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Fixed point theorem for mappings satisfying contractive condition of integral type on intuitionistic fuzzy metric space

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Abstract. In this paper, we shall establish some fixed point theorems for mappings with the contractive condition of integrable type on complete intuitionistic fuzzy metric spaces $(X, M, N, *, \Diamond)$. We also use Lebesgue-integrable mapping to obtain new results. Akram, Zafar, and Siddiqui introduced the notion of A-contraction mapping on metric space. In this paper by using the main idea of the work, we introduce the concept of A-fuzzy contractive mappings. Finally, we support our results by some examples.

Keywords: Intuitionistic fuzzy metric space, fixed point, A-fuzzy contractions.

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1. Introduction

In 2002, Branciari [3] analyzed the existence of fixed point for mapping T defined on a complete metric space (X, d) satisfying a general contractive condition of integral type. After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. In 2003, A fine work has been done by Rhoades extending the result by replacing new condition [12]. Akram et al. [1] introduced a new class of contraction maps, called A-contraction. which is a proper superclass of Kannan's, Reich's and Bianchini's type contractions [2, 7, 11]. In 2011, Dey et al. [4] proved some fixed point theorems for mixed type of contraction mappings of integral type in complete metric space .

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In this paper we consider $(X, M, N, *, \Diamond)$ intuitionistic fuzzy metric spaces in Park's sense [9] and by using their idea, we provide some fixed point results for the mappings f define on the space, satisfying a contractive condition of integral type.

2. Preliminaries

Definition 2.1 ([13]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm whenever it satisfies the following conditions:
(a) $*$ is commutative and associative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leqslant c * d$ for all $a, b, c, d \in [0,1]$ with $a \leqslant c$ and $b \leqslant d$.

For example, $a * b = ab$, $a * b = \min\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$ and

$$a * b = \frac{ab}{\max\{a, b, \lambda\}}$$

for $0 < \lambda < 1$ are continuous t-norms.

Definition 2.2 ([13]) A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-conorm whenever it satisfies the following conditions:
(a) $\Diamond$ is commutative and associative,
(b) $\Diamond$ is continuous,
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$.
(d) $a \Diamond b \leqslant c \Diamond d$ whenever $a \leqslant c$ and $b \leqslant d$, and $a, b, c, d \in [0,1]$.

For example, $a \Diamond b = \min\{a + b, 1\}$ and $a \Diamond b = \max\{a, b\}$ are continuous t-conorms.

Definition 2.3 ([9]) A 5-tuple $(X, M, N, *, \Diamond)$ is said to be intuitionistic fuzzy metric space whenever X is a set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:
(i) $M(x, y, t) + N(x, y, t) \leqslant 1$,
(ii) $M(x, y, 0) = 0$,
(iii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iv) $M(x, y, t) = M(y, x, t)$,
(v) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$ for all $x, y, z \in X,\ s, t > 0$,
(vi) $M(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is continuous,
(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$,
(viii) $N(x, y, 0) = 1$,
(ix) $N(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,
(x) $N(x, y, t) = N(y, x, t)$,
(xi) $N(x, y, t) \Diamond N(y, z, s) \geqslant N(x, z, t + s)$ for all $x, y, z \in X,\ s, t > 0$,
(xii) $N(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is continuous,
(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.
Then (M, N) is called an intuitionistic fuzzy metric on X.

***Example* 2.4** ([9]) Let (X, d) be a metric space. Denote $a*b = ab$ and $a \Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in \mathbb{R}^+$. If $h = k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric and $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space.

For an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, define

$$B(x, r, t) = \{y \in X \,:\, M(x, y, t) > 1 - r, N(x, y, t) < r\},$$

for all $t > 0$ and $0 < r < 1$. Denote the generated topology by the sets $B(x, r, t)$ by $\tau_{(M,N)}$. A sequence $\{x_n\}$ in $(X, M, N, *, \Diamond)$ is said to be Cauchy whenever for each $\varepsilon > 0$ and $t > 0$, there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geqslant n_0$. Also, $(X, M, N, *, \Diamond)$ is called complete whenever every Cauchy sequence is convergent with respect $\tau_{(M,N)}$.

Definition 2.5 ([5]) Let $(X, M, N, *, \Diamond)$ be a intuitionistic fuzzy metric space. The fuzzy metric M, N is triangular whenever

$$\frac{1}{M(x, y, t)} - 1 \leqslant \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1$$

and $N(x, y, t) \leqslant N(x, z, t) + N(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.

Definition 2.6 ([6]) A sequence $\{x_n\}$ in a intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is called intuitionistic fuzzy contractive sequence if there exists $0 < k < 1$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leqslant k \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)$$

and $N(x_{n+1}, x_{n+2}, t) \leqslant kN(x_n, x_{n+1}, t)$ for all n and $t > 0$.

Lemma 2.7 ([8]) Let $(X, M, N, *, \Diamond)$ be a triangular intuitionistic fuzzy metric space and $\{x_n\}$ an intuitionistic fuzzy contractive sequence in X. Then $\{x_n\}$ is a Cauchy sequence.

Definition 2.8 ([10]) Let $(X, M, N, *, \Diamond)$ be a intuitionistic fuzzy metric space. A self-map f on X is said to be intuitionistic fuzzy contractive whenever there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leqslant k \left(\frac{1}{M(x, y, t)} - 1 \right)$$

and $N(f(x), f(y), t) \leqslant kN(x, y, t)$ for all $x, y \in X$ and $t > 0$.

Definition 2.9 ([1]) Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and A be the set of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying
(A1) α is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}^3$),
(A2) $a \leqslant kb$ for some $k \in [0, 1)$ whenever $a \leqslant \alpha(a, b, b)$ or $a \leqslant \alpha(b, a, b)$ or $a \leqslant \alpha(b, b, a)$ for all a, b.

