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Some local fixed point results under C-class functions with applications to coupled elliptic systems

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Abstract. The main objective of the paper is to state newly fixed point theorems for set-valued mappings in the framework of 0-complete partial metric spaces which speak about a location of a fixed point with respect to an initial value of the set-valued mapping by using some C-class functions. The results proved herein generalize, modify and unify some recent results of the existing literature. As an application, we provide an existence theorem for a coupled elliptic system subject to various two-point boundary conditions.

Keywords: Fixed point, set-valued mapping, partial metric space, coupled elliptic system, C-class function.

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1. Introduction

In the study of differential inclusions or differential equations, the topological methods are used to give us the qualitative information about existence, stability, periodicity of solutions. The fixed point theorem and topological degree are the most topological techniques used, which are closely connected. In the present paper we are interesting about fixed point theory for set-valued mappings on 0-complete partial metric spaces.

Recall that the partial metric is an interesting distance function introduced by Matthews [22]. The motivation behind introducing the concept of a partial metric space

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is to solve certain problems arising in computer science. In [26], Romaguera introduced the notion of a 0-Cauchy sequence in a partial metric space and then the 0-complete partial metric space.

The aim of the paper is to improve the fixed point theorem mentioned on [8] to 0-complete partial metric spaces by using the concept of C-class function, which is introduced by Ansari in [4], and state an existence theorem for the following coupled elliptic system subject to various two-point boundary conditions

$$\begin{cases} -u_1'' = f(t,u_1,u_2) - \lambda & t \in (0,1) \\ -u_2'' = g(t,u_1,u_2) - \mu & t \in (0,1) \\ \alpha_i u_i(0) - \beta_i u_i'(0) = 0 & \text{i=1,2} \\ \gamma_i u_i(1) + \delta_i u_i'(1) = 0 & \text{i=1,2} \end{cases} \tag{1}$$

where $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The constants $\lambda, \mu, \alpha_i, \beta_i, \gamma_i$ and δ_i are such that, for each $i \in \{1,2\}$,

$$\lambda, \mu, \beta_i, \delta_i \geqslant 0, \quad \alpha_i + \beta_i > 0, \quad \gamma_i + \delta_i > 0, \quad k_i := \alpha_i\gamma_i + \alpha_i\delta_i + \beta_i\gamma_i > 0.$$

The paper is organized as follows: in section 2, we give some definitions and recall a few preliminary results. In section 3, we introduced a type of functions needed to prove our main results. Moreover, we give some related corollaries. To illustrate an application of our results, we prove the local existence of solutions to boundary value problem mentioned below in the last section.

2. Notations and preliminary results

We begin with the following definition.

Definition 2.1 Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

(P_1) $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$;
(P_2) $p(x,x) \leqslant p(x,y)$;
(P_3) $p(x,y) = p(y,x)$;
(P_4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is then called a partial metric space. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$. The closed p-ball of radius r centered at x is denoted by $\overline{B_p}(x,r)$ where

$$\overline{B_p}(x,r) = \{y \in X : p(x,y) \leqslant p(x,x) + r\}.$$

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Let (X, p) be a partial metric space. Then

- A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$.
- A sequence $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to +\infty} p(x_n, x_m)$. If $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$, then $\{x_n\}$ is said to be a 0-Cauchy sequence in (X, p).
- (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.
- (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.
- Each 0-Cauchy sequence in (X, p) is a Cauchy in (X, p^s).
- Every complete partial metric space is 0-complete.
- 0-complete partial metric space need not be complete.

Lemma 2.2 Let (X, p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore,

$$\lim_{n \to +\infty} p^s(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m),$$

where x is a limit of $\{x_n\}$ in (X, p^s).

Lemma 2.3 [22] Let (X, p) be a partial metric space. Then

(1) if $p(x, y) = 0$, then $x = y$. But if $x = y$, then $p(x, y)$ may not be zero;

(2) if $x \neq y$, then $p(x, y) > 0$.

Let $C^p(X)$ be the family of all nonempty and closed subsets of the partial metric space (X, p). For $x \in X$ and $A, B \in C^p(X)$, we define $p(x, A) = \inf\{p(x, a), a \in A\}$ and $\delta_p(A, B) = \sup\{p(a, B), a \in A\}$ with the convention

$$\delta_p(\emptyset, B) = 0. \tag{2}$$

Lemma 2.4 [3] Let (X, p) be a partial metric space and $A \subset X$. Then $p(a, A) = p(a, a) \Leftrightarrow a \in \overline{A}$. Moreover, $p(a, A) = 0 \Leftrightarrow p(a, a) = 0$ and $a \in \overline{A}$, where $\overline{A}$ denotes the closure of A with respect to the partial metric p.

