcal{T}_u^r f(y) - \mathcal{T}_u^r f(-y)\Big\|_Y \leqslant \alpha_u^r\varepsilon(x, y) \tag{12}$$

for all $x, y \in X_0$, $x+y \neq 0$, $x-y \neq 0$ and $r \in \mathbb{N}_0$. Indeed, if $r=0$, then (12) is simply (5). So, take $r \in \mathbb{N}$ and suppose that (12) holds for r and $x, y \in X_0$. Then, by using (9) and the triangle inequality, we have

$$\begin{aligned}
&\Big\|\mathcal{T}_u^{r+1} f(x+y) + \mathcal{T}_u^{r+1} f(x-y) - 2\mathcal{T}_u^{r+1} f(x) - \mathcal{T}_u^{r+1} f(y) - \mathcal{T}_u^{r+1} f(-y)\Big\|_Y \\
&= \Big\|2\mathcal{T}_u^{r} f\big(u'(x+y)\big) + \mathcal{T}_u^{r} f\big(u(x+y)\big) + \mathcal{T}_u^{r} f\big(-u(x+y)\big) - \mathcal{T}_u^{r} f\big((id_X-2u)(x+y)\big) \\
&\quad + 2\mathcal{T}_u^{r} f\big(u'(x-y)\big) + \mathcal{T}_u^{r} f\big(u(x-y)\big) + \mathcal{T}_u^{r} f\big(-u(x-y)\big) - \mathcal{T}_u^{r} f\big((id_X-2u)(x-y)\big) \\
&\quad - 4\mathcal{T}_u^{r} f\big(u'x\big) - 2\mathcal{T}_u^{r} f\big(ux\big) - 2\mathcal{T}_u^{r} f\big(-ux\big) + 2\mathcal{T}_u^{r} f\big((id_X-2u)x\big) \\
&\quad - 2\mathcal{T}_u^{r} f\big(u'y\big) - \mathcal{T}_u^{r} f\big(uy\big) - \mathcal{T}_u^{r} f\big(-uy\big) + \mathcal{T}_u^{r} f\big((id_X-2u)y\big) \\
&\quad + 2\mathcal{T}_u^{r} f\big(u'(-y)\big) + \mathcal{T}_u^{r} f\big(u(-y)\big) + \mathcal{T}_u^{r} f\big(-u(-y)\big) + \mathcal{T}_u^{r} f\big((id_X-2u)(-y)\big)\Big\|_Y \\
&\leqslant 2\,\Big\|\mathcal{T}_u^{r} f\big(u'(x+y)\big) + \mathcal{T}_u^{r} f\big(u'(x-y)\big) - 2\mathcal{T}_u^{r} f\big(u'x\big) - \mathcal{T}_u^{r} f\big(u'y\big) - \mathcal{T}_u^{r} f\big(u'(-y)\big)\Big\|_Y \\
&\quad + \Big\|\mathcal{T}_u^{r} f\big(u(x+y)\big) + \mathcal{T}_u^{r} f\big(u(x-y)\big) - 2\mathcal{T}_u^{r} f\big(ux\big) - \mathcal{T}_u^{r} f\big(uy\big) - \mathcal{T}_u^{r} f\big(u(-y)\big)\Big\|_Y \\
&\quad + \Big\|\mathcal{T}_u^{r} f\big(-u(x+y)\big) + \mathcal{T}_u^{r} f\big(-u(x-y)\big) - 2\mathcal{T}_u^{r} f\big(-ux\big) - \mathcal{T}_u^{r} f\big(-uy\big) - \mathcal{T}_u^{r} f\big(-u(-y)\big)\Big\|_Y \\
&\quad + \Big\|\mathcal{T}_u^{r} f\big((id_X-2u)(x+y)\big) + \mathcal{T}_u^{r} f\big((id_X-2u)(x-y)\big) - 2\mathcal{T}_u^{r} f\big((id_X-2u)x\big) \\
&\quad - \mathcal{T}_u^{r} f\big((id_X-2u)y\big) - \mathcal{T}_u^{r} f\big((id_X-2u)(-y)\big)\Big\|_Y \\
&\leqslant \alpha_u^r\,\Big(2\varepsilon(u'x,u'y) + \varepsilon(ux,uy) + \varepsilon(-ux,-uy) + \varepsilon\big((id_X-2u)x,(id_X-2u)y\big)\Big) \\
&\leqslant \alpha_u^r\,\Big(2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(id_X-2u)\Big)\varepsilon(x,y) \\
&\leqslant \alpha_u^{r+1}\varepsilon(x,y).
\end{aligned}$$