3. Main results

Theorem 3.1 Let $(X, M, N, *, \diamondsuit)$ be a complete intuitionistic fuzzy metric space, $c \in (0,1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$, $t > 0$,

$$\int_0^{\frac{1}{M(fx,fy,t)}-1} \varphi(s)ds \leqslant c \int_0^{\frac{1}{M(x,y,t)}-1} \varphi(s)ds, \tag{1}$$

$$\int_0^{N(fx,fy,t)} \varphi(s)ds \leqslant c \int_0^{N(x,y,t)} \varphi(s)ds, \tag{2}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)\ ds > 0.$$

Then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim\limits_{n\to+\infty} f^n x = a$.

Proof. Step 1. We have

$$\int_0^{\frac{1}{M(f^n x, f^{n+1}x,t)}-1} \varphi(s)ds \leqslant c^n \int_0^{\frac{1}{M(x,fx,t)}-1} \varphi(s)ds.$$

This follows immediately by iterating (1) n times:

$$\int_0^{\frac{1}{M(f^n x, f^{n+1}x,t)}-1} \varphi(s)ds \leqslant c \int_0^{\frac{1}{M(f^{n-1} x, f^{n}x,t)}-1} \varphi(s)ds \leqslant \cdots \leqslant c^n \int_0^{\frac{1}{M(x,fx,t)}-1} \varphi(s)ds.$$

As a consequence, since $c \in (0,1)$, we get

$$\int_0^{\frac{1}{M(f^n x, f^{n+1}x,t)}-1} \varphi(s)ds \leqslant c^n \int_0^{\frac{1}{M(x,fx,t)}-1} \varphi(s)ds \to 0^+.$$

Step 2. We have $\frac{1}{M(f^n x, f^{n+1}x,t)} - 1 \to 0$ as $n \to +\infty$. Suppose that

$$\lim_{n\to+\infty} \sup \left(\frac{1}{M(f^n x, f^{n+1}x, t)} - 1 \right) = \frac{\varepsilon}{t} > 0,$$

Then there exists a $m_\varepsilon \in \mathbb{N}$ and a sequence $\{f^{n_m}\}_{m \geqslant m_\varepsilon}$ such that

$$\left(\frac{1}{M(f^n x, f^{n+1}x, t)} - 1 \right) \to \frac{\varepsilon}{t} > 0$$

as $m \to +\infty$ and $\frac{1}{M(f^{n_m} x, f^{n_m+1}x,t)} - 1 \geqslant \frac{\varepsilon}{2t}$ for $m \geqslant m_\varepsilon$. Thus, by Step 1 and the sign of

φ, we have the following contradiction:

$$0=\lim_{m\to+\infty}\int_0^{\frac{1}{M(f^{n_m}x,f^{n_m+1}x,t)}-1}\varphi(s)ds\geqslant\int_0^{\frac{\varepsilon}{2t}}\varphi(s)ds>0.$$

Step 3. For each $x\in X$, $\{f^nx\}_{n\in\mathbb{N}}$ is a Cauchy sequence, that is

$$\forall\varepsilon>0\quad\exists m_\varepsilon\in\mathbb{N}:\forall m,n\in\mathbb{N},m>n>m_\varepsilon:\frac{1}{M(f^mx,f^nx,t)}-1<\frac{\varepsilon}{t}.$$

Suppose that there exists a $\varepsilon>0$ such that for each $l\in\mathbb{N}$ there are $m_l,n_l\in\mathbb{N}$ with $m_l>n_l>l$ such that $\frac{1}{M(f^{m_l}x,f^{n_l}x,t)}-1\geqslant\frac{\varepsilon}{t}$. Then we choose the sequences $\{m_l\}_{l\in\mathbb{N}}$ and $\{n_l\}_{l\in\mathbb{N}}$ such that for each $l\in\mathbb{N}$, m_l is "minimal" in $\frac{1}{M(f^{m_l}x,f^{n_l}x,t)}-1\geqslant\frac{\varepsilon}{t}$, but $\frac{1}{M(f^hx,f^{n_l}x,t)}-1<\frac{\varepsilon}{t}$ for each $h\in\{n_l+1,\cdots,m_l-1\}$. Now, we analyze the properties of $\frac{1}{M(f^{m_l}x,f^{n_l}x,t)}-1$ and $\frac{1}{M(f^{m_l+1}x,f^{n_l+1}x,t)}-1$. At the first, we have

$$\frac{1}{M(f^{m_l}x,f^{n_l}x,t)}-1\to\frac{\varepsilon^+}{t}$$

as $l\to+\infty$. Now, by the triangular inequality and Step 2

$$\begin{aligned}\frac{\varepsilon}{t}&\leqslant\frac{1}{M(f^{m_l}x,f^{n_l}x,t)}-1\\&\leqslant\frac{1}{M(f^{m_l}x,f^{m_l-1}x,t)}-1+\frac{1}{M(f^{m_l-1}x,f^{n_l}x,t)}-1\\&<\frac{1}{M(f^{m_l}x,f^{m_l-1}x,t)}-1+\frac{\varepsilon}{t}\to\frac{\varepsilon^+}{t}\text{ as }l\to+\infty.\end{aligned}$$

