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Some fixed point results for contractive type mappings in b-metric spaces

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Abstract. In this work, we prove some fixed point theorems by using wt-distance on b-metric spaces. Our results generalize some fixed point theorems in the literature. Moreover, we introduce wt_0-distance and by using the concept of wt_0-distance, we obtain some coupled fixed point results in complete b-metric spaces.

Keywords: Fixed Point, wt-distance, wt_0-distance, b-metric.

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1. Introduction and preliminaries

There has been numerous generalizations of metric spaces. One such well-known generalization is b-metric space defined by Czerwik [11]. After that many authors have obtained some fixed point theorems in b-metric spaces (see [10, 15, 19, 21–23, 28]). Hussain et al. [13] introduced the notion of wt-distance on b-metric spaces, which is a b-metric version of w-distance of Kada et al. [14] and they obtained some fixed point theorems in a partially ordered b-metric space by using wt-distance. Then, Mohanta [20] proved some fixed point theorems by using the wt-distance on a b-metric space. Saadati et al. [12] obtained some fixed point theorems for classes of contractive type multi-valued operators via wt-distances in the setting of a complete b-metric space. Mbarki et al. [18] introduced the probabilistic aspect of the b-metric spaces and they discussed some topological properties of these structures. Saadati et al. [1] defined the concept of rt-distance

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on a Menger probabilistic b-metric space and they investigated some fixed point theorems by using rt-distance which is a probabilistic version of wt-distance. In 2012, Samet et al. [26] introduced the concepts of α-ψ-contractive and α-admissible mappings. Then, many authors investigated some fixed point results by using this idea (see, [4]). Karapinar et al. [8] extended the results of Samet et al. [26] to the setting of b-metric space and they investigated Ulam-Hyers stability results for fixed point theorems by using α-ψ-contractive mapping of type-(b) in the sense of b-metric spaces. In this paper, we first prove some fixed point theorems by using wt-distance on complete b-metric spaces and we extend the results of Karapinar et al. [8]. Also, we introduce the notion of wt_0-distance and we obtain some coupled fixed point theorems via wt_0-distance on b-metric spaces.

Now, we recall some well known notions about b-metric space and wt-distance.

Definition 1.1 [11] Let X be a set. Let $D : X \times X \to [0, \infty)$ be a function which satisfies the following conditions:

(i) $D(x, y) = 0$ if and only if $x = y$;
(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$;
(iii) $D(x, y) \leqslant K[D(x, z) + D(z, y)]$ for all $x, y, z \in X$, for some constant $K \geqslant 1$.

Then, (X, D, K) is called a b-metric space.

***Example* 1.2** [13] Let $X = \mathbb{R}$ and define $D : X \times X \to [0, \infty)$ by $D(x, y) = |x - y|^2$. Then, $(X, D, 2)$ is a b-metric space, but not a metric space.

***Example* 1.3** Let (X, D, K) be a b-metric space. Then, the functional $D_p : X^2 \times X^2 \to [0, \infty)$ defined by $D_p((x, y), (z, t)) = D(x, z) + D(y, t)$ is a b-metric on X^2 with coefficient K.

***Example* 1.4** [8] Let X be a set with the cardinal $card(X) \geqslant 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geqslant 2$. Let $K > 1$ be arbitrary. Then the functional $D : X \times X \to [0, \infty)$ is defined by

$$D(x, y) = \begin{cases} 0 & x = y \\ 2K & x, y \in X_1 \\ 1 & otherwise \end{cases}$$

is a b-metric on X with the coefficient $K > 1$.

The concept of a wt-distance on a b-metric space has been introduced by Hussain et al. [13] by the following:

Definition 1.5 [13] Let (X, D, K) be a b-metric space. Then, a function $P : X \times X \to [0, \infty)$ is called a wt-distance on X if the following conditions are satisfied:
(wt-1) $P(x, z) \leqslant K[P(x, y) + P(y, z)]$ for any $x, y, z \in X$;
(wt-2) for any $x \in X$, $P(x, .) : X \to [0, \infty)$ is K-lower semi-continuous;
(wt-3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P(z, x) \leqslant \delta$ and $P(z, y) \leqslant \delta$ imply $D(x, y) \leqslant \varepsilon$.

Let us recall that a real-valued function f defined on a b-metric space X is said to be lower K-semi-continuous at a point $x_0 \in X$ if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leqslant \liminf_{x_n \to x_0} Kf(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$ (see [13]).

***Example* 1.6** [13] Let (X, D, K) be a b-metric space. Then the metric D is a wt-distance on X.

***Example* 1.7** [13] Let $X = \mathbb{R}$ and $D(x,y) = (x-y)^2$. Then the function $P : X \times X \to [0,\infty)$ defined by $P(x,y) = |x|^2 + |y|^2$ for every $x, y \in X$ is a wt-distance on X.

