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A representation for some groups; a geometric approach

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Abstract. In the present paper, we are going to use geometric and topological concepts, entities and properties of the integral curves of linear vector fields, and the theory of differential equations, to establish a representation for some groups on $R^n(n \geqslant 1)$. Among other things, we investigate the surjectivity and faithfulness of the representation. At the end, we give some applications.

Keywords: Group representation, exponential matrix, integral curve, vector field.

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1. Introduction and preliminaries

In the mathematical field of representation theory, group representations describe abstract groups as matrices, so that the group operation can be represented by matrix multiplication. The representation theory of groups, disparts into subtheories depending on the kind of the group being represented. Representation theory also depends on the type of the vector space on which the group acts, and the type of the field over which the vector space is defined [3, 4, 9, 12]. In [11] the author, using finite dimensional geometry [2], for obtaining a large number of representations of the full linear group over a Galois field. In the present paper, we establish a representation for some groups on $R^n(n \geqslant 1)$. As a follow-up to [11], we propose to use the geometric theory of vector fields and the topological properties of general linear groups. Let us start with recalling some definitions and the most essential concepts, needed in our work. Let U be an open subset of R^n. A smooth vector field on U, is a map $\chi(p) = (p, X(p))$ which assigns to each point

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of U, a vector at that point, such that the associated map $X : U \to R^n$ is smooth. A smooth parametric curve in R^n, is a map $\alpha : I \to R^n, \alpha(t) = (x_1(t), \ldots, x_n(t))$, where I is an open interval in R, and $x_i : I \to R$ is a smooth map for $1 \leqslant i \leqslant n$. A smooth parametric curve in U, is a smooth parametric curve in R^n such that $\alpha(I) \subseteq U$. The velocity vector at the time t of the smooth parametric curve α, is the vector at $\alpha(t)$ defined by $\dot{\alpha}(t) = (\alpha(t), x'_1(t), \ldots, x'_n(t))$. A parametric curve $\alpha : I \to U$ satisfying $\dot{\alpha}(t) = \chi(\alpha(t))$ for all $t \in I$, is called an integral curve of χ, the smooth vector field on U [10]. A representation of a group $(G, \star)$ on a real vector space V, is a group homomorphism from G to $GL(V)$, the general linear group on V. Here V is called the representation space, and the dimension of V is called the dimension of the representation. A faithful representation of a group G on a vector space V, is an injective group homomorphism $\rho : G \to GL(V)$. If G is a topological group, and V is a topological vector space, then a continuous representation of G on V, is a representation ρ such that the map $\Phi : G \times V \to V$ defined by $\Phi(g, v) = \rho(g)(v)$, from $G \times V$ with product topology, is continuous [3].

2. Linear vector fields

The following theorem is a reformulation of the fundamental existence and uniqueness theorem for solutions of systems of first order differential equations [6], and is an essential ingredient of this manuscript.

Theorem 2.1 [10] Let χ be a smooth vector field on an open set $U \subseteq R^n$, and let $p \in U$. Then, there exists an open interval I containing 0, and a smooth parametric curve $\alpha : I \to U$, such that

(*i*) $\alpha(0) = p$,

(*ii*) $\dot{\alpha}(t) = \chi(\alpha(t))$,

(*iii*) If $\beta : \tilde{I} \to U$ is any other parametric curve satisfying (*i*) and (*ii*), then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

The unique α satisfying the above conditions, is called the maximal integral curve of χ through p. A smooth vector field on an open set $U \subseteq R^n$, is said to be complete if for each $p \in U$, the domain of the maximal integral curve of χ through p, is equal to R [10]. If $U = R^n$ and χ satisfies the relations $X(p+q) = X(p)+X(q), X(\lambda p) = \lambda X(p)$, for all $\lambda \in R$ and all $p, q \in U$, then χ is called a linear vector field. The associated map of any linear vector field $\chi : R^n \to R^n \times R^n$ can be written by $X(p) = (\Sigma_{j=1}^n a_{1j}p_j, \ldots, \Sigma_{j=1}^n a_{nj}p_j)$, where $A = [a_{ij}]$ is an $n \times n$ constant real matrix, called the matrix of χ, and $p = (p_1, \ldots, p_n)$. Therefore, χ is a smooth vector field, and the integral curve of χ through p, is given by the initial value problem

$$\alpha'(t) = A\alpha(t) \quad \forall t \in R, \quad \alpha(0) = p. \tag{1}$$

The following theorem, gives an explicit formula for the integral curve of a linear vector field.

Theorem 2.2 Let χ be a linear vector field and $p \in R^n$, then χ has a unique integral curve through p.