Further there exists $\mu\in\mathbb{N}$ such that for each natural number $\nu>\mu$,

$$\frac{1}{M(f^{m_\nu+1}x,f^{n_\nu+1}x,t)}-1<\frac{\varepsilon}{t}.$$

In fact, if there exists a subsequence $\{\nu_k\}_{k\in\mathbb{N}}\subseteq\mathbb{N}$ such that

$$\frac{1}{M(f^{m_{\nu_k}+1}x,f^{n_{\nu_k}+1}x,t)}-1\geqslant\frac{\varepsilon}{t},$$

then

$$\begin{aligned}\frac{\varepsilon}{t}\leqslant&\frac{1}{M(f^{m_{\nu_k}+1}x,f^{n_{\nu_k}+1}x,t)}-1\\\leqslant&\frac{1}{M(f^{m_{\nu_k}+1}x,f^{m_{\nu_k}}x,t)}-1\\&+\frac{1}{M(f^{m_{\nu_k}}x,f^{n_{\nu_k}}x,t)}-1+\frac{1}{M(f^{n_{\nu_k}}x,f^{n_{\nu_k}+1}x,t)}-1\to\frac{\varepsilon}{t}\text{ as }k\to+\infty\end{aligned}$$

and from (1),

$$\int_0^{\frac{1}{M(f^{m_{\nu_k}+1}x,f^{n_{\nu_k}+1}x,t)}-1}\varphi(s)ds \leqslant c\int_0^{\frac{1}{M(f^{m_{\nu_k}}x,f^{n_{\nu_k}}x,t)}-1}\varphi(s)ds. \tag{3}$$

Letting now $k\to+\infty$ in both sides of (3), we have

$$\int_0^{\varepsilon}\varphi(s)ds \leqslant c\int_0^{\varepsilon}\varphi(s)ds$$

which is a contradiction being $c\in(0,1)$ and the integral being positive. Therefore, for a certain $\mu\in\mathbb{N}$,

$$\frac{1}{M(f^{m_\nu+1}x,f^{n_\nu+1}x,t)}-1<\frac{\varepsilon}{t}$$

for all $\nu>\mu$. Finally, we prove the stronger property that there exist a $\sigma_\varepsilon\in(0,\varepsilon)$ and a $\nu_\varepsilon\in\mathbb{N}$ such that for each $\nu>\nu_\varepsilon$ ($\nu\in\mathbb{N}$), we have

$$\frac{1}{M(f^{m_\nu+1}x,f^{n_\nu+1}x,t)}-1<\frac{\varepsilon-\sigma_\varepsilon}{t}.$$

Suppose the existence of a subsequence $\{\nu_k\}_{k\in\mathbb{N}}\subseteq\mathbb{N}$ such that

$$\frac{1}{M(f^{m_{\nu_k}+1}x,f^{n_{\nu_k}+1}x,t)}-1\to\frac{\varepsilon}{t}$$

as $k\to+\infty$. Also, we have

$$\int_0^{\frac{1}{M(f^{m_{\nu_k}+1}x,f^{n_{\nu_k}+1}x,t)}-1}\varphi(s)ds \leqslant c\int_0^{\frac{1}{M(f^{m_{\nu_k}}x,f^{n_{\nu_k}}x,t)}-1}\varphi(s)ds.$$

Letting $k\to+\infty$, we have again the contradiction that

$$\int_0^{\frac{\varepsilon}{t}}\varphi(s)ds \leqslant c\int_0^{\frac{\varepsilon}{t}}\varphi(s)ds.$$

In conclusion of this step, we can prove the Cauchy character of $\{f^nx\}_{n\in\mathbb{N}}$ ($x\in X$). In fact, for each natural number $\nu>\nu_\varepsilon$ (ν_ε as above), we have

$$\begin{aligned}
\frac{\varepsilon}{t} \leqslant&\frac{1}{M(f^{m_\nu}x,f^{n_\nu}x,t)}-1\\
\leqslant&\frac{1}{M(f^{m_\nu}x,f^{m_\nu+1}x,t)}-1\\
&+\frac{1}{M(f^{m_\nu+1}x,f^{n_\nu+1}x,t)}-1+\frac{1}{M(f^{n_\nu+1}x,f^{n_\nu}x,t)}-1\\
<&\frac{1}{M(f^{m_\nu}x,f^{n_\nu+1}x,t)}-1+(\varepsilon-\sigma_\varepsilon)+\frac{1}{M(f^{n_\nu}x,f^{n_\nu+1}x,t)}-1,
\end{aligned}$$

when $\nu \to +\infty$, we have $\varepsilon < \varepsilon - \sigma_\varepsilon$, which is a contradiction. This prove Step 3.
Step 4. Existence of a fixed point. Since $(X, M, N, *, \Diamond)$ is a complete intuitionistic fuzzy metric space, there exists a point $a \in X$ such that $a = \lim_{n\to+\infty} f^n x$. Further a is a fixed point. In fact, suppose that $\frac{1}{M(a,fa,t)} - 1 > 0$. Then

$$0 < \frac{1}{M(a, fa, t)} - 1 \leqslant \frac{1}{M(a, f^{n+1}x, t)} - 1 + \frac{1}{M(f^{n+1}x, fa, t)} - 1 \to 0 \text{ as } n \to +\infty, \quad (4)$$

Becuase $M(a, f^{n+1}x, t)$ and $M(f^{n+1}x, fa, t)$ converge to 1 as $n \to +\infty$. For the first one it is obvious, while the second one we have

$$\int_0^{\frac{1}{M(f^{n+1}x,fa,t)}-1} \varphi(s)ds \leqslant c \int_0^{\frac{1}{M(f^n x,a,t)}-1} \varphi(s)ds \to 0 \text{ as } n \to +\infty.$$

Now, if $M(f^{n+1}x, fa, t)$ does not converge to 1 as $n \to +\infty$, then there exists a subsequence $\{f^{n_\nu+1}x\}_{\nu\in\mathbb{N}} \subseteq \{f^{n+1}x\}_{n\in\mathbb{N}}$ such that $\frac{1}{M(f^{n_\nu+1}x,a,t)} - 1 \geqslant \frac{\varepsilon}{t}$ for a certain $\varepsilon > 0$. Thus, we have following contradictions:

$$0 < \int_0^{\frac{\varepsilon}{t}} \varphi(s)ds \leqslant \int_0^{\frac{1}{M(f^{n_\nu+1}x,fa,t)}-1} \varphi(s)ds \to 0 \text{ as } \nu \to +\infty.$$