***Example* 1.8** [13] Let $X = \mathbb{R}$ and $D(x,y) = (x-y)^2$. Then the function $P : X \times X \to [0,\infty)$ defined by $P(x,y) = |y|^2$ for every $x, y \in X$ is a wt-distance on X.

Following lemma has been proved by Hussain et al. [13] and it is necessary to prove our main theorem.

Lemma 1.9 [13] Let (X, D, K) be a b-metric space and P be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero, and let $x, y, z \in X$. Then, the following hold:

(i) if $P(x_n, y) \leqslant \alpha_n$ and $P(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.
(ii) if $P(x_n, y_n) \leqslant \alpha_n$ and $P(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $D(y_n, z) \to 0$.
(iii) if $P(x_n, x_m) \leqslant \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.
(iv) if $P(y, x_n) \leqslant \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

We denote by Ψ the family of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(1) ψ is nondecreasing,
(2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

Remark 1 *[17] For each $\psi \in \Psi$, we have*

(1) $\lim_{n\to\infty} \psi^n(t) = 0$ *for all* $t > 0$.
(2) $\psi(t) < t$ *for all* $t > 0$.
(3) $\psi(0) = 0$.

In the following definition, Berinde [6] introduced the notion of (b)-comparison function in order to extend some fixed point results to the class of b-metric spaces.

Definition 1.10 [6] Let $s \geqslant 1$ be a real number. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called (b)-comparison function if the following conditions satisfy:

(1) φ is monotonically increasing;
(2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leqslant a s^k \varphi^k(t) + v_k$, for $k \geqslant k_0$ and any $t \in [0,\infty)$.

In this paper, we will denote by Ψ_b the family of all (b)-comparison functions.

Lemma 1.11 [5] If $\varphi : [0,\infty) \to [0,\infty)$ is a (b)-comparison function, then the following are true:

(i) the series $\sum_{k=1}^{\infty} s^k \varphi^k(t)$ converges for any $t \in [0,\infty)$.
(ii) the function $b_s : [0,\infty) \to [0,\infty)$ defined by $b_s(t) = \sum_{k=1}^{\infty} s^k \varphi^k(t)$, $t \in [0,\infty)$, is increasing and continuous at 0.

Samet et al. [26] introduced the concept of $\alpha-\psi$-contractive and α-admissible mappings as follows.

Definition 1.12 [26] Let (X, d) be a metric space and $f : X \to X$ a given mapping. Then, f is called $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that $\alpha(x,y) d(f(x), f(y)) \leqslant \psi(d(x,y))$ for all $x, y \in X$.

Definition 1.13 [26] Let $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then, f is called α-admissible mapping if $\alpha(x,y) \geqslant 1$ for all $x, y \in X$, then $\alpha(f(x), f(y)) \geqslant 1$.

Samet et al. [26] obtained some fixed point theorems for $\alpha-\psi$-contractive mappings satisfying α-admissibility condition in complete metric spaces. Then many authors extended the concepts of $\alpha-\psi$-contractive and α-admissible mappings. (see [2, 3, 9, 17, 27–29]).

Karapinar et al. [8] extended the concept of $\alpha-\psi$-contractive and α-admissible mappings to the b-metric spaces. They introduced the concept of $\alpha-\psi$-contractive mapping of type-(b) and obtained the following results.

Definition 1.14 [8] Let (X, d) be a b-metric space and $f : X \to X$ be a given mapping. Then f is called $\alpha-\psi$-contractive mapping of type-(b) if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that $\alpha(x, y)d(f(x), f(y)) \leqslant \psi(d(x, y))$ for all $x, y \in X$.

Theorem 1.15 [8] Let (X, d) be a complete b-metric space with constant $s > 1$. Let $f : X \to X$ be an $\alpha-\psi$-contractive mapping of type-(b) satisfying the following conditions:

(i) f is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geqslant 1$;
(iii) f is continuous.

Then, f has a fixed point.

Theorem 1.16 [8] Let (X, d) be a complete b-metric space with constant $s > 1$. Let $f : X \to X$ be an $\alpha-\psi$-contractive mapping of type-(b) satisfying the following conditions:

(i) f is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geqslant 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all n.

Then, f has a fixed point.

