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Preclosure operator and its applications in general topology

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Abstract. In this paper, we show that a pointwise symmetric pre-isotonic preclosure function is uniquely determined the pairs of sets it separates. We then show that when the preclosure function of the domain is pre-isotonic and the preclosure function of the codomain is pre-isotonic and pointwise-pre-symmetric, functions which separate only those pairs of sets which are already separated are precontinuous.

Keywords: Preclosure-separated, preclosure functions, precontinuous functions.

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1. Introduction

Generalized open sets play a very important role in general topology and they are now the research topics of many topologist worldwide. Indeed a significant there in general topology and real analysis concerns the variously modified forms of continuity, separation axioms, compactness etc by utilizing generalized open sets. One of the most well-known

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notions and also an inspiration source is the notion of preopen sets introduced by Moshhour et al. [7]. Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset of X. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively. $A \subset X$ is called a preopen [6,7] or nearly open [8] or locally dense [2] set of X if $A \subset Int(Cl(A))$. The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing a set A is called the preclosure [3] of A and is denoted by $pCl(A)$. Notions and notations not described in this paper are standard and usual. This paper is closely related to [1].

Definition 1.1 (1) A generalized preclosure space is a pair (X, pCl) consisting of a set X and a preclosure function pCl, a function from the power set of X to itself.
(2) The preclosure of a subset A of X, denoted pCl, is the image of A under pCl.
(3) The pre-exterior of A is $pExt(A) = X \backslash pCl(A)$, and the pre-interior of A is $pInt(A) = X \backslash pCl(X \backslash A)$.
(4) A is preclosed if $A = pCl(A)$, A is preopen if $A = pInt(A)$ and N is a preneighborhood of a point $x \in X$ [4], [5]if $x \in pInt(N)$.

Definition 1.2 A preclosure function pCl defined on X is:
(1) pre-grounded if $pCl(\phi) = \phi$.
(2) pre-isotonic if $pCl(A) \subseteq pCl(B)$ whenever $A \subseteq B$.
(3) pre-enlarging if $A \subseteq pCl(A)$ for each subset A of X.
(4) pre-idempotent if $pCl(A) = pCl(pCl(A))$ for each subset A of X.
(5) pre-sub-linear if $pCl(A \cup B) \subseteq pCl(A) \cup pCl(B)$ for all $A, B \subseteq X$.

Definition 1.3 (1) Subsets A and B of X are said to be preclosure-separated in a generalized preclosure space (X, pCl) (or simply, $pCl-$ separated) if $A \cap pCl(B) = \phi$ and $B \cap pCl(A) = \phi$,or equivalently, if $A \subseteq pExt(B)$ and $B \subseteq pExt(A)$.
(2) Pre-Exterior points are said to be preclosure-separated in a generalized preclosure space (X, pCl) if for each $A \subseteq X$ and for each $x \in pExt(A)$, $\{x\}$ and A are $pCl-separated.$

Theorem 1.4 Let (X, pCl) be a generalized preclosure space in which pre-Exterior points are pCl-separated and let S be the pairs of pCl-separated sets in X. Then, for each subset A of X, the preclosure of A is $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$.

Proof. In any generalized preclosure space $pCl(A) \subseteq \{x \in X : \{\{x\}, A\} \notin S\}$. Suppose that $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$; that is, $\{\{y\}, A\} \in S$. Then $\{y\} \cap pCl(A) = \phi$, and so $y \notin pCl(A)$. Now, let $y \notin pCl(A)$. By hypothesis, $\{\{y\}, A\} \in S$. Therefore, $y \notin \{x \in X : \{\{x\}, A\} \notin S\}$. ■

2. Some basic properties

Definition 2.1 A preclosure function pCl defined on a set X is said to be pointwise pre-symmetric when, for all $x, y \in X$, if $x \in pCl(\{y\})$, then $y \in pCl(\{x\})$.
A generalized preclosure space (X, pCl) is said to be pre-R_0 when, for all $x, y \in X$, if x is in each preneighborhood of y, then y is in each preneighborhood of x.

Corollary 2.2 Let (X, pCl) be a generalized preclosure space in which pExterior points are pCl-separated. Then pCl is pointwise pre-symmetric and (X, pCl) is pre-R_0.