Step 5. Uniqueness of the fixed point. Suppose that there are two distinct points $a, b \in X$ such that $fa = a$ and $fb = b$. Then, by (1), we have the contradiction

$$0 < \int_0^{\frac{1}{M(a,b,t)}-1} \varphi(s)ds = \int_0^{\frac{1}{M(fa,fb,t)}-1} \varphi(s)ds \leqslant c \int_0^{\frac{1}{M(a,b,t)}-1} \varphi(s)ds < \int_0^{\frac{1}{M(a,b,t)}-1} \varphi(s)ds.$$

The final step also proves that for each $x \in X$, $\lim_{n\to+\infty} f^n x = a = fa$. The proof is completed. ■

Now we give remark and examples concerning these contractive mappings of integral type, which clarify the connection between our result and the classical ones.

Remark 1 *Theorem 3.1 is a generalization of the Banach principle, letting $\varphi(s) = 1$ for each $s \geqslant 0$ in (1), we have*

$$\int_0^{\frac{1}{M(fx,fy,t)}-1} \varphi(s)ds = \frac{1}{M(fx, fy, t)} - 1 \leqslant c\left(\frac{1}{M(x, y, t)} - 1\right) = c\int_0^{\frac{1}{M(x,y,t)}-1} \varphi(s)ds.$$

Thus, a Banach fuzzy contraction also satisfies (1). The converse is not true as we will see in follow examples.

Example 3.2 Let $X := \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$, $M(x, y, t) = \frac{t}{t+d(x,y)}$ and $N(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$ with metric induced by $\mathbb{R} : d(x, y) := |x - y|$, thus, since X is a closed subset of $\mathbb{R}$, it is a intuitionistic complete fuzzy metric space. We consider now a mapping $f : X \to X$ defined by

$$fx := \begin{cases} \frac{1}{n+1} & x = \frac{1}{n}, n \in \mathbb{N}, \\ 0 & x = 0, \end{cases}$$

Then it satisfies (1) with $\varphi(t) = t^{\frac{1}{t}-2}[1 - \log t]$ for $t > 0$, $\varphi(0) = 0$, and $c = \frac{1}{2}$. In this context $\int_0^r \varphi(s)ds = r^{\frac{1}{r}}$, so that (1), for $x \neq y$ is equivalent to

$$\left(\frac{1}{M(fx,fy,t)} - 1\right)^{1/\left(\frac{1}{M(fx,fy,t)}-1\right)} \leqslant c\left(\frac{1}{M(x,y,t)} - 1\right)^{1/\left(\frac{1}{M(x,y,t)}-1\right)}.$$

If $m, n \in \mathbb{N}$ with $m > n$ and $x = \frac{1}{n}$, $y = \frac{1}{m}$, then we have

$$\begin{aligned}\left(\frac{1}{M(fx,fy,t)} - 1\right)^{1/\left(\frac{1}{M(fx,fy,t)}-1\right)} &= \left|\frac{1}{n+1} - \frac{1}{m+1}\right|^{1/\left|\frac{1}{n+1}-\frac{1}{m+1}\right|} \\ &= \left[\frac{m-n}{(n+1)(m+1)}\right]^{\frac{(n+1)(m+1)}{m-n}}.\end{aligned}$$

On the other hand,

$$\left(\frac{1}{M(x,y,t)} - 1\right)^{1/\left(\frac{1}{M(x,y,t)}-1\right)} \left|\frac{1}{n} - \frac{1}{m}\right|^{1/\left|\frac{1}{n}-\frac{1}{m}\right|} = \left[\frac{m-n}{nm}\right]^{\frac{nm}{m-n}}.$$

Now, we show that

$$\left[\frac{m-n}{(n+1)(m+1)}\right]^{\frac{(n+1)(m+1)}{m-n}} \leqslant \frac{1}{2}\left[\frac{m-n}{nm}\right]^{\frac{nm}{m-n}}$$

or equivalently,

$$\left[\frac{m-n}{(n+1)(m+1)}\right]^{\frac{(n+m+1)}{m-n}} \cdot \left[\frac{nm}{(n+1)(m+1)}\right]^{\frac{nm}{m-n}} \leqslant \frac{1}{2}.$$

Since $nm < (n+1)(m+1)$ and $\frac{nm}{m-n} > 0$, we have $\left[\frac{nm}{(n+1)(m+1)}\right]^{\frac{nm}{m-n}} \leqslant 1$. In addition to, since for all $m, n \in \mathbb{N}$, we have $m \leqslant 3n + nm + 1$, and so $2(m-n) \leqslant (n+1)(m+1)$, we have

$$\left[\frac{m-n}{(n+1)(m+1)}\right]^{\frac{(n+m+1)}{m-n}} \leqslant \frac{1}{2}.$$

On the other hand, taking $x = \frac{1}{n}$ and $y = 0$. For each $n \in \mathbb{N}$, we have $\left[\frac{n}{n+1}\right]^n . \frac{1}{n+1} \leqslant \frac{1}{2}$ and so,

$$\left(\frac{1}{M(fx,fy,t)} - 1\right)^{1/\left(\frac{1}{M(fx,fy,t)}-1\right)} = \left[\frac{1}{n+1}\right]^{n+1} \leqslant \frac{1}{2}\left[\frac{1}{n}\right]^n = \frac{1}{2}\left(\frac{1}{M(x,y,t)} - 1\right)^{1/\left(\frac{1}{M(x,y,t)}-1\right)}$$

Therefore, such mapping f satisfies condition (3.2) with $c = \frac{1}{2}$ and therefore (1) with the same c and for defined by $\varphi(t) = t^{\frac{1}{t}-2}[1-\log t]$ for $t > 0$ and $\varphi(0) = 0$, but

$$\sup_{\{x,y\in X|x\neq y\}} \frac{\frac{1}{M(fx,fy,t)}-1}{\frac{1}{M(x,y,t)}-1} = 1.$$

Thus, it is not a Banach contraction.