2. Main Results

We now prove some new fixed point results for generalized (α, ψ, P)-contractive mappings with wt-distances in b-metric spaces. Before starting our main theorem, we introduce a new notion as follows:

Definition 2.1 Let (X, D, K) be a b-metric space with the wt-distance P and $f : X \to X$ a given mapping. We say that f is (α, ψ, P)-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that for all $x, y \in X$,

$$\alpha(x, y)P(f(x), f(y)) \leqslant \psi(P(x, y)) \tag{1}$$

We can give the following example to illustrate the notion of (α, ψ, P)-contractive mapping.

***Example* 2.2** Let $X = [0, \infty)$ and $D(x, y) = |x - y|^2$ be a b-metric on X and consider the wt-distance $P(x, y) = |x|^2 + |y|^2$ on $(X, D, 2)$. Let $f : X \to X$ defined by $f(x) = \frac{x}{2}$. Moreover, let the function $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x \text{ or } y \in [0, 1] \\ 1 & \text{otherwise} \end{cases}$$

Then, f is an $(\alpha, \psi, 2)$-contractive for $\psi : [0, \infty) \to [0, \infty)$ which is defined by $\psi(t) = \dfrac{t}{2}$.

Now, we give our main result.

Theorem 2.3 Let P be a wt-distance on a complete b-metric space (X, D, K) and let $f : X \to X$ be an (α, ψ, P)-contractive mapping. Suppose that the following hold:

(i) f is an α-admissible mapping;
(ii) there exists a point $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geqslant 1$;
(iii) f is continuous.

Then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geqslant 1$. We define a sequence x_n in X by $x_{n+1} = f(x_n) = f^{n+1}(x_0)$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = x$ is a fixed point of f. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since f is α-admissible mapping, we have

$$\alpha(x_0, x_1) = \alpha(x_0, f(x_0)) \geqslant 1 \Rightarrow \alpha(f(x_0), f(x_1)) = \alpha(x_1, x_2) \geqslant 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geqslant 1 \tag{2}$$

for all $n \in \mathbb{N}$. By (1) and (2), we have

$$P(x_n, x_{n+1}) = P(f(x_{n-1}), f(x_n)) \leqslant \alpha(x_{n-1}, x_n) P(f(x_{n-1}), f(x_n)) \leqslant \psi(P(x_{n-1}, x_n))$$

for all $n \in \mathbb{N}$. Iteratively, we get that

$$P(x_n, x_{n+1}) \leqslant \psi^n(P(x_0, x_1)) \text{ for all n} \in \mathbb{N}. \tag{3}$$

From (3) and using triangle inequality, for all $p \geqslant 1$, we have

$$\begin{aligned} P(x_n, x_{n+p}) &\leqslant KP(x_n, x_{n+1}) + K^2 P(x_{n+1}, x_{n+2}) + \cdots + K^p P(x_{n+p-1}, x_{n+p}) \\ &\leqslant K\psi^n(P(x_0, x_1)) + K^2\psi^{n+1}(P(x_0, x_1)) + \cdots + K^p\psi^{n+p-1}(P(x_0, x_1)) \\ &= \frac{1}{K^{n-1}}[K^n\psi^n(P(x_0, x_1)) + K^{n+1}\psi^{n+1}P(x_0, x_1) + \cdots \\ &\quad + K^{n+p-1}\psi^{n+p-1}(P(x_0, x_1))]. \end{aligned}$$

Let us say $T_n = \sum_{k=0}^{n} K^k \psi^k(P(x_0, x_1))$ for $n \geqslant 1$. Therefore, we get that

$$P(x_n, x_{n+p}) \leqslant \frac{1}{K^{n-1}}[T_{n+p-1} - T_{n-1}], \; n \geqslant 1, p \geqslant 1. \tag{4}$$

From Lemma 1.11, we have $\sum_{k=0}^{\infty} K^k \psi^k(P(x_0, x_1))$ is convergent. Also, from Lemma 1.9, we get that x_n is a Cauchy sequence in (X, D, K). Since X is complete, there exists x^* such that $x_n \to x^*$ as $n \to \infty$. From the continuity of f, we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_{n+1}) = f(\lim_{n\to\infty} x_n) = f(x^*).$$

Thus, x^* is a fixed point of f. ■

In the next theorem, we omit the continuity hypothesis of f.

Theorem 2.4 Let P be a wt-distance on a complete b-metric (X, D, K) and let $f : X \to X$ be an (α, ψ, P)-contractive mapping. Suppose that the following conditions hold:

(i) f is an α-admissible;
(ii) there exists a point $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geqslant 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all n.

Then, f has a fixed point.