Proof. It suffices to show that the initial value problem (1) has a unique solution defined on R. Let α_p be defined by $\alpha_p(t) = e^{tA}p$, where $t \in R$ and e^{tA} is the matrix exponential map. Then $\alpha'_p(t) = Ae^{tA}p = A\alpha_p(t)$ for all $t \in R$ and $\alpha_p(0) = p$ [5, 7, 8]. Now Theorem 2.1 implies the uniqueness. ■

Corollary 2.3 Let $p \in R^n$ and χ be a linear vector field, then χ is a complete vector field with the maximal integral curve α_p.

The following theorem is a consequence of Theorem 2.2.

Theorem 2.4 Let $t \in R$ and $\varphi_t : R^n \to R^n$ be defined by $\varphi_t(p) = \alpha_p(t)$. Then

(1) $\varphi_0 = I_{R^n}$,
(2) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in R$,
(3) $(\varphi_s \circ \varphi_t) \circ \varphi_q = \varphi_s \circ (\varphi_t \circ \varphi_q)$ for all $s, t, q \in R$,
(4) $\varphi_{-t} = \varphi_t^{-1}$ for all $t \in R$,
(5) $\varphi_t(p + q) = \varphi_t(p) + \varphi_t(q)$ for all $t \in R$ and $p, q \in R^n$,
(6) $\varphi_t(\lambda p) = \lambda\varphi_t(p)$ for all $\lambda, t \in R$ and all $p \in R^n$,
(7) φ_t and φ_t^{-1} are continuous isomorphisms for all $t \in R$,
(8) The function $\Phi : R \times R^n \to R^n$ defined by $\Phi(t, p) = \varphi_t(p)$ is continuous.

3. Some groups representation

In the case where $V = R^n$, it is common to choose a basis for R^n and identify $GL(R^n)$ with $GL(n, R)$, the general linear group over the field of real numbers, which is the group of $n \times n$ invertible matrices on the field R. In this section, we use the consequences of the forehand section to establish an n dimensional representation for some groups. We begin with the following result, as a consequence of Theorem 2.4.

Theorem 3.1 Let χ be a linear vector field. Then the map $\rho_\chi : R \to GL(n, R)$ given by $\rho_\chi(t) = \varphi_t$, is a continuous representation of the real additive group on R^n.

3.1 *Faithfulness*

The following theorem shows that ρ_χ, however, in general is not faithful.

Theorem 3.2 For $n > 1$, there exist infinitely many linear vector fields χ on R^n, such that the map ρ_χ, is a faithful group representation. There exist also some linear vector fields χ, such that the map ρ_χ, is a non-faithful group representation. For $n = 1$, and any linear vector field χ on R, the map ρ_χ is a faithful representation.

Proof. Let $R(2, t) = \begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix}$ for $t \in R$, $O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2\times1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $M_a = [a]$ for some $a \in R$, $O_{1\times2} = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $diag(c_{11}, c_{22}, \ldots, c_{nn})$ be the $n \times n$ diagonal matrix with diagonal entries $c_{11}, c_{22}, \ldots, c_{nn} \in R$.
1. If $n = 2$ and $A = R(2, \frac{\pi}{2})$, then a computation yields that φ_t is the matrix $B(t) = R(2, t)$. Therefore $B(t + 2\pi) = B(t)$ for all $t \in R$.
2. If $n = 2k(k > 1)$, and $A = [A_{ij}](1 \leqslant i, j \leqslant k)$ be an $n \times n$ block matrix with k^2 partitions A_{ij}, such that

$$A_{ij} = \begin{cases} R(2, \frac{\pi}{2}), if & i = j, \\ O_{2\times2}, \;\; if & i \neq j, \end{cases}$$

then a computation yields that φ_t is an $n \times n$ block matrix $B(t) = [B_{ij}(t)]$, with k^2

partitions $B_{ij}(t)$, such that

$$B_{ij}(t) = \begin{cases} R(2,t), if & i = j, \\ O_{2\times 2}, \ if & i \neq j. \end{cases}$$

Therefore $B(t + 2\pi) = B(t)$ for all $t \in R$.
3. If $l \in R$, $n = 2k + 1 (k \geqslant 0)$, and $A(l) = [A_{ij}(l)](1 \leqslant i, j \leqslant k + 1)$ be an $n \times n$ block matrix with $(k + 1)^2$ partitions $A_{ij}(l)$, such that

$$A_{ij}(l) = \begin{cases} R(2, \frac{\pi}{2}), if & 1 \leqslant i = j \leqslant k, \\ O_{2\times 2}, \ \ if & 1 \leqslant i \neq j \leqslant k, \\ M_l, \ \ \ \ if & i = j = k + 1, \\ O_{2\times 1}, \ \ if & i < j = k + 1, \\ O_{1\times 2}, \ \ if & j < i = k + 1, \end{cases}$$

then φ_t is an $n \times n$ block matrix $B(t, l) = [B_{ij}(t, l)]$, with $(k + 1)^2$ partitions $B_{ij}(t, l)$, such that

$$B_{ij}(t, l) = \begin{cases} R(2,t), if & 1 \leqslant i = j \leqslant k, \\ O_{2\times 2}, \ if & 1 \leqslant i \neq j \leqslant k, \\ M_{e^{tl}}, \ if & i = j = k + 1, \\ O_{2\times 1}, \ if & i < j = k + 1, \\ O_{1\times 2}, \ if & j < i = k + 1. \end{cases}$$