***Example* 3.3** Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $fx := x+2$, $\varphi \equiv -2$, $M(x,y,t) = \frac{t}{t+d(x,y)}$, $N(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ and d be the Euclidean distance function. Then, for an arbitrary $c \in (0,1)$, we have

$$\begin{aligned}\int_0^{\frac{1}{M(fx,fy,t)}-1} \varphi(s)ds =& -2\left(\frac{1}{M(fx,fy,t)}-1\right) = -2\left(\frac{1}{M(x,y,t)}-1\right)\\ \leqslant& -2c\left(\frac{1}{M(x,y,t)}-1\right) = c\int_0^{\frac{1}{M(x,y,t)}-1} \varphi(s)ds.\end{aligned}$$

Thus (1) is satisfied with $\varphi \equiv -2$ and for all $c \in (0,1)$, but f, being a translation on $\mathbb{R}^+$, has no fixed points.

Definition 3.4 A self-map f on a intuitionistic fuzzy metric spaces $(X, M, N, *, \lozenge)$ is said to be A-fuzzy contraction if it satisfies the condition

$$\frac{1}{M(fx,fy,t)}-1 \leqslant \alpha\left(\frac{1}{M(x,y,t)}-1, \frac{1}{M(x,fx,t)}-1, \frac{1}{M(y,fy,t)}-1\right)$$

for all $x, y \in X$ and some $\alpha \in A$.

Lemma 3.5 Let a self-map f on a intuitionistic fuzzy metric spaces $(X, M, N, *, \lozenge)$, for all $x, y \in X$, $t > 0$ and some $\beta \in [0, \frac{1}{2})$ satisfying

$$\begin{aligned}\frac{1}{M(fx,fy,t)}-1 \leqslant \beta\max\Bigg\{&\frac{1}{M(fx,x,t)}+\frac{1}{M(fy,y,t)}-2,\\ &\frac{1}{M(fy,y,t)}+\frac{1}{M(x,y,t)}-2, \frac{1}{M(fx,x,t)}+\frac{1}{M(x,y,t)}-2\Bigg\},\end{aligned}$$

is a A-fuzzy contraction.

Proof. Define the map $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ as

$$\alpha(u,v,w) = \beta\max\{u+v, v+w, u+w\}$$

for all $u, v, w \in \mathbb{R}_+$, where β is any fixed number in $[0, \frac{1}{2})$. Then $\alpha \in A$ ([1]). first note that α is continuous, second for

$$u \leqslant \alpha(u,v,v) = \beta\max\{u+v, v+u, v+v\},$$

we consider the following cases.
Case I. $\max\{u+v, v+u, v+v\} = u+v$. In this case, $u \leqslant \frac{\beta}{1-\beta}v$, with $k = \frac{\beta}{1-\beta} \in [0,1)$.

Case II. $\max\{u+v, v+u, v+v\} = 2v$. In this case, $u \leqslant kv$, with $k = 2\beta \in [0,1)$. Similarly, for $u \leqslant \alpha(v,u,v)$ or $u \leqslant \alpha(v,v,u)$ we have $u \leqslant kv$ for some $k \in [0,1)$. Hence,

$$\begin{aligned}\frac{1}{M(fx,fy,t)} - 1 \leqslant &\beta \max\Big\{\frac{1}{M(fx,x,t)} + \frac{1}{M(fy,y,t)} - 2,\\ &\frac{1}{M(fy,y,t)} + \frac{1}{M(x,y,t)} - 2, \frac{1}{M(fx,x,t)} + \frac{1}{M(x,y,t)} - 2\Big\},\\ =&\alpha\left(\frac{1}{M(x,y,t)} - 1, \frac{1}{M(x,x,t)} - 1, \frac{1}{M(fy,y,t)} - 1\right),\end{aligned}$$

by the construction of α. Thus, f is an A-contraction. ■

***Example* 3.6** Let $X = \{0, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$, $M(x,y,t) = \frac{t}{t+d(x,y)}$ and $N(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ with usual metric relative to real line. f be a self-map on X, given by

$$fx = \begin{cases} 12 & x = 0,\\ 11 & otherwise.\end{cases}$$

One can easily verify that f satisfies

$$\begin{aligned}\frac{1}{M(fx,fy,t)} - 1 \leqslant \beta \max\Big\{&\frac{1}{M(fx,x,t)} + \frac{1}{M(fy,y,t)} - 2,\\ &\frac{1}{M(fy,y,t)} + \frac{1}{M(x,y,t)} - 2, \frac{1}{M(fx,x,t)} + \frac{1}{M(x,y,t)} - 2\Big\},\end{aligned}$$

for all $x, y \in X$ and some $\beta \in [0, \frac{1}{2})$. Hence, by Lemma 3.5, f is a A-fuzzy contraction.

Theorem 3.7 Let f be a self-map of a complete intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ satisfying the following condition:

$$\int_0^{\frac{1}{M(fx,fy,t)}-1} \varphi(s)ds \leqslant \alpha\left(\int_0^{\frac{1}{M(x,y,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x,fx,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(y,fy,t)}-1} \varphi(s)ds\right) \tag{5}$$

for each $x, y \in X$ and $t > 0$ with some $\alpha \in A$, where $\varphi : [0,+\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,+\infty)$, nonnegative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)\ ds > 0. \tag{6}$$

Then f has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_{n\to+\infty} f^n x = z$.