Proof. Let pre-Exterior points be pCl-separated in (X, pCl). If $x \in pCl(\{y\})$, then $\{x\}$ and $\{y\}$ are not pCl-separated. This means that $y \in pCl(\{x\})$. Hence, pCl is pointwise pre-symmetric. Suppose that x belongs to every preneighborhood of y; that is, $x \in M$

whenever $y \in pInt(M)$. Letting $A = X\backslash M$ and rewriting contrapositively, $y \in pCl(A)$ whenever $x \in A$. Let $x \in pInt(N)$ consequently $x \notin pCl(X\backslash N)$, so x is pCl-separated from $X\backslash N$. Hence $pCl(\{x\}) \subseteq N, x \in \{x\}$, so $y \in pCl(\{x\}) \subseteq N$. Hence, (X, pCl) is pre-R_0. ■

Observe that these three axioms are not equivalent in general, but they are equivalent when the preclosure function is pre-isotonic.

Theorem 2.3 Let (X, pCl) be a generalized preclosure space with pCl pre-isotonic. Then the following are equivalent:
(1) $pExterior$ points are pCl-separated.
(2) pCl is pointwise pre-symmetric.
(3) (X, pCl) is pre-R_0.

Proof. Suppose that (2) is true. Let $A \subseteq X$, and let $x \in pExt(A)$. Then, as pCl is pre-isotonic, for each $y \in A, x \notin pCl(\{y\})$, and thus, $y \notin pCl(\{x\})$. Hence $A \cap pCl(\{x\}) = \phi$. Therefore (2) implies (1). Moreover, by the previous corollary, (1) implies (2).

Suppose now that (2) is true and let $x, y \in X$ such that x is in every preneighborhood of y, i.e. $x \in N$ whenever $y \in pInt(N)$. Then $y \in pCl(A)$ whenever $x \in A$, and in particular, since $x \in \{x\}, y \in pCl(\{x\})$. It follows that $x \in pCl(\{y\})$. Thus if $y \in B$, then $x \in pCl(\{y\}) \subseteq pCl(B)$, as pCl is pre-isotonic. Therefore, if $x \in pInt(C)$, then $y \in C$, that is, y is in every preneighborhood of x. Hence, (2) implies (3).

Now, let (X, pCl) be pre-R_0 and $x \in pCl(\{y\})$. Since pCl is pre-isotonic, $x \in pCl(B)$ whenever $y \in B$, or equivalently, y is in every preneighborhood of x. Since (X, pCl) is pre-$R_0, x \in N$ whenever $y \in pInt(N)$. Therefore, $y \in pCl(\{A\})$ whenever $x \in A$, and in particular, since $x \in \{x\}, y \in pCl(\{x\})$. It follows that (3) implies (2). ■

Theorem 2.4 Let S be a set of unordered pairs of subsets of a set X such that, for all $A, B, C \subseteq X$,
(1) if $A \subseteq B$ and $\{B, C\} \in S$, then $\{A, C\} \in S$ and
(2) if $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$, then $\{A, B\} \in S$.
Then there exists a unique pointwise pre-symmetric pre-isotonic preclosure function pCl on X which preclosure-separates the elements of S.

Proof. Define pCl by $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. If $A \subseteq B \subseteq X$ and $x \in pCl(A)$, then $\{\{x\}, A\} \notin S$. Thus $\{\{x\}, B\} \notin S$, that is, $x \in pCl(B)$. Hence pCl is pre-isotonic. Moreover $x \in pCl(\{y\})$ if and only if $\{\{x\}, \{y\}\} \notin S$ if and only if $y \in pCl(\{x\})$. Thus pCl is pointwise pre-symmetric. Suppose that $\{A, B\} \in S$. Then $A \cap pCl(B) = A \cap \{x \in X : \{\{x\}, B\} \notin S\} = \{x \in A : \{\{x\}, A\} \notin S\} = \phi$. Similarly, $pCl(A) \cap B = \phi$. Therefore, if $\{A, B\} \in S$, then A and B are pCl-separated.

Now suppose that A and B are pCl-separated. Then $\{x \in A : \{\{x\}, B\} \notin S\} = A \cap pCl(B) = \phi$ and $\{x \in B : \{\{x\}, A\} \notin S\} = pCl(A) \cap B = \phi$. Hence, $\{\{x\}, B\} \in S$ for each $x \in A$ and $\{\{y\}, A\} \in S$ for each $y \in B$. Therefore, $\{A, B\} \in S$. ■

In the following we show that many properties of preclosure functions can be expressed in terms of the sets they separate.