By induction, we have shown that (12) holds for all $x, y \in X_0$, $x+y \neq 0$ and $x-y \neq 0$. Letting $r \to \infty$ in (12), we get $D_u(x+y) + D_u(x-y) = 2D_u(x) + D_u(y) + D_u(-y)$ for all $x, y \in X_0$. Thus, we have proved that for every $u \in \mathcal{U}$ there exists a function $D_u : X_0 \to Y$ which is a solution of the functional equation (3) on X_0 and satisfies

$$\Big\|f(x) - D_u(x)\Big\|_Y \leqslant \frac{\varepsilon(u'x,ux)}{1-\alpha_u}$$

for all $x \in X_0$. Next, we prove that each solution $D\colon X_0 \to Y$ of (3) satisfying the inequality

$$\|f(x) - D(x)\|_Y \leqslant L\varepsilon(v'x,vx), \qquad (x \in X_0) \tag{13}$$

with some $L > 0$ and $v \in \mathcal{U}$, is equal to D_w for each $w \in \mathcal{U}$. So, fix $v, w \in \mathcal{U}$, $L > 0$ and $D\colon X_0 \to Y$ a solution of (3) satisfying (13). Note that, by (11) and (13), there is $L_0 > 0$

such that

$$\begin{aligned}\|D(x)-D_w(x)\|_Y &\leqslant \|D(x)-f(x)\|_Y+\|f(x)-D_w(x)\|_Y\\ &\leqslant L_0\ \big(\varepsilon(v'x,vx)+\varepsilon(w'x,wx)\big)\cdot\sum_{r=0}^{\infty}\alpha_w^r\end{aligned} \tag{14}$$

for $x\in X_0$. In other side, D and D_w are solutions of (10) because they are satisfy (3). We show that, for each $j\in\mathbb{N}$,

$$\|D(x)-D_w(x)\|_Y\leqslant L_0\big(\varepsilon(v'x,vx)+\varepsilon(w'x,wx)\big)\cdot\sum_{r=j}^{\infty}\alpha_w^r,\qquad (x\in X_0). \tag{15}$$

The case $j=0$ is exactly (14). So fix $\gamma\in\mathbb{N}_0$ and assume that (15) holds for $j=\gamma$. Then, in view of definition of $\lambda(u)$,

$$\begin{aligned}&\big\|D(x)-D_w(x)\big\|_Y\\ &=\Big\|2D(w'x)+D(wx)+D(-wx)-D\big((id_X-2w)x\big)\\ &\quad-2D_w(w'x)-D_w(wx)-D_w(-wx)+D_w\big((id_X-2w)x\big)\Big\|_Y\\ &\leqslant 2\big\|D(w'x)-D_w(w'x)\big\|+\big\|D(wx)-D_w(wx)\big\|+\big\|D(-wx)-D_w(-wx)\big\|_Y\\ &\quad+\big\|D\big((id_X-2w)x\big)-D_w\big((id_X-2w)x\big)\big\|_Y\\ &\leqslant 2L_0\ \Big(\varepsilon(w'v'x,w'vx)+\varepsilon(w'w'x,w'wx)\Big)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\ &\quad+L_0\ \Big(\varepsilon(wv'x,wvx)+\varepsilon(ww'x,wwx)\Big)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\ &\quad+L_0\ \Big(\varepsilon(-wv'x,-wvx)+\varepsilon(-ww'x,-wwx)\Big)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\ &\quad+L_0\ \Big(\varepsilon\big((id_X-2w)v'x,(id_X-2w)vx\big)+\varepsilon\big((id_X-2w)w'x,(id_X-2w)wx\big)\Big)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\ &\leqslant L_0\big(\varepsilon(v'x,vx)+\varepsilon(w'x,wx)\big)\ \Big(2\lambda(w')+\lambda(w)+\lambda(-w)+\lambda(id_X-2w)\Big)\cdot\sum_{r=\gamma}^{\infty}\alpha_w^r\\ &=L_0\Big(\varepsilon(v'x,vx)+\varepsilon(w'x,wx)\Big)\cdot\sum_{r=\gamma+1}^{\infty}\alpha_w^r.\end{aligned}$$

Hence, we have shown (15). Now, letting $j\to\infty$ in (15), we get

$$D(x)=D_w(x),\qquad (x\in X_0). \tag{16}$$

By similar method, we also prove that $D_u=D_w$ for each $u\in\mathcal{U}$, which yields

$$\|f(x) - D_w(x)\|_Y \leqslant \tfrac{\varepsilon(u'x,ux)}{1-\alpha_u}, \qquad (x \in X_0, \quad u \in \mathcal{U}).$$

This implies (7) with $D = D_w$ and the uniqueness of D is given by (16). ∎

In the following theorem, we prove the hyperstability of equation (3) in Banach spaces.