Proof. For the second part, we argue by contradiction. Let $a \in X$ and $A \subset X$ where $p(a, A) = 0$ such that $p(a, a) \neq 0$ or $a \notin \overline{A}$. Let $z \in \overline{A}$ such that $p(a, \overline{A}) = p(a, z)$. If $p(a, a) \neq 0$ then the use of property (P_2) of partial metric gives

$$0 < p(a, a) \leqslant p(a, z) = p(a, \overline{A}) \leqslant p(a, A) = 0$$

which is a contradiction. If $a \notin \overline{A}$ then $a \neq z$ and, by using Lemma 2.3, we have

$$0 < p(a, z) = p(a, \overline{A}) \leqslant p(a, A) = 0$$

which is also a contradiction. Hence $p(a, A) = 0$ implies that $p(a, a) = 0$ and $a \in \overline{A}$. ■

Note that A is closed in (X, p) if and only if $\overline{A} = A$.

Lemma 2.5 [8] Let (X,p) be a partial metric space. Let $x \in X$ and $A \in C^p(X)$. If $p(x,A) < \mu$ $(\mu > 0)$ then there exists $a \in A$ such that $p(x,a) < \mu$.

Note that the mapping $\delta_p : C^p(X) \times C^p(X) \to [0,+\infty]$ satisfying the following properties.

Proposition 2.6 [5] Let (X,p) be a partial metric space. For any $A,\ B,\ C \in\ C^p(X)$, we have the following:

(i) $\delta_p(A,A) = \sup\{p(a,a), a \in A\}$;
(ii) $\delta_p(A,A) \leqslant \delta_p(A,B)$;
(iii) $\delta_p(A,B) = 0 \Rightarrow A \subseteq B$;
(iv) $\delta_p(A,B) \leqslant \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

In the sequel, J and J' will denote intervals on $\mathbb{R}^+$ containing 0, that are intervals of the form $[0,a)$, $[0,a]$ or $[0,+\infty)$.

Definition 2.7 [9] A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function (or (c)-comparison) on J if

$$s(t) := \sum_{n=0}^{\infty} \varphi^n(t) \text{ is convergent for all } t \in J,$$

where φ^n denotes the n-th iteration of the function φ and $\varphi^0(t) = t$ i.e.

$$\varphi^0(t) = t,\ \varphi^1(t) = \varphi(t),\ \varphi^2(t) = \varphi(\varphi(t)),\ \dots,\ \varphi^n(t) = \varphi(\varphi^{n-1}(t)).$$

In [4], Ansari introduced the concept of C-class functions as follows:

Definition 2.8 [4] Let $F : J \times J' \to \mathbb{R}$ be a continuous mapping. We say that F is a C-class function if it satisfies the following conditions:

(F_1) $F(s,t) \leqslant s$, for all $(s,t) \in J \times J'$.
(F_2) $F(s,t) = s$ implies that $s = 0$ or $t = 0$.

Note that $F(0,0) = 0$. We denote by $\mathcal{C}$ the set of all C-class functions on $J \times J'$

***Example* 2.9** The following functions $F : J \times J' \to \mathbb{R}$ are elements of $\mathcal{C}$:

(1) $F(s,t) = s - t$, $J = J' = [0,\infty)$;
(2) $F(s,t) = ms$, $0\leqslant m<1$, $J = J' = [0,\infty)$;
(3) $F(s,t) = s - \varphi(t)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$ and $J = J' = [0,\infty)$;
(4) $F(s,t) = s^\alpha$, where $\alpha > 1$ and $J \times J' = [0,1) \times [0,\infty)$;.
(5) $F(s,t) = st^k$, where $k > 1$ and $J \times J' = [0,\infty) \times [0,1)$

For more examples on C-class functions one can refer to [4, 11, 18].

Definition 2.10 We denote by $\mathcal{C}_I$ the collection of C-class functions satisfying the following

- $F(s,t)$ is nondecreasing in s and in t
- for any fixed $t \in J'$ we have

$$w(s,t) := \sum_{n=0}^{\infty} F^n(s,t) \text{ is convergent for all } s \in J,$$

where F^n denotes the n-th iteration of the function F satisfying the following:

$$F^0(s,t) = s,\ F^1(s,t) = F(s,t) \text{ and } F^{n+1}(s,t) = F(F^n(s,t),t).$$

Definition 2.11 We denote by $\mathcal{C}_{II}$ the collection of C-class functions satisfying the following

- $F(s,t)$ is nondecreasing in s and nonincreasing in t
- for any fixed $t \in J'$ we have

$$w(s,t) := \sum_{n=0}^{\infty} F^n(s,t) \text{ is convergent for all } s \in J,$$

where F^n denotes the n-th iteration of the function F satisfying the following:

$$F^0(s,t) = s,\ F^1(s,t) = F(s,t) \text{ and } F^{n+1}(s,t) = F(F^n(s,t), F^n(s,t)).$$

Remark 1 *The functions w and F satisfy the functional equation*

$$w(s,t) = s + w(F(s,t),t), \qquad if \quad F \in \mathcal{C}_I$$

and

$$w(s,t) = s + w(F(s,t),F(s,t)), \qquad if \quad F \in \mathcal{C}_{II}.$$

The following examples illustrate Definition 2.10 and 2.11.