Therefore $B(t+2\pi, 0) = B(t, 0)$ for all $t \in R$, and $B(t_1, l) \neq B(t_2, l)$ for $l \neq 0$ and $t_1 \neq t_2$.
4. If $n \geqslant 1$, and $A = diag(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn})$, then φ_t is the matrix $B(t) = diag(e^{\lambda_{11}t}, e^{\lambda_{22}t}, \ldots, e^{\lambda_{nn}t})$.
Therefore, if $\Pi_{i=1}^{i=n} \lambda_{ii} \neq 0$ and $t_1 \neq t_2$, then $B(t_1) \neq B(t_2)$. ■

3.2 *Surjectivity*

For any linear vector field χ, and any $t \in R$, the linear map $\rho_\chi(t)$ has always a positive determinant, while there exist matrices in $GL(n, R)$ with a negative determinant. Therefore, we have the following lemma.

Lemma 3.3 For any linear vector field χ, the map ρ_χ is not surjective.

Remark 1 *The set of all elements of $GL(n, R)$ with positive determinants which is denoted by $GL^+(n, R)$, is a subgroup of $GL(n, R)$. An argument, using the properties of the matrix exponential [5, 7, 8], shows that not only the map $\rho_\chi : R \to GL(n, R)$, is not surjective, but also as a map to $GL^+(n, R)$.*

Let χ be a linear vector field, G be a group and $J : G \to R$ be a group homomorphism. Then the map $\rho_{(\chi,J)} : G \to GL(n, R)$ given by $\rho_{(\chi,J)}(g) = \varphi_{J(g)}$, is simply denoted by ρ.

Corollary 3.4 For any linear vector field χ, ρ is a non-surjective representation of G on R^n.

4. Applications

The multiplicative group of positive real numbers $(R^+, \times)$, is an abelian group with 1 being its neutral element. The logarithm is a group isomorphism from this group to the real additive group.

Corollary 4.1 For any linear vector field χ, the map $\rho : R^+ \to GL(n, R)$ given by $\rho(t) = \varphi_{\log t}$, is a continuous non-surjective representation of multiplicative group R^+ on R^n.

The map $J : UT(2, R) \to R$, $J(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}) = t$, gives an isomorphism between the multiplicative group of upper triangular unipotent 2×2 real matrices $UT(2, R)$, and additive group R.

Corollary 4.2 The map $\rho : UT(2, R) \to GL(n, R)$ given by $\rho(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}) = \varphi_t$, is a continuous non-surjective representation of multiplicative group $UT(2, R)$ on R^n.

The additive group $(R^m, +)$ for $m \geqslant 1$, with usual topology, is a connected topological group. The same fact is true for $((R^+)^m, \times)$, the direct product of the multiplicative group of positive real numbers $(R^+, \times)$. The general linear group $GL(n, R)$, can be also viewed as a non-connected topological group with the topology defined by viewing $GL(n, R)$ as a subspace of Euclidean space R^{n^2} [1].

Corollary 4.3 Let $G_i (1 \leqslant i \leqslant m)$ are (topological) groups, $J_i : G_i \to R (1 \leqslant i \leqslant m)$ are (topological) group homomorphisms and $H = \Pi_{i=1}^{i=m} G_i$ be their direct product. Then the map $\rho : H \to GL(n, R)$ given by $\rho(g_1, \ldots, g_m) = \varphi_{\Sigma_{i=1}^{i=m} J_i(g_i)}$, is a (continuous) non-surjective representation of H on R^n.

Corollary 4.4 The map $\rho : R^m \to GL(n, R)$ (res. $\rho : (R^+)^m \to GL(n, R)$) given by $\rho(t_1, \ldots, t_m) = \varphi_{\Sigma_{i=1}^{i=m} t_i}$ (res. $\rho(t_1, \ldots, t_m) = \varphi_{\Sigma_{i=1}^{i=m} \log t_i}$), is a (continuous) non-surjective representation of R^m (res. $(R^+)^m$) on R^n. For $m > 1$, it is also non-faithful.

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