Theorem 2.5 Let S be the pairs of pCl-separated sets of a generalized preclosure space (X, pCl) in which pre-exterior points are preclosure-separates. Then pCl is
(1) pre-grounded if and only if for all $x \in X, \{\{x\}, \phi\} \in S$.
(2) pre-enlarging if and only if for all $\{A, B\} \in S, A$ and B are disjoint.
(3) pre-sub-linear if and only if $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$.
Furthermore, if pCl is pre-enlarging and for all $A, B \subseteq X, \{\{x\}, A\} \notin S$ whenever

$\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$, then pCl is pre-idempotent. Now, if pCl is pre-isotonic and pre-idempotent, then $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$.

Proof. (1) By Theorem 1.4, $pCl(A) = \{x \in X : \{\{x\}, A\} \notin S\}$ for every $A \subseteq X$. Suppose that for all $x \in X, \{\{x\}, \phi\} \in S$. Then $pCl(\phi) = \{x \in X : \{\{x\}, \phi\} \notin S\} = \phi$. Hence pCl is pre-grounded. Conversely, if $\phi = pCl(\phi) = \{x \in X : \{\{x\}, \phi\} \notin S\}$, then $\{\{x\}, \phi\} \in S$, for all $x \in X$.

(2) Assume that for all $\{A, B\} \in S$, A and B are disjoint. Since $\{\{a\}, A\} \notin S$ if $a \in A, A \subseteq pCl(A)$ for each $A \subseteq X$. Therefore, pCl is pre-enlarging. Conversely, let pCl be pre-enlarging and $\{A, B\} \in S$. Then $A \cap B \subseteq pCl(A) \cap B = \phi$.

(3) Suppose that $\{A, B \cup C\} \in S$ whenever $\{A, B\} \in S$ and $\{A, C\} \in S$. Let $x \in X$ and $B, C \subseteq X$ such that $\{\{x\}, B \cup C\} \notin S$. Then $\{\{x\}, B\} \notin S$ or $\{\{x\}, C\} \notin S$. Hence $pCl(B \cup C) \subseteq pCl(B) \cup pCl(C)$. Therefore, pCl is pre-sub-linear. Conversely, suppose that pCl is pre-sub-linear and let $\{A, B\}, \{A, C\} \in S$. Then $pCl(B \cup C) \cap A \subseteq (pCl(B) \cup pCl(C)) \cap A = (pCl(B) \cap A) \cup (pCl(C)) \cap A) = \phi$ and $(B \cup C) \cap pCl(A) = (B \cap pCl(A)) \cup (C \cap pCl(A)) = \phi$.

Let pCl be pre-enlarging and suppose that $\{\{x\}, A\} \notin S$ whenever $\{\{x\}, B\} \notin S$ and $\{\{y\}, A\} \notin S$ for each $y \in B$. Then $pCl(pCl(A)) \subseteq pCl(A)$. If $x \in pCl(pCl(A))$, then $\{\{x\}, pCl(A)\} \notin S$. $\{\{y\}, A\} \notin S$, for each $y \in pCl(A)$; hence $\{\{x\}, A\} \notin S$. Since pCl is pre-enlarging, then $pCl(A) \subseteq pCl(pCl(A))$. Therefore, $pCl(pCl(A)) = pCl(A)$ for each $A \subseteq X$. Suppose that pCl is pre-isotonic and pre-idempotent. Let $x \in X$ and $A, B \subseteq X$ such that $\{\{x\}, B\} \notin S$ and for each $y \in B, \{\{y\}, A\} \notin S$. Then $x \in pCl(B)$ and for each $y \in B, y \in pCl(A)$, i.e. $B \subseteq pCl(A)$. Therefore, $x \in pCl(B) \subseteq pCl(pCl(A)) = pCl(A)$. ■

Definition 2.6 If $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ are generalized preclosure spaces, then a function $f : X \to Y$ is said to be
(1) preclosure preserving if $f((pCl)_X(A)) \subseteq (pCl)_Y f(A))$ for each $A \subseteq X$.
(2) precontinuous if $(pCl)_X(f^{-1}(B)) \subseteq f^{-1}((pCl)_Y(B))$ for each $B \subseteq Y$.

Observe that in general, neither condition implies the other. Now, we have the following result:

Theorem 2.7 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces and let $f : X \to Y$ be a function.
(1) If f is preclosure preserving and $(pCl)_Y$ is pre-isotonic, then f is precontinuous.
(2) If f is precontinuous and $(pCl)_X$ is pre-isotonic, then f is preclosure preserving.