***Example* 2.12**

- $F(s,t) = s - t \Rightarrow w(s,t) = 2s - t$ and then $F \in \mathcal{C}_{II}$.
- $F(s,t) = \lambda s \Rightarrow w(s,t) = \dfrac{s}{1-\lambda}$ for $\lambda \in [0,1)$ and then $F \in \mathcal{C}_I \cap \mathcal{C}_{II}$.
- $F(s,t) = \varphi(s)$ where φ is a Bianchini-Grandolfi gauge function and then $F \in \mathcal{C}_I \cap \mathcal{C}_{II}$.
- $F(s,t) = \frac{s^2}{2\sqrt{s^2+a^2}}$, where $a \geqslant 0 \Rightarrow w(s,t) = s + \sqrt{s^2+a^2} - a$ for $s,t \geqslant 0$ and then $F \in \mathcal{C}_I \cap \mathcal{C}_{II}$.
- $F(s,t) = st^k \Rightarrow w(s,t) = \dfrac{s}{1-t^k}$, where $k > 1$ and then $F \in \mathcal{C}_I$.

3. The main results

In this section, we denote

$$M(x,y) := \max\left\{p(x,y), p(x,\phi(x)), p(y,\phi(y)), \frac{p(x,\phi(y)) + p(y,\phi(x))}{2}\right\}.$$

At first, we introduce the following concept needed on the rest.

Definition 3.1 We denote by Ξ the class of functions $\tau : X^2 \times (2^X)^2 \to J'$ satisfying the following: $\tau(x,y,A,C) = 0$ implies at least $p(x,y) = 0$ or $x = y$, for any $x,y \in X$ and $A, C \in 2^X$

***Example* 3.2**

- $\tau(x,y,A,C) = L(\delta_p(A,C) + p(x,y))$ where $L > 0$.
- $\tau(x,y,A_x,C_y) = p(x,y) - \min\{p(x,x), p(y,y)\}$.
- $\tau(x,y,A_x,C_y) = p^s(x,y) + \min\{p(x,A_x), p(y,C_y), p(x,C_y), p(y,A_x), \delta_p(A_x,C_y)\}$.
- $\tau(x,y,A_x,C_y) = \int_0^{p(x,y)} f(s) \ \mathrm{d}s$ where f be positive function.

Definition 3.3 We say that $\tau \in \Xi$ is a nondecreasing on (X,p) if

$$p(x,y) \leqslant p(a,b) \Rightarrow \tau(x,y,A_x,C_y) \leqslant \tau(a,b,A_a,C_b) \quad \forall A_x, A_a, C_y, C_b \in 2^X.$$

Now, we are ready to state and prove our main result.

Theorem 3.4 Let (X, p) be a partial metric space such that, for $\overline{x} \in X$ and $r > 0$, $\overline{B_p}(\overline{x}, r)$ be a 0-complete subspace of X. Let $\phi : \overline{B_p}(\overline{x}, r) \to C^p(X)$ be a set-valued mapping. Let $F \in \mathcal{C}$, $\tau \in \Xi$ and $\alpha \in J$ satisfying one of the following

- $F \in \mathcal{C}_I$ and τ is nondecreasing,
- $F \in \mathcal{C}_{II}$ and $\tau(x, y, \phi(x), \phi(y)) \geqslant \alpha$ where $x, y \in \overline{B_p}(\overline{x}, r)$.

We assume that the following two conditions hold:

(a) $\delta_p(\phi(x) \cap \overline{B_p}(\overline{x}, r), \phi(y)) \leqslant F(M(x, y), \tau(x, y, \phi(x), \phi(y))) \;\; \forall \;\; x, y \in \overline{B_p}(\overline{x}, r)$,
(b) $p(\overline{x}, \phi(\overline{x})) < \alpha$ where $w(\alpha, \cdot) \leqslant p(\overline{x}, \overline{x}) + r$.

Then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$ then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x}, r)$.

Proof. If $\overline{x} \in \phi(\overline{x})$ or $F \equiv 0$ the proof is finished. So we assume that $\overline{x} \notin \phi(\overline{x})$ and $F \not\equiv 0$. According to the second condition, and using lemma 2.5, there exists $x_1 \in \phi(\overline{x}) \cap \overline{B}_p(\overline{x}, r)$ such that

$$p(\overline{x}, x_1) = \begin{cases} F^0(p(\overline{x}, x_1), \tau(\overline{x}, x_1, \phi(\overline{x}), \phi(x_1))) < \alpha \text{ or} \\ F^0(p(\overline{x}, x_1), p(\overline{x}, x_1)) < \alpha, \end{cases}$$

By induction we construct a sequence $\{x_k\}$ satisfying:

$$\begin{aligned} x_0 &= \overline{x} \\ x_{k+1} &\in \phi(x_k) \cap \overline{B_p}(x_0, r) \\ p(x_k, x_{k+1}) &\leqslant \psi^k(p(x_0, x_1)) \leqslant p(x_0, x_1) \\ \psi^k(p(x_0, x_1)) &= \begin{cases} F^k(p(x_0, x_1), \tau(x_0, x_1, \phi(x_0), \phi(x_1))), \text{if } F \in \mathcal{C}_I \text{ and } \tau \text{ is nondecreasing;} \\ F^k(p(x_0, x_1), p(x_0, x_1)), \text{if } F \in \mathcal{C}_{II} \text{ and } \tau(x_k, x_{k+1}, \phi(x_k), \phi(x_{k+1}) \geqslant \alpha \end{cases} \end{aligned} \tag{3}$$