Theorem 2.2 Let X, Y and ε be as in Theorem 2.1. Suppose that there exists a nonempty set $\mathcal{U} \in l(X)$ such that $u \circ v = v \circ u$ for all $u, v \in \mathcal{U}$ and

$$\begin{cases} \inf\limits_{u\in\mathcal{U}} \varepsilon(u'x, ux) = 0, & (x \in X_0), \\ \sup\limits_{u\in\mathcal{U}} \alpha_u < 1. \end{cases} \tag{17}$$

Then every $f\colon X \to Y$ satisfying (5) is a solution of (3) on X_0.

Proof. Suppose that $f\colon X \to Y$ satisfies (5). Then, by Theorem 2.1, there exists a mapping $D\colon X \to Y$ satisfies (3) and $\|f(x) - D(x)\|_Y \leqslant \tilde{\varepsilon}(x)$ for all $x \in X_0$.
Since, in view of (17), $\tilde{\varepsilon}(x) = 0$ for all $x \in X_0$. This means that $f(x) = D(x)$ for all $x \in X_0$, where

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \qquad (x, y \in X_0)$$

which implies that f satisfies the functional equation (3) on X_0. ∎

From Theorems 2.1 and 2.2, we can obtain the following corollaries as natural results.

Corollary 2.3 Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be a normed space and a Banach space, respectively. Assume that $p, q \in \mathbb{R}$, $p < 0$, $q < 0$ and $\theta \geqslant 0$. If $f\colon X \to Y$ satisfies

$$\Big\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\Big\|_Y \leqslant \theta\big(\|x\|_X^p + \|y\|_X^q \big) \tag{18}$$

for all $x, y \in X_0$, then f satisfies the Drygas functional equation (3) on X_0.

Proof. The proof follows from Theorem 2.2 by taking $\varepsilon(x,y) = \theta\Big(\|x\|_X^p + \|y\|_X^q \Big)$ for all $x, y \in X_0$ with some real numbers $\theta \geqslant 0$, $p < 0$ and $q < 0$. For each $m \in \mathbb{N}$, define $u_m\colon X_0 \to X_0$ by $u_m x := u_m(x) = -mx$ and $u'_m\colon X_0 \to X_0$ by $u'_m x := u'_m(x) = (1+m)x$. Then

$$\begin{aligned} \varepsilon(u_m x, u_k y) &= \varepsilon(-mx, -ky) \\ &= \theta\Big(\|-mx\|_X^p + \|-ky\|_X^q \Big) \\ &= \theta m^p \|x\|_X^p + \theta k^q \|y\|_X^q \\ &\leqslant (m^p + k^q)\, \theta\Big(\|x\|_X^p + \|y\|_X^q \Big) \\ &= (m^p + k^q)\, \varepsilon(x,y) \end{aligned}$$

for all $x, y \in X_0$ and $k, m \in \mathbb{N}$. Hence,

$$\lim_{m\to\infty} \varepsilon(u'_m x, u_m y) \leqslant \lim_{m\to\infty} \Big((1+m)^p + m^q\Big)\varepsilon(x,y) = 0$$

for all $x, y \in X_0$. Then (17) is valid with $\lambda(u_m) = m^p + m^q$ for $m \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $m \geqslant n_0$ and

$$\alpha_{u_m} = 2\Big((1+m)^p + (1+m)^q\Big) + 2m^p + 2m^q + (1+2m)^p + (1+2m)^q < 1.$$

So it easily seen that (4) is fulfilled with $\mathcal{U} := \{u_m \in Aut(X) : m \in \mathbb{N}_{n_0}\}$. Therefore, by Theorem 2.2, every $f\colon X \to Y$ satisfying (18) is a solution of the functional equation (3) on X_0. ■

In the following corollary we find the main result of [21].

Corollary 2.4 ([21, Theorem 2]) Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be a normed space and a Banach space, respectively, $\theta \geqslant 0$, and $p < 0$. Assume that $f\colon X \to Y$ satisfies

$$\Big\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\Big\|_Y \leqslant \theta\big(\,\|x\|_X^p + \|y\|_X^p\,\big) \tag{19}$$

for all $x, y \in X_0$. Then f satisfies the Drygas functional equation on X_0.