Proof. Let $x_0 \in X$ be a arbitrary and define $x_{n+1} = fx_n$. From (5), for each integer

$n \geqslant 1$, we get

$$\begin{aligned}
\int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds &= \int_0^{\frac{1}{M(fx_{n-1},fx_n,t)}-1} \varphi(s)ds \\
&\leqslant \alpha\left(\int_0^{\frac{1}{M(x_{n-1},x_n,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x_{n-1},fx_{n-1},t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x_n,fx_n,t)}-1} \varphi(s)ds\right) \\
&= \alpha\left(\int_0^{\frac{1}{M(x_{n-1},x_n,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x_{n-1},x_n,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds\right).
\end{aligned}$$

Then, by the axiom (A2) of function α,

$$\int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds \leqslant k \int_0^{\frac{1}{M(x_{n-1},x_n,t)}-1} \varphi(s)ds \tag{7}$$

for some $k \in [0, 1)$ as $\alpha \in A$. In this fashion, one can obtain

$$\begin{aligned}
\int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds \leqslant k \int_0^{\frac{1}{M(x_{n-1},x_n,t)}-1} \varphi(s)ds \\
\leqslant k^2 \int_0^{\frac{1}{M(x_{n-2},x_{n-1},t)}-1} \varphi(s)ds \\
\cdots \\
\leqslant k^n \int_0^{\frac{1}{M(x_0,x_1,t)}-1} \varphi(s)ds.
\end{aligned}$$

Taking limit as $n \to +\infty$, we get $\lim_n \int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds = 0$ as $k \in [0, 1)$. Which, from (6) implies that

$$\lim_n \frac{1}{M(x_n, x_{n+1}, 1)} - 1 = 0. \tag{8}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{m_i\}$ and $\{n_i\}$ such that $m_i < n_i < m_{i+1}$ with

$$\frac{1}{M(x_{m_i}, x_{n_i}, t)} - 1 \geqslant \frac{\varepsilon}{t}, \quad \frac{1}{M(x_{m_i}, x_{n_i-1}, t)} - 1 < \frac{\varepsilon}{t}. \tag{9}$$

Now, we have

$$\begin{aligned}
\frac{1}{M(x_{m_i-1}, x_{n_i}, t)} - 1 &\leqslant \frac{1}{M(x_{m_i-1}, x_{m_i}, t)} - 1 + \frac{1}{M(x_{m_i}, x_{n_i-1}, t)} - 1 \\
&< \frac{1}{M(x_{m_i-1}, x_{m_i}, t)} - 1 + \frac{\varepsilon}{t}.
\end{aligned} \tag{10}$$

So, by (8) and (10), we get

$$\lim_i \int_0^{\frac{1}{M(x_{m_i-1},x_{n_i-1},t)}-1} \varphi(s)ds \leqslant \int_0^{\varepsilon} \varphi(s)ds. \tag{11}$$

Using (7), (9) and (11), we have

$$\int_0^{\varepsilon} \varphi(s)ds \leqslant \int_0^{\frac{1}{M(x_{m_i},x_{n_i},t)}-1} \varphi(s)ds \leqslant k \int_0^{\frac{1}{M(x_{m_i-1},x_{n_i-1},t)}-1} \varphi(s)ds \leqslant k \int_0^{\varepsilon} \varphi(s)ds,$$

which is a contradiction (since $k \in [0,1)$). Thus, $\{x_n\}$ is Cauchy and hence, is convergent. Call the limit z. From (5), we get

$$\begin{aligned}\int_0^{\frac{1}{M(fz,x_{n+1},t)}-1} \varphi(s)ds =& \int_0^{\frac{1}{M(fz,fx_n,t)}-1} \varphi(s)ds \\ \leqslant&\alpha\left(\int_0^{\frac{1}{M(z,x_n,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(z,fz,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds\right).\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{\frac{1}{M(fz,z,t)}-1} \varphi(s)ds \leqslant \alpha\left(0, \int_0^{\frac{1}{M(z,fz,t)}-1} \varphi(s)ds, 0\right).$$

So, by the axiom (A2) of function α,

$$\int_0^{\frac{1}{M(fz,z,t)}-1} \varphi(s)ds \leqslant k \cdot 0 = 0,$$

which implies that $\frac{1}{M(fz,z,t)} = 1$ or $fz = z$ (by (6)). Next, suppose that $w \neq z$ be another fixed point of f. From (5) we have

$$\begin{aligned}\int_0^{\frac{1}{M(z,w,t)}-1} \varphi(s)ds =& \int_0^{\frac{1}{M(fz,fw,t)}-1} \varphi(s)ds \\ \leqslant&\alpha\left(\int_0^{\frac{1}{M(z,w,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(z,fz,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(w,fw,t)}-1} \varphi(s)ds\right) \\ =&\alpha\left(\int_0^{\frac{1}{M(z,w,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(z,z,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(w,w,t)}-1} \varphi(s)ds\right) \\ =&\alpha\left(\int_0^{\frac{1}{M(z,w,t)}-1} \varphi(s)ds, 0, 0\right).\end{aligned}$$

So, by axiom (A2) of function α,

$$\int_0^{\frac{1}{M(z,w,t)}-1} \varphi(s)ds = 0$$

, whice implies that $\frac{1}{M(z,w,t)} = 1$ or $z = w$ (by (6)). Hence, the fixed point is unique. ■

Next theorem describes common fixed point of two self-maps on X having two related metrics in integral setting.