If $x_k = x_{k+1}$ or $x_k \in \phi(x_k)$ for some $k \in \mathbb{N}$, we are done. So we suppose that, for all $k \in \mathbb{N}$, $x_k \notin \phi(x_k)$ and $x_k \neq x_{k+1}$ and then $p(x_k, x_{k+1}) > 0$.

First, we show that the sequence $\{x_k\}$ satisfying (3) gives $M(x_{k-1}, x_k) \leqslant p(x_{k-1}, x_k) \in J$. Indeed, let $k \in \mathbb{N}^*$. Then we have

$$\begin{aligned} M(x_{k-1}, x_k) &= \max \Big\{ p(x_{k-1}, x_k), p(x_{k-1}, \phi(x_{k-1})), p(x_k, \phi(x_k)), \\ &\qquad \frac{p(x_{k-1}, \phi(x_k)) + p(x_k, \phi(x_{k-1}))}{2} \Big\} \\ &= \max \left\{ p(x_{k-1}, x_k), p(x_k, \phi(x_k)), \frac{p(x_{k-1}, \phi(x_k)) + p(x_k, x_k)}{2} \right\} \\ &\leqslant \max \left\{ p(x_{k-1}, x_k), p(x_k, x_{k+1}), \frac{p(x_{k-1}, x_{k+1}) + p(x_k, x_k)}{2} \right\} \\ &\leqslant \max \left\{ p(x_{k-1}, x_k), p(x_k, x_{k+1}), \frac{p(x_{k-1}, x_k) + p(x_k, x_{k+1})}{2} \right\} \\ &= \max \left\{ p(x_{k-1}, x_k), p(x_k, x_{k+1}) \right\}. \end{aligned}$$

If $\max\{p(x_{k-1},x_k),p(x_k,x_{k+1})\}=p(x_k,x_{k+1})$, then by condition (a) and the definition of F yields a contradiction. Therefore, we must have $M(x_{k-1},x_k)\leqslant p(x_{k-1},x_k)\in J$. We start then by using assumption (a) and we have

$$\begin{aligned}p(x_{k+1},\phi(x_{k+1})) &\leqslant \delta_p(\phi(x_k)\cap\overline{B_p}(x_0,r),\phi(x_{k+1}))\\ &\leqslant F(M(x_k,x_{k+1}),\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1})))\\ &\leqslant M(x_k,x_{k+1}).\end{aligned}$$

If we assume that $M(x_k,x_{k+1})\leqslant p(x_{k+1},\phi(x_{k+1}))$ or $\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1}))=0$ for some $k\in\mathbb{N}$ then we have $F(M(x_k,x_{k+1}),\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1})))=M(x_k,x_{k+1})$, which implies that $M(x_k,x_{k+1})=0$ or $\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1}))=0$ and then $x_k=x_{k+1}$ or $p(x_k,x_{k+1})=0$ which is a contradiction.

So we assume that $p(x_{k+1},\phi(x_{k+1}))<M(x_k,x_{k+1})$ and $\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1}))\neq 0$ for all $k\in\mathbb{N}$ and then there exists $x_{k+2}\in\phi(x_{k+1})$ such that

$$p(x_{k+1},x_{k+2})<M(x_k,x_{k+1})\leqslant p(x_k,x_{k+1}).$$

Moreover, if $F\in\mathcal{C}_I$ and τ is nondecreasing then we have

$$\begin{aligned}p(x_{k+1},x_{k+2}) &\leqslant \delta_p(\phi(x_k)\cap\overline{B_p}(x_0,r),\phi(x_{k+1}))\\ &\leqslant F(M(x_k,x_{k+1}),\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1})))\\ &\leqslant F(p(x_k,x_{k+1}),\tau(x_0,x_1,\phi(x_0),\phi(x_1)))\\ &\leqslant F(\psi^k(p(x_0,x_1)),\tau(x_0,x_1,\phi(x_0),\phi(x_1)))\\ &\leqslant \psi^{k+1}(p(x_0,x_1))\end{aligned}$$

else if $F\in\mathcal{C}_{II}$ and $\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1})\geqslant\alpha$, then

$$\begin{aligned}p(x_{k+1},x_{k+2}) &\leqslant \delta_p(\phi(x_k)\cap\overline{B_p}(x_0,r),\phi(x_{k+1}))\\ &\leqslant F(M(x_k,x_{k+1}),\tau(x_k,x_{k+1},\phi(x_k),\phi(x_{k+1})))\\ &\leqslant F(p(x_k,x_{k+1}),\alpha)\\ &\leqslant F(\psi^k(p(x_0,x_1)),p(x_0,x_1))\\ &\leqslant F(\psi^k(p(x_0,x_1)),\psi^k(p(x_0,x_1)))\\ &\leqslant \psi^{k+1}(p(x_0,x_1)).\end{aligned}$$