Proof. Following the proof of Theorem 2.3, we have that x_n is a Cauchy sequence in the complete b-metric space (X, D, K). Then, there exists $x^* \in X$ such that $x_n \to x^*$. Moreover, from (2) and the hypothesis (iii), we have $\alpha(x_n, x^*) \geqslant 1$ for all $n \in \mathbb{N}$. Since f is α-admissible, $\alpha(f(x_n), f(x^*)) \geqslant 1$. From, (wt-2) and (4), we get

$$P(x_n, x^*) \leqslant \liminf_{p\to\infty} KP(x_n, x_{n+p})$$

for all $n \in \mathbb{N}$. Thus, we have

$$\lim_{n\to\infty} P(x_n, x^*) = 0. \tag{5}$$

Then,

$$P(x_{n+1}, f(x^*)) = P(f(x_n), f(x^*)) \leqslant \alpha(x_n, x^*)P(f(x_n), f(x^*)) \leqslant \psi(P(x_n, x^*))$$

for all $n \in \mathbb{N}$. Using (5) in the above inequality we obtain that $\lim_{n\to\infty} P(x_{n+1}, f(x^*)) = 0$. By the triangle inequality, we have that

$$P(x_n, f(x^*)) \leqslant K[P(x_n, x_{n+1}) + P(x_{n+1}, f(x^*))].$$

Hence,

$$\lim_{n\to\infty} P(x_n, f(x^*)) = 0. \tag{6}$$

Hence by (i) of the Lemma1.9, (5) and (6) we conclude that $f(x^*) = x^*$. ■

Next example shows that, setting $P = D$, Theorem 2.3 and Theorem 2.4 are generalizations of Theorem 17 and Theorem 18 in [8] respectively.

***Example* 2.5** Consider $X = [0, \infty)$ with the b-metric $D(x, y) = |y-x|^2$ and wt-distance $P : X \times X \to [0, \infty)$ is defined by $P(x, y) = |y|^2$. Let $f : X \to X$ be a function defined by $f(x) = \frac{x}{\sqrt{2}}$ and $\alpha : X \times X \to [0, \infty)$ is defined by

$$\alpha(x, y) = \begin{cases} 1 & x \geqslant y \\ 0 & x < y \end{cases}$$

It is clear that f is α-admissible. Moreover, f is (α, ψ, P)-contractive mapping with respect to $\psi(t) = \frac{t}{2}$. Indeed, let $x < y$. Then, $\alpha(x, y) = 0$. Thus, it is obvious that

$$\alpha(x, y)P(\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}) = 0 \leqslant \psi(P(x, y)) = \frac{y^2}{2}.$$

Now, suppose that $x \geqslant y$. Then, $\alpha(x, y) = 1$ and we have

$$\alpha(x, y)P(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}) = \frac{y^2}{2} \leqslant \psi(P(x, y)) = \frac{y^2}{2}.$$

Also, there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geqslant 1$. Indeed, we have $\alpha(x_0, f(x_0)) \geqslant 1$ for $x_0 = 0$. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. By the definition of the function α, we have that $\{x_n\}$ is a decreasing sequence. Then, it is clear that $x_n \geqslant x$ and $\alpha(x_n, x) \geqslant 1$. Therefore, all the hypotheses of Theorem 2.3 and Theorem 2.4 are satisfied. 0 is the fixed point of f.

Our main results does not guarantee the uniqueness of the fixed point.

***Example* 2.6** Let $X = [0, \infty)$ and $D(x, y) = |x - y|^2$ be a b-metric on X and consider the wt-distance $P(x, y) = |x|^2+|y|^2$ on $(X, D, 2)$. Let $f : X \to X$ defined by $f(x) = \sqrt{x}$. Moreover, let the function $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x \text{ or } y \in [0, 1] \\ 1 & \text{otherwise} \end{cases}$$

Then f is a (α, ψ, P)-contractive mapping, where $\psi(t) = \dfrac{t}{2}$. All the hypotheses of Theorem 2.3 holds, but f has not a unique fixed point.

To assure the uniqueness of the fixed point, we will consider the following hypothesis: (H) $\forall x, y \in X$ there exists $z \in X$ such that $\alpha(x, z) \geqslant 1$, $\alpha(y, z) \geqslant 1$.

Theorem 2.7 Adding property (H) to the hypothesis of Theorem 2.3 we obtain the uniqueness of the fixed point of f.