Proof. It is easily seen that the function ε given by $\varepsilon(x,y) = \theta\Big(\,\|x\|_X^p + \|y\|_X^p\,\Big)$ for all $x, y \in X_0$ satisfies (17), since

$$\varepsilon(mx, ky) = \theta\,\|mx\|_X^p + \theta\,\|ky\|_X^p \leqslant \theta\big(m^p + k^p\big)\Big(\,\|x\|_X^p + \|y\|_X^p\,\Big) = \big(m^p + k^p\big)\varepsilon(x,y)$$

for all $x, y \in X_0$, $k, m \in \mathbb{N}$, and $km \neq 0$. The remainder of the proof is similar to the proof of Corollary 2.3. ■

If X is a normed space and $f\colon X \to Y$ satisfies (19) for $x, y \in X_0$, with $p < 0$, then by Theorem 2.2 we know that f satisfies the Drygas equation on X_0. It is easy to see that if $f(0) = 0$, then f satisfies the Drygas equation on the whole X. So we have the following corollary.

Corollary 2.5 Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be a normed space and a Banach space, respectively, $\theta \geqslant 0$, and $p < 0$. Assume that $f\colon X \to Y$ satisfies $f(0) = 0$ and fulfills the inequality

$$\Big\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\Big\|_Y \leqslant \theta\big(\,\|x\|_X^p + \|y\|_X^p\,\big)$$

for all $x, y \in X_0$. Then f satisfies the Drygas functional equation on X.

Corollary 2.6 Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be a normed space and a Banach space, respectively. Assume that $p, q \in \mathbb{R}$, $p + q < 0$ and $\theta \geqslant 0$. If $f\colon X \to Y$ satisfies

$$\Big\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\Big\|_Y \leqslant \theta\,\|x\|_X^p\,\|y\|_X^q$$

for all $x, y \in X_0$. Then f satisfies the Drygas functional equation (3) on X_0.

Proof. It is easily seen that the function ε given by $\varepsilon(x,y) = \theta\,\|x\|_X^p\,\|y\|_X^q$ for $x, y \in X_0$ satisfies (17), since

$$\varepsilon(mx, ky) = \theta\,\|mx\|_X^p\,\|ky\|_X^q = \theta m^p k^q\,\|x\|_X^p\,\|y\|_X^q = m^p k^q \varepsilon(x,y)$$

for all $x, y \in X_0$, $k, m \in \mathbb{N}$, and $km \neq 0$. The rest of the proof is similar to the proof of Corollary 2.3. ■

By an analogous conclusion, the function ε given by

$$\varepsilon(x,y) = \theta\Big(\,\|x\|_X^p + \|y\|_X^q + \|x\|_X^p\,\|y\|_X^q\,\Big)$$

fro all $x, y \in X_0$ satisfies (17), since

$$\begin{aligned}\varepsilon(mx, ky) &= \theta\Big(\|mx\|_X^p + \|ky\|_X^q + \|mx\|_X^p \|ky\|_X^q \Big)\\ &= \theta\Big(m^p \|x\|_X^p + k^q \|y\|_X^p + m^p k^q \|x\|_X^p \|y\|_X^q \Big)\\ &\leqslant \Big(m^p + k^q + m^p k^q\Big)\varepsilon(x, y)\end{aligned}$$

for all $x, y \in X_0$, $k, m \in \mathbb{Z}$, and $km \neq 0$. So we have the following corollary.

Corollary 2.7 Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be a normed space and a Banach space, respectively. Assume that $p < 0$, $q < 0$, $p + q < 0$ and $\theta \geqslant 0$. If $f \colon X \to Y$ satisfies

$$\Big\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\Big\|_Y \leqslant \theta\left(\|x\|_X^p + \|y\|_X^q + \|x\|_X^p \|y\|_X^q\right)$$

for all $x, y \in X_0$, then f satisfies the functional equation (3) on X_0.

The next corollary corresponds to the results on the inhomogeneous Drygas functional equation (20).

Corollary 2.8 Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ and ε be as in Theorem 2.1 and $G \colon X^2 \to Y$. Suppose that $\big\|G(x, y)\big\|_Y \leqslant \varepsilon(x, y)$ for all $x, y \in X_0$, where $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X_0$, and there exists a nonempty $\mathcal{U} \subset l(X)$ such that (6) and (17) hold. Then the inhomogeneous equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) + G(x, y) \tag{20}$$

for all $x, y \in X_0$, has no solutions in the class of functions $f \colon X \to Y$.

Proof. Suppose that $f \colon X \to Y$ is a solution to (20). Then

$$\begin{aligned}\Big\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\Big\|_Y &= \Big\|2f(x) + f(y) + f(-y) + G(x, y)\\ &\quad - 2f(x) - f(y) - f(-y)\Big\|_Y\\ &= \big\|G(x, y)\big\|_Y\\ &\leqslant \varepsilon(x, y)\end{aligned}$$

for all $x, y \in X_0$. Consequently, by Theorem 2.2, f is a solution of (3), whence

$$G(x_0, y_0) = f(x_0 + y_0) + f(x_0 - y_0) - 2f(x_0) - f(y_0) - f(-y_0) = 0,$$

which is contradiction. ■

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