On the other hand, x_{k+2} be an element of the closed p-ball $\overline{B_p}(x_0,r)$. Indeed,

$$\begin{aligned}p(x_{k+2},x_0) &\leqslant \sum_{j=0}^{k+1}p(x_{j+1},x_j)-\sum_{j=1}^{k+1}p(x_j,x_j)\\ &\leqslant \sum_{j=0}^{+\infty}\psi^j(p(x_1,x_0))\\ &\leqslant w(\alpha,\cdot)\\ &\leqslant p(x_0,x_0)+r.\end{aligned}$$

For all integers n and m such that $n > m$, we have

$$\begin{aligned} p(x_n, x_m) &\leqslant \sum_{k=m}^{n-1} p(x_k, x_{k+1}) - \sum_{k=m+1}^{n-1} p(x_k, x_k) \\ &\leqslant \sum_{k=m}^{n-1} \psi^k(p(x_0, x_1)) \\ &\leqslant \sum_{k=0}^{+\infty} \psi^k(p(x_0, x_1)) \\ &\leqslant w(\alpha, \cdot). \end{aligned}$$

Since $w(s, \cdot)$ is convergent for each $s \in J$, we obtain that $\{x_n\}$ is a 0-Cauchy sequence in $\overline{B_p}(x_0, r)$. Since $\overline{B_p}(x_0, r)$ is 0-complete subspace then $\{x_n\}$ converges, with respect to τ_p, to a point $x^* \in \overline{B_p}(\overline{x}, r)$ such that

$$p(x^*, x^*) = \lim_{n \to +\infty} p(x_n, x^*) = 0.$$

We assert now that $x^* \in \phi(x^*)$. The modified triangle inequality and assumption (a) give

$$\begin{aligned} p(x^*, \phi(x^*)) &\leqslant p(x^*, x_k) + p(x_k, \phi(x^*)) - p(x_k, x_k) \\ &\leqslant p(x^*, x_k) + \delta_p(\phi(x_{k-1}) \cap \overline{B_p}(x_0, r), \phi(x^*)) \\ &\leqslant p(x^*, x_k) + F(M(x_{k-1}, x^*), \tau(x_k, x^*, \phi(x_k), \phi(x^*)))) \\ &\leqslant p(x^*, x_k) + M(x_{k-1}, x^*) \\ &\leqslant p(x^*, x_k) + p(x_{k-1}, x^*) \end{aligned}$$

Taking limit as $k \to +\infty$, we obtain $p(x^*, \phi(x^*)) = 0 = p(x^*, x^*)$ which from Lemma 2.4 implies that $x^* \in \overline{\phi(x^*)} = \phi(x^*)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, we suppose that there exist two fixed points $x^*, x^{**} \in \overline{B_p}(x_0, r)$. Then, we have

$$\begin{aligned} M(x^*, x^{**}) &\leqslant p(x^*, x^{**}) \\ &\leqslant p(x^*, x_0) + p(x_0, x^{**}) - p(x_0, x_0) \\ &\leqslant p(x_0, x_0) + 2r \in J \end{aligned}$$

and

$$\begin{aligned} M(x^*, x^{**}) &\leqslant p(x^*, x^{**}) \\ &= p(x^*, \phi(x^{**})) \\ &\leqslant \delta_p(\phi(x^*) \cap \overline{B_p}(x_0, r), \phi(x^{**})) \\ &\leqslant F(M(x^*, x^{**}), \tau(x^*, x^{**}, \phi(x^*), \phi(x^{**}))) \\ &\leqslant M(x^*, x^{**}). \end{aligned}$$

Hence, we have

$$F(M(x^*, x^{**}), \tau(x^*, x^{**}, \phi(x^*), \phi(x^{**}))) = M(x^*, x^{**})$$

which implies that $M(x^*, x^{**}) = 0$ or $\tau(x^*, x^{**}, \phi(x^*), \phi(x^{**})) = 0$, thus $p(x^*, x^{**}) = 0$ or $x^* = x^{**}$ which is a contradiction and the proof is completed. ■

For $F(s,t) = \varphi(s)$ a Bianchini-Grandolfi gauge function we get the following corollary.

Corollary 3.5 Let (X, p) be a partial metric space. Let $\overline{x} \in X$ and $r > 0$ such that $\overline{B_p}(\overline{x}, r)$ be a 0-complete subspace of X. Let $\phi : \overline{B_p}(\overline{x}, r) \to C^p(X)$ be a set-valued mapping and let φ a Bianchini-Grandolfi-gauge function on J. If there exists $\alpha \in J$ such that the following two conditions hold:

(a) $p(\overline{x}, \phi(\overline{x})) < \alpha$ where $s(\alpha) \leqslant p(\overline{x}, \overline{x}) + r$,
(b) $\delta_p(\phi(x) \cap \overline{B_p}(\overline{x}, r), \phi(y)) \leqslant \varphi(M(x, y)) \quad \forall x, y \in \overline{B_p}(\overline{x}, r)$,

then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x}, r)$.