Proof. Suppose that x^* and y^* are two fixed points of f. By property (H), there exists $z^* \in X$ such that $\alpha(z^*, x^*) \geqslant 1$ and $\alpha(z^*, y^*) \geqslant 1$. Since f is α-admissible, we get that $\alpha(f^n(z^*), f^n(x^*)) \geqslant 1$ and $\alpha(f^n(z^*), f^n(y^*)) \geqslant 1$. Since f is (α, ψ, P)-contraction, we have that

$$\begin{aligned} P(f^{n+1}(z^*), x^*) &= P(f(f^n(z^*)), f(x^*)) \\ &\leqslant \alpha(f^n(z^*), f^n(x^*))P(f(f^n(z^*)), f(x^*)) \\ &\leqslant \psi(P(f^n(z^*), x^*)) \end{aligned}$$

for each $n \in \mathbb{N}$. By induction, we get $P(f^{n+1}(z^*), x^*) \leqslant \psi^n(P(z^*, x^*))$ for all $n \in \mathbb{N}$. In a similar way, we get that $P(f^{n+1}(z^*), y^*) \leqslant \psi^n(P(z^*, y^*))$. Then, we have $\lim_{n\to\infty} \psi^n P(z^*, x^*) = 0$ and $\lim_{n\to\infty} \psi^n P(z^*, y^*) = 0$. From Lemma 1.9, we obtain $y^* = x^*$ ■

The next two theorems generalize the results of Ran and Reurings [25] and Nieto-Rodrigues-Lopez [24].

Theorem 2.8 Let (X, D, K) be a complete b-metric space such that $(X, \preceq)$ is a partially ordered set. Let $f : X \to X$ be a nondecreasing mapping with respect to " $\preceq$ ". Suppose that the following conditions hold:

(i) There exists $k \in [0, 1)$ such that

$$D(f(x), f(y)) \leqslant kD(x, y) \text{ for each } x, y \in X \text{ such that } x \preceq y;$$

(ii) There exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$;

(iii) f is continuous.

Then f has a fixed point.

Proof. Consider the mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

We will show that the contractive condition (1) is satisfied with respect to the wt-distance D on the b-metric space (X, D, K). By (i), we have that $\alpha(x,y)D(f(x), f(y)) \leqslant kD(x,y)$ for all $x, y \in X$. Then, f is (α, ψ, D)-contractive mapping with $\psi(t) = kt$ for all $t > 0$. Now, we assume that $\alpha(x,y) \geqslant 1$. Then, $x \preceq y$. Since f is nondecreasing with respect to "$\preceq$", we get that $f(x) \preceq f(y)$ and so $\alpha(f(x), f(y)) \geqslant 1$. Therefore, f is α-admissible. From (ii), there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$. This implies that $\alpha(x_0, f(x_0)) \geqslant 1$. Then, the hypotheses of Theorem 2.3 are satisfied and f has a fixed point. ■

Theorem 2.9 Let (X, D, K) be a complete b-metric space such that $(X, \preceq)$ is a partially ordered set. Let $f : X \to X$ be a nondecreasing mapping with respect to "$\preceq$". Suppose that the following conditions hold:

(i) There exists $k \in [0, 1)$ such that

$$D(f(x), f(y)) \leqslant kD(x,y) \text{ for each } x, y \in X \text{ such that } x \preceq y;$$

(ii) There exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$;

(iii) If $\{x_n\}$ is a nondecreasing sequence in X such that $x \in X$ $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all n.

Then f has a fixed point.

Proof. Define the mapping $\alpha : X \times X \to X$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

Then, f is (α, ψ, D)-contractive, where $\psi(t) = kt$ and $k \in [0, 1)$. Moreover, f is α-admissible. Let x_n be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Then, $\alpha(x_n, x) = 1$. Thus, all the hypotheses of Theorem 2.4 are satisfied and f has a fixed point. ■

Theorem 2.10 Adding the condition (H'):

$$\text{For all } x, y \in X \text{ there exists } z \in X \text{ such that } x \preceq z \text{ and } y \preceq z$$

to the Theorem 2.8 and Theorem 2.9, we obtain the uniqueness.

Proof. Suppose that x^* and y^* are two fixed point of f. Then, there exists $z \in X$ such that $x^* \preceq z$ and $y^* \preceq z$. Then, $\alpha(x^*, z) \geqslant 1$ and $\alpha(y^*, z) \geqslant 1$. Then the hypothesis (H) is satisfied and f has a unique fixed point. ■

3. Some coupled fixed point results and wt_0-distance

In [16], Radenović et al. introduced the notion of w_0-distance to obtain some fixed point results. In this section, we will introduce wt_0-distance which is a b-metric version of w_0-distance. Then, we will show that our previous results help us to obtain some coupled fixed point theorems in complete b-metric spaces.

Definition 3.1 Let (X, D, K) be a b-metric space. Then, a function $P : X \times X \to [0, \infty)$ is called a wt_0-distance on X if the following are satisfied:

(wt_0)-1 $P(x,y) \leqslant K[P(x,z) + P(z,y)]$;

(wt_0)-2 for any $x \in X$, the functions $P(x,.), P(.,x) : X \to [0,\infty)$ are K-lower semi-continuous;

(wt_0)-3 for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P(z,x) \leqslant \delta$ and $P(z,y) \leqslant \delta$ imply $D(x,y) \leqslant \varepsilon$.