Theorem 3.8 Let $(X, M_d, N_d, *, \diamondsuit)$ and $(X, M_\delta, N_\delta, *, \diamondsuit)$ be intuitionistic fuzzy metric spaces with two fuzzy metric $M_d(x,y,t) = \frac{t}{t+d(x,y)}$, $N_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ and $M_\delta(x,y,t) = \frac{t}{t+\delta(x,y)}$, $N_\delta(x,y,t) = \frac{\delta(x,y)}{t+\delta(x,y)}$ satisfying the following conditions:
(i) for all $x, y \in X$,

$$\int_0^{\frac{1}{M_d(x,y,t)}-1} \varphi(s)ds \leqslant \int_0^{\frac{1}{M_\delta(x,y,t)}-1} \varphi(s)ds \text{ and } \int_0^{N_d(x,y,t)} \varphi(s)ds \leqslant \int_0^{N_\delta(x,y,t)} \varphi(s)ds,$$

(ii) $(X, M_d, N_d, *\diamondsuit)$ is complete,
(iii) S, T are self-maps on X such that T is continuous with respect to d and

$$\int_0^{\frac{1}{M_\delta(Tx,Sy,t)}-1} \varphi(s)ds \leqslant \alpha\left(\int_0^{\frac{1}{M_\delta(x,y,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(x,Tx,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(y,Sy,t)}-1} \varphi(s)ds\right) \tag{12}$$

for each $x, y \in X$ and $t > 0$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)\ ds > 0. \tag{13}$$

Then T and S have a unique common fixed point $z \in X$.

Proof. For each integer $n \geqslant 0$, we define $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Then, from (12), we get

$$\begin{aligned}\int_0^{\frac{1}{M_\delta(x_1,x_2,t)}-1} \varphi(s)ds &= \int_0^{\frac{1}{M_\delta(Tx_0,Sx_1,t)}-1} \varphi(s)ds \\ &\leqslant \alpha\left(\int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(x_0,Tx_0,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(x_1,Sx_1,t)}-1} \varphi(s)ds\right) \\ &\leqslant \alpha\left(\int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(x_1,x_2,t)}-1} \varphi(s)ds\right).\end{aligned}$$

Then, by the axiom (A2) function α,

$$\int_0^{\frac{1}{M_\delta(x_1,x_2,t)}-1} \varphi(s)ds \leqslant k \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds$$

for some $k \in [0,1)$. Similarly, one can show that

$$\int_0^{\frac{1}{M_\delta(x_2,x_3,t)}-1} \varphi(s)ds \leqslant k \int_0^{\frac{1}{M_\delta(x_1,x_2,t)}-1} \varphi(s)ds$$

for some $k \in [0,1)$. In general, for any $r \in \mathbb{N}$ odd or even,

$$\int_0^{\frac{1}{M_\delta(x_r,x_{r+1},t)}-1} \varphi(s)ds \leqslant k \int_0^{\frac{1}{M_\delta(x_{r-1},x_r,t)}-1} \varphi(s)ds.$$

Thus, for any $n \in \mathbb{N}$ odd or even, one can easily obtain that

$$\int_0^{\frac{1}{M_\delta(x_n,x_{n+1},t)}-1} \varphi(s)ds \leqslant k^n \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds.$$

Then, by the condition (i) of the theorem, we obtain

$$\int_0^{\frac{1}{M_d(x_n,x_{n+1},t)}-1} \varphi(s)ds \leqslant \int_0^{\frac{1}{M_\delta(x_n,x_{n+1},t)}-1} \varphi(s)ds \leqslant k^n \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds.$$

Taking limit as $n \to \infty$, we get

$$\lim_n \int_0^{\frac{1}{M(x_n,x_{n+1},t)}-1} \varphi(s)ds = 0$$

as $k \in [0,1)$, which from (13) implies that $\lim_n \frac{1}{M(x_n,x_{n+1},t)} - 1 = 0$ or $M(x_n, x_{n+1}, t) = 1$. We now show that $\{x_n\}$ is a Cauchy sequence with respect to $(X, M_d, N_d, *, \Diamond)$. For any integer $p > 0$,

$$\begin{aligned}
\int_0^{\frac{1}{M(x_n,x_{n+p},t)}-1} \varphi(s)ds \leqslant & \int_0^{\frac{1}{M_\delta(x_n,x_{n+p},t)}-1} \varphi(s)ds \\
\leqslant & \int_0^{\frac{1}{M_\delta(x_n,x_{n+1},t)}-1} \varphi(s)ds + \int_0^{\frac{1}{M_\delta(x_{n+1},x_{n+2},t)}-1} \varphi(s)ds \\
& + \cdots + \int_0^{\frac{1}{M_\delta(x_{n+p-1},x_{n+p},t)}-1} \varphi(s)ds \\
\leqslant & k^n \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds + k^{n+1} \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds \\
& + \cdots + k^{n+p-1} \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds \\
\leqslant & \frac{k^n}{1-k} \int_0^{\frac{1}{M_\delta(x_0,x_1,t)}-1} \varphi(s)ds \to 0 \quad \text{as} \quad n \to +\infty,
\end{aligned}$$

since $k \in [0,1)$. Therefore, $\{x_n\}$ is Cauchy. Hence, by completeness of X, $\{x_n\}$ converges to some $z \in X$, i.e. $\frac{1}{M_d(x_n,z,t)} - 1 \to 0$ or $M_d(x_n, z, t) = 1$ as $n \to +\infty$ for some $z \in X$.