Proof. Since $F(s,t) = \varphi(s)$ be a C-class function does not depend on second variable t, it can be choose any $\tau \in \Xi$ such that τ is nondecreasing or greater than α and then apply Theorem 3.4. ■

Remark 2 *Corollary 3.5 extends [8, Theorem 3.2] on 0-complete partial metric spaces and then the results in [3, 5–7, 14–17, 19–22, 24, 27]).*

As special case, for $F(s,t) = \lambda s$ we have

Corollary 3.6 Let (X, p) be a partial metric space. Let $\overline{x} \in X$, $\lambda \in [0, 1)$ and $r > 0$ such that $\overline{B_p}(\overline{x}, r)$ be a 0-complete subspace of X. Let $\phi : \overline{B_p}(\overline{x}, r) \to C^p(X)$ be a set-valued mapping such that the following two conditions hold:

(a) $p(\overline{x}, \phi(\overline{x})) < (p(\overline{x}, \overline{x}) + r)(1 - \lambda)$,
(b) $\delta_p(\phi(x) \cap \overline{B_p}(\overline{x}, r), \phi(y)) \leqslant \lambda M(x, y) \quad \forall x, y \in \overline{B_p}(\overline{x}, r)$,

then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping, then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x}, r)$.

Proof. We apply Corollary 3.6 for $\varphi(t) = \lambda t$ which is a Bianchini-Grandolfi gauge function on $J = [0, +\infty)$ and $s(t) = \dfrac{t}{1 - \lambda}$. Take $\alpha = (p(\overline{x}, \overline{x}) + r)(1 - \lambda) \in J$ and complete the proof. ■

For $F(s,t) = s - t$ and $\tau(x, y, \phi(x), \phi(y)) = \alpha + \psi(x, y, \phi(x), \phi(y))$ such that $\psi : X \times X \times C^p(X) \times C^p(X) \to [0, +\infty)$ a function, then we get the following corollary.

Corollary 3.7 Let (X, p) be a partial metric space. Let $\overline{x} \in X$ and $r > 0$ such that $\overline{B_p}(\overline{x}, r)$ be a 0-complete subspace of X. Let $\phi : \overline{B_p}(\overline{x}, r) \to C^p(X)$ be a set-valued mapping. If there exists $\alpha \geqslant 0$ such that the following two conditions hold:

(a) $p(\overline{x}, \phi(\overline{x})) < \alpha \leqslant \dfrac{1}{2}\,(p(\overline{x}, \overline{x}) + r)$,
(b) $\delta_p(\phi(x) \cap \overline{B_p}(\overline{x}, r), \phi(y)) + \alpha \leqslant M(x, y) - \psi(x, y, \phi(x), \phi(y)) \quad \forall x, y \in \overline{B_p}(\overline{x}, r)$,

then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x}, r)$.

Proof. Since $F(s,t) = s - t$ be a C-class function for $J = J' = [0, +\infty)$ then we get

$$w(\alpha, t) = 2\alpha - t \leqslant 2\alpha$$

for each $t \in J'$. As $p(\overline{x}, \overline{x}) + 2r \in J$, it sufficient to take $2\alpha \leqslant p(\overline{x}, \overline{x}) + r$ and apply Theorem 3.4 to complete the proof. ■

Similarly, for $F(s,t) = s - t$ and $\tau(x, y, \phi(x), \phi(y)) = \dfrac{\alpha}{\psi(x, y, \phi(x), \phi(y))}$ such that $\psi : X \times X \times C^p(X) \times C^p(X) \to (0, 1]$ then we have the following corollary.

Corollary 3.8 Let (X, p) be a partial metric space. Let $\overline{x} \in X$ and $r > 0$ such that $\overline{B_p}(\overline{x}, r)$ be a 0-complete subspace of X. Let $\phi : \overline{B_p}(\overline{x}, r) \to C^p(X)$ be a set-valued mapping. If there exists $\alpha \geqslant 0$ such that the following two conditions hold:

(a) $p(\overline{x}, \phi(\overline{x})) < \alpha \leqslant \frac{1}{2}\left(p(\overline{x}, \overline{x}) + r\right)$,

(b) $\psi(x, y, \phi(x), \phi(y))\delta_p(\phi(x) \cap \overline{B_p}(\overline{x}, r), \phi(y)) \leqslant M(x, y) - p(\overline{x}, \phi(\overline{x})) \quad \forall x, y \in \overline{B_p}(\overline{x}, r)$,

then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x}, r)$.