***Example* 3.2** Let consider the b-metric space $(\mathbb{R}, D, 2)$, where $D(x,y) = (x-y)^2$ for all $x, y \in \mathbb{R}$. Then, the function $P : X \times X \to [0,\infty)$ defined by $P(x,y) = |x|^2 + |y|^2$. Then, P is a wt_0 distance on $(\mathbb{R}, D, 2)$, but not a b-metric.

***Example* 3.3** Let $X = [0,\infty)$ and consider the b-metric $(X, D, 2)$, where $D(x,y) = (x-y)^2$ for all $x, y \in X$ and wt-distance function $P : X \times X \to [0,\infty)$ defined by $P(x,y) = |y|^2$. Inspired by the Example 1.3 given in [16], we will construct the following wt-distance. Let $\alpha : X \to [0,\infty)$ defined by

$$\alpha(x) = \begin{cases} e^{-x} & x > 0 \\ 3 & x = 0 \end{cases}$$

The function $P' : X \times X \to [0,\infty)$ defined by $P'(x,y) = max\{\alpha(x), P(x,y)\}$. Then, P' is a $wt-$distance on $(X, D, 2)$. However, P' is not a wt_0-distance on X. Indeed, consider the sequence $\{x_n\}$ in X, where $x_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Then, x_n converges to 0 in $(X, D, 2)$. But, for $x = 0$, we have the following $\liminf_{n\to\infty} 2\max\{e^{-\frac{1}{n}}, 0\} = 2 \leqslant P'(0,0) = 3$. Thus, the function $P'(.,0)$ is not $2-$lower semi-continuous. Hence, P' is not a wt_0 distance on $(X, D, 2)$.

Lemma 3.4 Let (X, D, K) be a complete b-metric space and P be a wt_0-distance on X. Then the function $\delta : X^2 \times X^2 \to [0,\infty)$ defined by

$$\delta((x,y),(z,t)) = \max\{P(x,z) + P(y,t), P(z,x) + P(t,y)\}$$

for all $(x,y),(z,t) \in X^2$ is a symmetric wt_0-distance on the complete b-metric space (X^2, D_p, K), where D_p is defined on X^2 by $D_p((x,y),(z,t)) = D(x,z) + D(y,t)$.

Proof. (wt_0)-1 Let $(x,y),(z,t),(u,v) \in X^2$. Then, we have

$$\begin{aligned}
&K[\delta((x,y),(u,v)) + \delta((u,v),(z,t))] \\
&= K[\max\{P(x,u) + P(y,v), P(u,x) + P(v,y)\} + \max\{P(u,z) + P(v,t), P(z,u) + P(t,v)\}] \\
&\geqslant K[\max\{P(x,u) + P(y,v) + P(u,z) + P(v,t), P(u,x) + P(v,y) + P(z,u) + P(t,v)\}] \\
&= \max\{K[P(x,u) + P(y,v) + P(u,z) + P(v,t)], K[P(u,x) + P(v,y) + P(z,u) + P(t,v)]\} \\
&\geqslant \max\{P(x,z) + P(y,t), P(z,x) + P(t,y)\} = \delta((x,y),(z,t)).
\end{aligned}$$

(wt_0)-2 Let (x,y) be a point of X^2. Now we show that the function $\delta((x,y),.) : X^2 \to [0,\infty)$ is K-lower semi-continuous. To this end, let (x_n, y_n) be a sequence in X^2 and there exists a point $(a,b) \in X^2$ such that $\lim_{n\to\infty} D_p((x_n,y_n),(a,b)) = 0$. Thus, we have $\lim_{n\to\infty} D(x_n,a) = 0$ and $\lim_{n\to\infty} D(y_n,b) = 0$. Since P is a wt_0-distance, we have the following inequalities from (wt_0)-2 condition:

$$P(x,a) \leqslant \liminf_{n\to\infty} KP(x,x_n), \tag{7}$$

$$P(a,x) \leqslant \liminf_{n\to\infty} KP(x_n,x), \tag{8}$$

$$P(y,b) \leqslant \liminf_{n\to\infty} KP(y,y_n), \tag{9}$$

$$P(b,y) \leqslant \liminf_{n\to\infty} KP(y_n,y). \tag{10}$$

Adding (7) to (9) and (8) to (10), we get the following:

$$\begin{aligned} P(x,a)+P(y,b) &\leqslant \liminf_{n\to\infty} KP(x,x_n) + \liminf_{n\to\infty} KP(y,y_n) \\ &\leqslant \liminf_{n\to\infty} K[P(x,x_n)+P(y,y_n)] \\ &\leqslant \liminf_{n\to\infty} [\max\{K[P(x,x_n)+P(y,y_n)], K[P(x_n,x)+P(y_n,y)]\} \end{aligned}$$

and

$$\begin{aligned} P(a,x)+P(b,y) &\leqslant \liminf_{n\to\infty} KP(x_n,x) + \liminf_{n\to\infty} KP(y_n,y) \\ &\leqslant \liminf_{n\to\infty} K[P(x_n,x)+P(y_n,y)] \\ &\leqslant \liminf_{n\to\infty} [\max\{K[P(x_n,x)+P(y_n,y)], K[P(x,x_n)+P(y,y_n)]\}]. \end{aligned}$$

Thus, we have

$$\begin{aligned} \max\{P(x,a)+P(y,b), P(a,x)+P(b,y)\} \leqslant \liminf_{n\to\infty} K\max\{&P(x,x_n)+P(y,y_n) \\ &, P(x_n,x)+P(y_n,y)\}. \end{aligned}$$

Therefore, we get that $\delta((x,y),(a,b)) \leqslant \liminf_{n\to\infty} K\delta((x,y),(x_n,y_n))$, which implies $\delta((x,y),.)$ is K-lower semi-continuous function. Also, in a similar way, $\delta(.,(x,y))$ is K-lower semi-continuous function.
(wt_0)-3 Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)$ be points of X^2 and $\varepsilon > 0$. Since P is wt_0 distance, there exist $\delta_1 > 0, \delta_2 > 0$ such that $P(z_1,x_1) \leqslant \delta_1$ and $P(z_1,y_1) \leqslant \delta_1$ imply that $D(x_1,y_1) \leqslant \frac{\varepsilon}{2}$. Also, $P(z_2,x_2) \leqslant \delta_2$ and $P(z_2,y_2) \leqslant \delta_1$ imply that $D(x_2,y_2) \leqslant \frac{\varepsilon}{2}$. Let us say $\delta_0 = \min\{\delta_1,\delta_2\}$. Then, $\delta((z_1,z_2),(x_1,x_2)) \leqslant \delta_0$ and $\delta((z_1,z_2),(y_1,y_2)) \leqslant \delta_0$ imply that $D_p((x_1,x_2),(y_1,y_2)) \leqslant \varepsilon$. Moreover, it is clear that δ is a symmetric distance. Therefore, we obtain that δ is a symmetric wt_0-distance on (X^2,D_p,K). ■

Now, we recall some well known notions about coupled fixed points.

Definition 3.5 [7] Let $F: X\times X \to X$ be a given mapping. We say that (x,y) is a coupled fixed point of F if $F(x,y)=x$ and $F(y,x)=y$.

Lemma 3.6 [26] Let $F: X\times X \to X$ be a given mapping. Define the mapping $T: X\times X \to X\times X$ by $T(x,y)=(F(x,y),F(y,x))$ for all $(x,y)\in X\times X$. Then (x,y) is a coupled fixed point of F iff (x,y) is a fixed point of T.

Theorem 3.7 Let (X, D, K) be a complete b-metric space and P be a wt_0-distance on X. Let $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi_b$ and a function $\alpha : X^2 \times X^2 \to [0, \infty)$ such that

$$\alpha((x,y),(u,v))[P(F(x,y),F(u,v)) + P(F(y,x),F(v,u))] \leqslant \frac{1}{2}\psi(P(x,u) + P(y,v)) \quad (11)$$

for all $(x, y), (u, v) \in X \times X$. Suppose also that

(i) For all $(x, y), (u, v) \in X \times X$, we have

$$\alpha((x,y),(u,v)) \geqslant 1 \Rightarrow \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u))) \geqslant 1;$$

(ii) There exists $(x_0, y_0) \in X \times X$ such that

$$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geqslant 1, \quad \alpha((F(y_0,x_0),F(x_0,y_0)),(y_0,x_0)) \geqslant 1;$$

(iii) F is continuous.

Then F has a coupled fixed point.