Proof. Let f be preclosure preserving and $(pCl)_Y$ is pre-isotonic. Let $B \subseteq Y$. $f((pCl)_X(f^{-1}(B)) \subseteq (pCl)_Y(f(f^{-1}(B))) \subseteq (pCl)_Y(B)$ and hence, $(pCl)_X(f^{-1}(B)) \subseteq f^{-1}(f((pCl)_X(f^{-1}(B)))) \subseteq f^{-1}((pCl)_Y(B))$. Suppose that f is precontinuous and $(pCl)_X$ is pre-isotonic. Let $A \subseteq X$. $(pCl)_X(A) \subseteq (pCl)_X(A)(f^{-1}(f(A))) \subseteq f^{-1}((pCl)_Y(f(A)))$. Therefore, $f((pCl)_X(A)) \subseteq f(f^{-1}((pCl)_Y(f(A)))) \subseteq (pCl)_Y(f(A))$. ■

Definition 2.8 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces and let $f : X \to Y$ be a function. If for all $A, B \subseteq X$, $f(A)$ and $f(B)$ are not $(pCl)_Y$-separated whenever A and B are not $(pCl)_X$-separated, then we say that f is non-pre-separating. Observe that f is non-pre-separating if and only if A and B are not $(pCl)_X$-separated whenever $f(A)$ and $f(B)$ are $(pCl)_Y$-separated.

Theorem 2.9 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces and let $f : X \to Y$ be a function.

(1) If $(pCl)_Y$ is pre-isotonic and f is non-pre-separating. then $f^{-1}(C)$ and $f^{-1}(D)$ are $(pCl)_X$-separated whenever C and D are $(pCl)_Y$-separated.
(2) If $(pCl)_X$ is pre-isotonic and $f^{-1}(C)$ and $f^{-1}(D)$ are $(pCl)_X$-separated whenever C and D are $(pCl)_Y$-separated, then f is non-pre-separating.

Proof. Suppose that C and D are $(pCl)_Y$-separated subsets, where $(pCl)_Y$ is pre-isotonic. Let $A = f^{-1}(C)$ and $B = f^{-1}(D)$. $f(A) \subseteq C$ and $f(B) \subseteq D$ and since $(pCl)_Y$ is pre-isotonic, $f(A)$ and $f(B)$ are also $(pCl)_Y$-separated. It follows now that A and B are $(pCl)_X$-separated in X. Suppose that $(pCl)_X$ is pre-isotonic and let $A, B \subseteq X$ such that $C = f(A)$ and $D = f(B)$ are $(pCl)_X$-separated. Then $f^{-1}(C)$ and $f^{-1}(D)$ are $(pCl)_X$-separated and since $(pCl)_X$ is pre-isotonic, $A \subseteq f^{-1}(f(A)) = f^{-1}(C)$ and $B \subseteq f^{-1}(f(B)) = f^{-1}(D)$ are $(pCl)_X$-separated as well. ■

Theorem 2.10 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces and let $f : X \to Y$ be a function. If f is preclosure preserving, then f is non-pre-separating.

Proof. Suppose that f is preclosure preserving and $A, B \subseteq X$ are not $(pCl)_X$-separated. Suppose that $(pCl)_X(A) \cap B \neq \phi$. Then $\phi \neq f((pCl)_X(A) \cap B) \subseteq f((pCl)_X(A)) \cap f(B) \subseteq (pCl)_Y(f(A)) \cap f(B)$. Similarly, if $A \cap (pCl)_X(B) \neq \phi$, then $f(A) \cap (pCl)_Y(f(B)) \neq \phi$. Hence $f(A)$ and $f(B)$ are not $(pCl)_Y$-separated. ■

Corollary 2.11 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces with $(pCl)_X$ pre-isotonic and let $f : X \to Y$ be a function. If f is precontinuous, then f is non-pre-separating.

Proof. If f is precontinuous and $(pCl)_X)$ pre-isotonic, then by Theorem 2.9 (2) f is pre-closure-preserving. Now, by Theorem 2.10, f is non-pre-separating. ■

Theorem 2.12 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces which pre-Exterior points $(pCl)_Y$-separated in Y and let $f : X \to Y$ be a function. Then f is preclosure-preserving if and only if Y is non-pre-separating.