4. Application to coupled elliptic systems

In this section, we consider $X = C([0, 1])$ the space of all continuous functions defined on $I = [0, 1]$ endowed with the maximum norm $\|u\| = \sup_{t \in I} |u(t)|$. Let us consider the cartesian product $X \times X$ endowed with the partial metric

$$p((x, y), (u, v)) = \|x - u\| + \|y - v\| + c$$

where c is a nonnegative constant and then $X \times X$ is a 0-complete partial metric. Note that the coupled elliptic system (1) contains several problems with different choices on the constants and the functions as [1, 10, 12, 13, 23, 25, 28–30]. We can see that coupled elliptic system (1) is equivalent to the following system of integral equations

$$\begin{cases} u_1(t) = \int_0^1 G_1(t, s)[f(s, u_1(s), u_2(s)) - \lambda]ds := A(u_1, u_2)(t) & t \in I \\ u_2(t) = \int_0^1 G_2(t, s)[g(s, u_1(s), u_2(s)) - \mu]ds := B(u_1, u_2)(t) & t \in I \end{cases} \tag{4}$$

where $G_i(t, s)$, for $i \in \{1, 2\}$, is the Green function of the second-order Sturm-Liouville boundary value problem

$$\begin{cases} -z''(t) = 0, & t \in (0, 1); \\ \alpha_i z(0) - \beta_i z'(0) = 0, \quad \gamma_i z(1) + \delta_i z'(1) = 0 \end{cases}$$

It is known that [2, 31]

$$G_i(t, s) = \frac{1}{k_i} \begin{cases} (\beta_i + \alpha_i s)[\delta_i + \gamma_i(1 - t)], & 0 \leqslant s \leqslant t \leqslant 1, \\ (\beta_i + \alpha_i t)[\delta_i + \gamma_i(1 - s)], & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

and then for all $t \in I$ and for each $i \in \{1, 2\}$, we have

$$\begin{aligned}\int_0^1 G_i(t,s)ds &= \int_0^t G_i(t,s)ds + \int_t^1 G_i(t,s)ds \\ &= \frac{1}{k_i}\left(\int_0^t (\beta_i+\alpha_i s)[\delta_i+\gamma_i(1-t)]ds + \int_t^1 (\beta_i+\alpha_i t)[\delta_i+\gamma_i(1-s)]ds\right) \\ &= \frac{1}{2k_i}\left(\beta_i\gamma_i + 2\beta_i\delta_i + (\alpha_i\gamma_i+2\alpha_i\delta_i)t - k_i t^2\right),\end{aligned}$$

which implies that $\sup_{t\in I} \int_0^1 G_i(t,s)ds = \frac{1}{8k_i^2}\left(4k_i(\beta_i\gamma_i+2\beta_i\delta_i) + (\alpha_i\gamma_i+2\alpha_i\delta_i)^2\right) = M_i \neq 0$ and we denote by $M := \max\{M_1, M_2\}$.

Let $\phi(u_1,u_2)(t) = (A(u_1,u_2)(t), B(u_1,u_2)(t))$. Then the system (4) is equivalent to the fixed point equation $\phi(u_1,u_2) = (u_1,u_2)$.

Now, we consider the following conditions:

(1) There exist a constant $C \geqslant 0$ and $K(\lambda,\mu)$ a positive continuous function defined, w.l.o.g., for $\lambda \geqslant \mu \geqslant C$,

(2) There exists $F \in \mathcal{C}$, $\tau \in \Xi$ and $\alpha \in J$ satisfying one of the following
- $F \in \mathcal{C}_I$ and τ is nondecreasing,
- $F \in \mathcal{C}_{II}$ and $\tau_{x,y,u,v} := \tau((x,y),(u,v),\phi(x,y),\phi(u,v)) \geqslant \alpha$ where $(x,y),(u,v) \in X \times X$,

(3) $w(\alpha,\cdot) \leqslant c + K(\lambda,\mu)$,

(4) $\|f(\cdot,0,0)-\lambda\| + \|g(\cdot,0,0)-\mu\| < \dfrac{\alpha - c}{M}$,

(5) $\dfrac{1}{2M}\left(F(|a-b|+|a'-b'|+c, \tau_{a,a',b,b'}) - c\right) \geqslant \begin{cases} |f(\cdot,a,a')-f(\cdot,b,b')|, \\ |g(\cdot,a,a')-g(\cdot,b,b')|, \end{cases}$ for all $\begin{cases} a,a',b,b' \in \mathbb{R}, \\ |a-b|,|a'-b'| \leqslant K(\lambda,\mu), \end{cases}$

Theorem 4.1 For a fixed $c \geqslant 0$, suppose that conditions (1)-(5) holds. Then (1) has at least one solution (u^*,v^*) in $(C([0,1]) \cap C^2((0,1)))^2$ such that $\|u^*\| + \|v^*\| \leqslant K(\lambda,\mu)$. Moreover, if $c + 2K(\lambda,\mu) \in J$ then the solution is unique.

Proof. Let us define a sample set-valued mapping $\phi : X \times X \to C^p(X \times X)$ by

$$\begin{aligned}\phi(u,v)(t) &= (A(u,v)(t), B(u,v)(t)) \\ &= \left(\int_0^1 G_1(t,s)[f(s,u(s),v(s))-\lambda]ds, \int_0^1 G_2(t,s)[g(s,u(s),v(s))-\mu]ds\right)\end{aligned}$$

for all $u, v \in X$.