Proof. From (11), we have

$$\alpha((x,y),(u,v))[P(F(x,y),F(u,v)) + P(F(y,x),F(v,u))] \leqslant \frac{1}{2}\psi(P(x,u) + P(y,v)),$$

$$\alpha((v,u),(y,x))[P(F(v,u),F(y,x)) + P(F(u,v),F(x,y))] \leqslant \frac{1}{2}\psi(P(v,y) + P(u,x).$$

Since ψ is monotonically increasing, we get that

$$\alpha((x,y),(u,v))[P(F(x,y),F(u,v)) + P(F(y,x),F(v,u))] \leqslant \frac{1}{2}\psi(\delta((x,y),(u,v))), \quad (12)$$

$$\alpha((v,u),(y,x))[P(F(v,u),F(y,x)) + P(F(u,v),F(x,y))] \leqslant \frac{1}{2}\psi(\delta((x,y),(u,v))), \quad (13)$$

where δ is defined by

$$\delta((x,y),(u,v)) = \max\{P(x,u) + P(y,v), P(u,x) + P(v,y)\}.$$

From Lemma 3.4, we know that δ is a symmetric wt_0-distance. Adding (12) to (13), we get that $\theta((z,t))\delta((T(z),T(t))) \leqslant \psi(\delta(z,t))$ for all $z = (z_1, z_2)$, $t = (t_1, t_2) \in Y$, where $\theta : Y \times Y \to [0, \infty)$ is a function defined by

$$\theta((z_1,z_2),(t_1,t_2)) = \min\{\alpha((z_1,z_2),(t_1,t_2)), \alpha((t_2,t_1),(z_2,z_1))\}$$

and $T : Y \to Y$ is defined by $T(x, y) = (F(x, y), F(y, x))$. Thus, T is continuous and (θ, ψ, δ)-contractive mapping. Moreover, let $\theta((z_1, z_2), (t_1, t_2)) \geqslant 1$. By using (i), we obtain that $\theta(T(z_1, z_2), T(t_1, t_2)) \geqslant 1$. Thus, T is θ-admissible. From condition (ii), we have that there exists $(x_0, y_0) \in Y$ such that $\theta((x_0, y_0), T(x_0, y_0)) \geqslant 1$. Thus all the hypotheses of Theorem 2.3 are satisfied and T has a fixed point. By using Lemma 3.6, F has a coupled fixed point. ■

In the next theorem, we omit the continuity hypothesis of F.

Theorem 3.8 Let (X, D, K) be a complete b-metric space and P be a wt_0-distance on X. Let $F : X \times X \to X$ be a function. Suppose that there exists $\psi \in \Psi_b$ and $\alpha : X^2 \times X^2 \to [0, \infty)$ such that

$$\alpha((x,y),(u,v))[P(F(x,y),F(u,v)) + P(F(y,x) + F(v,u))] \leqslant \frac{1}{2}\psi(P(x,u) + P(y,v))$$

for all $(x, y), (u, v) \in X \times X$. Suppose that

(i) For all $(x, y), (u, v) \in X \times X$, we have

$$\alpha((x,y),(u,v)) \geqslant 1 \Rightarrow \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u))) \geqslant 1;$$

(ii) There exists $(x_0, y_0) \in X \times X$ such that

$$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))) \geqslant 1, \alpha((F(y_0,x_0),F(x_0,y_0)),(y_0,x_0)) \geqslant 1;$$

(iii) If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geqslant 1$ and $\alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geqslant 1$, $x_n \to x \in X$ and $y_n \to y \in X$ as $n \to \infty$, then $\alpha((x_n, y_n), (x, y)) \geqslant 1$ and $\alpha((y, x), (y_n, x_n)) \geqslant 1$.

Then F has a coupled fixed point.

Proof. We will use the similar arguments given in the proof of Theorem 3.7. Let $\{(x_n, y_n)\}$ be a sequence in Y such that $\theta((x_n, y_n), (x_{n+1}, y_{n+1})) \geqslant 1$ and $(x_n, y_n) \to (x, y)$ as $n \to \infty$. By the condition (iii), we obtain that $\theta((x_n, y_n), (x, y)) \geqslant 1$. Thus, all the hypotheses of Theorem 2.4 are satisfied. Therefore, T has a fixed point. Whence, F has a coupled fixed point. ■

For the uniqueness of the coupled fixed point, we consider the following hypothesis:
(H") For all $(x, y), (u, v) \in X \times X$, there exists $(w_1, w_2) \in X \times X$ such that

$$\alpha((x,y),(w_1,w_2)) \geqslant 1, \alpha((w_2,w_1),(y,x)) \geqslant 1,$$
$$\alpha((u,v),(w_1,w_2)) \geqslant 1, \alpha((w_2,w_1),(v,u)) \geqslant 1.$$

Theorem 3.9 Adding condition (H") to the hypothesis of the Theorem 3.7, we obtain the uniqueness of the coupled fixed point of F.

Proof. It is clear that θ satisfy the condition (H). Thus, the proof follows from Theorem 2.7. ■

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