Since T is continuous with the respect to d, we get

$$0=\lim_n \int_0^{\frac{1}{M_d(x_{2n+1},z,t)}-1}\varphi(s)ds=\lim_n \int_0^{\frac{1}{M_d(Tx_{2n},z,t)}-1}\varphi(s)ds=\lim_n \int_0^{\frac{1}{M_d(Tz,z,t)}-1}\varphi(s)ds.$$

So, by (13), $\frac{1}{M_d(Tz,z,t)}-1=0$ or $M_d(Tz,z,t)=1$ i.e. $Tz=z$. Now, by (12), we have

$$\begin{aligned}\int_0^{\frac{1}{M_\delta(z,Sz,t)}-1}\varphi(s)ds &= \int_0^{\frac{1}{M_\delta(Tz,Sz,t)}-1}\varphi(s)ds\\ &\leqslant \alpha\left(\int_0^{\frac{1}{M_\delta(z,z,t)}-1}\varphi(s)ds,\ \int_0^{\frac{1}{M_\delta(z,Tz,t)}-1}\varphi(s)ds, \int_0^{\frac{1}{M_\delta(z,Sz,t)}-1}\varphi(s)ds\right)\\ &\leqslant \alpha\left(0,0,\int_0^{\frac{1}{M_\delta(z,Sz,t)}-1}\varphi(s)ds\right).\end{aligned}$$

Then, by the axiom (A2) of function α,

$$\int_0^{\frac{1}{M_\delta(z,Sz,t)}-1}\varphi(s)ds\leqslant k\cdot 0=0$$

and by (13), $M_\delta(z,Sz,t)=1$ or $Sz=z$. Thus z is a common fixed point of S and T.

Let $w\neq z$ be another common fixed point of S and T in X. Then by (12)

$$\begin{aligned}\int_0^{\frac{1}{M_\delta(z,w,t)}-1}\varphi(s)ds &= \int_0^{\frac{1}{M_\delta(Tz,Sw,t)}-1}\varphi(s)ds\\ &\leqslant \alpha\left(\int_0^{\frac{1}{M_\delta(z,w,t)}-1}\varphi(s)ds,\ \int_0^{\frac{1}{M_\delta(z,Tz,t)}-1}\varphi(s)ds, \int_0^{\frac{1}{M_\delta(w,Sw,t)}-1}\varphi(s)ds\right)\\ &\leqslant \alpha\left(\int_0^{\frac{1}{M_\delta(z,w,t)}-1}\varphi(s)ds,0,0\right)\\ &\leqslant k\cdot 0=0 \quad \text{as} \quad \alpha\in A.\end{aligned}$$

Then by (13) we have $\frac{1}{M_\delta(z,w,t)}-1=0$ or $M_\delta(z,w,t)=1$, hence $z=w$. ■

If $S=T$, then the Theorem 3.8 gives as follow.

Corollary 3.9 Let $(X,M_d,N_d,*,\Diamond)$ and $(X,M_\delta,N_\delta,*,\Diamond)$ be intuitionistic fuzzy metric spaces with two fuzzy metric $M_d(x,y,t)=\frac{t}{t+d(x,y)}$, $N_d(x,y,t)=\frac{d(x,y)}{t+d(x,y)}$ and $M_\delta(x,y,t)=\frac{t}{t+\delta(x,y)}$, $N_\delta(x,y,t)=\frac{\delta(x,y)}{t+\delta(x,y)}$ satisfying the following conditions:
(i) for all $x,y\in X$,

$$\int_0^{\frac{1}{M_d(x,y,t)}-1}\varphi(s)ds\leqslant \int_0^{\frac{1}{M_\delta(x,y,t)}-1}\varphi(s)ds,$$

$$\int_0^{N_d(x,y,t)} \varphi(s)ds \leqslant \int_0^{N_\delta(x,y,t)} \varphi(s)ds,$$

(ii) $(X, M_d, N_d, *, \Diamond)$ is complete,
(iii) T is self-map on X such that T is continuous with respect to d and

$$\int_0^{\frac{1}{M_\delta(Tx,Ty,t)}-1} \varphi(s)ds \leqslant \alpha \left(\int_0^{\frac{1}{M_\delta(x,y,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(x,Tx,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M_\delta(y,Ty,t)}-1} \varphi(s)ds \right)$$

for each $x, y \in X$ and $t > 0$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(s)\ ds > 0.$$

Then T has a unique fixed point $z \in X$.

***Example* 3.10** Consider X as Example 3.6, $M(x, y, t) = \frac{t}{t+d(x,y)}$ and $N(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$ with usual metric relative to real line. Define f on X by

$$fx = \begin{cases} 12 & x = 0, \\ 11 & otherwise. \end{cases}$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $\varphi(s) = \frac{s-1}{s}$ for all $s \in \mathbb{R}_+$. Then $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative, and such that for each $\varphi > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$. Now, as we know from Example 3.6, a self-map f satisfying

$$\frac{1}{M(fx, fy, t)} - 1 \leqslant \beta \max\left\{ \frac{1}{M(fx, x, t)} + \frac{1}{M(fy, y, t)} - 2, \frac{1}{M(fy, y, t)} + \frac{1}{M(x, y, t)} - 2, \frac{1}{M(fx, x, t)} + \frac{1}{M(x, y, t)} - 2 \right\}$$

for all $x, y \in X$, $t > 0$ and some $\beta \in [0, \frac{1}{2})$, is an A-fuzzy contraction. We have

$$\begin{aligned} \int_0^{\frac{1}{M(fx,fy,t)}-1} \varphi(s)ds \leqslant& \alpha \left(\int_0^{\frac{1}{M(x,y,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(x,fx,t)}-1} \varphi(s)ds, \int_0^{\frac{1}{M(y,fy,t)}-1} \varphi(s)ds \right) \\ =& \beta \max\left\{ \int_0^{\frac{1}{M(fx,x,t)}+\frac{1}{M(x,y,t)}-2} \varphi(s)ds, \right. \\ & \left. \int_0^{\frac{1}{M(fx,x,t)}+\frac{1}{M(fy,y,t)}-2} \varphi(s)ds, \int_0^{\frac{1}{M(fy,y,t)}+\frac{1}{M(x,y,t)}-2} \varphi(s)ds \right\}, \end{aligned}$$

which is satisfied for all $x, y \in X$, $t > 0$ and some $\beta \in [0, \frac{1}{2})$.

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