Now, we check that ϕ satisfies all assumptions of Theorem 3.4 on the closed p-ball of radius $K(\lambda,\mu)$ centered at $(0_X,0_X)$ which denotes by $\overline{B_p}((0_X,0_X),K(\lambda,\mu))$ where 0_X be the null function of X.

First, the use of assumptions (1)-(4) give the following

$$
\begin{aligned}
& p((0_X,0_X),\phi(0_X,0_X)) \\
& = p((0_X,0_X),(A(0_X,0_X),B(0_X,0_X))) \\
& = \|A(0_X,0_X)\| + \|B(0_X,0_X)\| + c \\
& = \sup_{t\in I}\left|\int_0^1 G_1(t,s)[f(s,0_X(s),0_X(s))-\lambda]ds\right| + \sup_{t\in I}\left|\int_0^1 G_2(t,s)[g(s,0_X(s),0_X(s))-\mu]ds\right| + c \\
& \leqslant \left(\sup_{t\in I}\int_0^1 G_1(t,s)ds\right)\|f(\cdot,0,0)-\lambda\| + \left(\sup_{t\in I}\int_0^1 G_2(t,s)ds\right)\|g(\cdot,0,0)-\mu\| + c \\
& \leqslant M_1\|f(\cdot,0,0)-\lambda\| + M_2\|g(\cdot,0,0)-\mu\| + c \\
& \leqslant M(\|f(\cdot,0,0)-\lambda\| + \|g(\cdot,0,0)-\mu\|) + c \\
& < M\frac{\alpha-c}{M} + c = \alpha
\end{aligned}
$$

and $w(\alpha,\cdot) \leqslant c + K(\lambda,\mu) = p((0_X,0_X),(0_X,0_X)) + K(\lambda,\mu)$. Thus the condition (b) of Theorem 3.4 is satisfied.

Let $(x,y),(u,v) \in \overline{B_p}((0_X,0_X),K(\lambda,\mu))$ then we have two cases. The first one, if $\phi(x,y) \notin \overline{B_p}((0_X,0_X),K(\lambda,\mu))$ then according to convention (2) we have

$$
\begin{aligned}
0 &= \delta_p(\phi(x,y)\cap\overline{B_p}((0_X,0_X),K(\lambda,\mu)),\phi(u,v)) \\
&\leqslant F(M((x,y),(u,v)),\tau((x,y),(u,v),\phi(x,y),\phi(u,v))).
\end{aligned}
$$

So we assume that $\phi(x,y) \in \overline{B_p}((0_X,0_X),K(\lambda,\mu))$ and, from condition (5), we have

$$
\begin{aligned}
& \delta_p(\phi(x,y)\cap\overline{B_p}((0_X,0_X),K(\lambda,\mu)),\phi(u,v)) \\
& = p(\phi(x,y),\phi(u,v)) \\
& = \|A(x,y)-A(u,v)\| + \|B(x,y)-B(u,v)\| + c \\
& = \sup_{t\in I}\left|\int_0^1 G_1(t,s)(f(s,x(s),y(s))-f(s,u(s),v(s)))ds\right| \\
& + \sup_{t\in I}\left|\int_0^1 G_2(t,s)(g(s,x(s),y(s))-g(s,u(s),v(s)))ds\right| + c \\
& \leqslant \sup_{t\in I}\int_0^1 G_1(t,s)\left|f(s,x(s),y(s))-f(s,u(s),v(s))\right|ds \\
& + \sup_{t\in I}\int_0^1 G_2(t,s)\left|g(s,x(s),y(s))-g(s,u(s),v(s))\right|ds + c
\end{aligned}
$$

by using the positivity of G_i, the increasing of F at the first variable and the condition (5) we have

$$\begin{aligned}
&\delta_p(\phi(x,y)\cap \overline{B_p}((0_X,0_X),K(\lambda,\mu)),\phi(u,v))\\
&\leqslant \frac{M_1+M_2}{2M}\left(F(\|x-u\|+\|y-v\|+c,\tau_{x,y,u,v})-c\right)+c\\
&\leqslant F(p((x,y),(u,v)),\tau_{x,y,u,v})-c+c\\
&\leqslant F(M((x,y),(u,v)),\tau_{x,y,u,v}).
\end{aligned}$$

Thus all conditions are satisfied and then A has a fixed point (u^*,v^*) in $\overline{B_p}((0_X,0_X),K(\lambda,\mu))$ i.e.,

$$p((u^*,v^*),(0_X,0_X))\leqslant p((0_X,0_X),(0_X,0_X))+K(\lambda,\mu)\Leftrightarrow \|u^*\|+\|v^*\|\leqslant K(\lambda,\mu).$$

If $c+2K(\lambda,\mu)\in J$, i.e. $p((0_X,0_X),(0_X,0_X))+2K(\lambda)\in J$ and since ϕ is a single-valued then (u^*,v^*) is unique. ■

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