Proof. By Theorem 2.10, if f is preclosure-preserving, then f is non-pre-separating. Suppose that f is non-pre-separating and let $A \subseteq X$. If $(pCl)_X = \phi$, then
$f((pCl)_X(A)) = \phi \subseteq (pCl)_Y(f(A))$.
Suppose $(pCl)_X(A) \neq \phi$. Let S_X and S_Y denote the pairs of $(pCl)_X$-separated subsets of X and the pairs of $(pCl)_Y$-separated subsets of Y, respectively. Let $y \in f((pCl)_X(A))$ and let $x \in (pCl)_X(A) \cap f^{-1}(\{y\})$. Since $x \in (pCl)_X(A)$, $\{\{x\}, A\} \notin S_X$ and since f non-pre-separating, $\{\{y\}, f(A)\} \notin S_Y$. Since pre-Exterior points are $(pCl)_Y$-separated, $y \in (pCl)_Y(f(A))$. Thus $f((pCl)_X(A)) \subseteq (pCl)_Y(f(A))$ for each $A \subseteq X$. ■

Corollary 2.13 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces which pre-isotonic closure functions and with $(pCl)_Y$-pointwise-pre-symmetric and let $f : X \to Y$ be a function. Then f is precontinuous if and only if f non-pre-separating.

Proof. Since $(pCl)_Y$ is pre-isotonic and pointwise-pre-symmetric, pre-Exterior points are preclosure separated in $(Y, (pCl)_Y)$ (Theorem 2.3 (1)). Since both pre-closure functions are pre-isotonic, f is preclosure-preserving if and only if f is precontinuous. Hence, we can apply the Theorem 2.12. ■

3. Preconnected generalized preclosure spaces

Definition 3.1 Let (X, pCl) be generalized preclosure space. X is said to be preconnected if X is not a union of disjoint nontrivial preclosure-separated pair of sets.

Theorem 3.2 Let (X, pCl) be generalized preclosure space with pre-grounded pre-isotonic pre-enlarging pCl. Then, the following are equivalent:
(1) (X, pCl) is preconnected,
(2) X can not be a union of nonempty disjoint preopen sets.

Proof. (1) $\Rightarrow$ (2): Let X be a union of nonempty disjoint preopen sets A and B. Then, $X = A \cup B$ and this implies that $B = X \backslash A$ and A is a preopen set. Thus, B is preclosed and hence $A \cap pCl(B) = A \cap B = \phi$. By using similar way, we obtain $B \cap pCl(A) = \phi$. Hence, A and B are preclosure-separated and hence X is not preconnected. This is a contradiction.
(2) $\Rightarrow$ (1): Suppose that X is not preconnected. Then $X = A \cup B$, where A, B are disjoin preclosure-separated sets, i.e. $A \cup pCl(B) = pCl(A) \cap B = \phi$. We have $pCl(B) \subseteq X \setminus A \subseteq B$. Since pCl is pre-enlarging, we obtain $pCl(B) = B$ and hence, B is preclosed. By using $pCl(A) \cap B = \phi$ and similar way, it is obvious that A is preclosed. But this is a contradiction. ■

Definition 3.3 Let (X, pCl) be a generalized preclosure space with pre-grounded pre-isotonic pCl. Then, (X, pCl) is called a T_1-pre-grounded pre-isotonic space if $pCl(\{x\}) \subset \{x\}$ for all $x \in X$.

Theorem 3.4 Let (X, pCl) be a generalized preclosure space with λ-grounded pre-isotonic pCl. Then, the following are equivalent:
(1) (X, pCl) is preconnected,
(2) Any precontinuous function $f : X \to Y$ is constant for all T_1-pre-grounded pre-isotonic spaces $Y = \{0, 1\}$.

Proof. (1) $\Rightarrow$ (2): Let X be preconnected. Suppose that $f : X \to Y$ is pre-continuous and it is not constant. Then there exists a set $U \subset X$ such that $U = f^{-1}(\{0\})$ and $X \setminus U = f^{-1}(\{1\})$. Since f is precontinuous and Y is T_1-λ-grounded pre-isotonic space, then we have $Cl_\lambda(U) = pCl(f^{-1}(\{0\})) \subset f^{-1}(pCl(\{0\}) \subset f^{-1}(\{0\}) = U$ and hence $pCl(U) \cap (X \setminus U) = \phi$. By using similar way we have $U \cap pCL(X \setminus U) = \phi$. This is a contradiction. Thus, f is constant.
(2) $\Rightarrow$ (1): Suppose that X is not preconnected. Then there exist preclosure-separated sets U and V such that $U \cup V = X$. We have $pCl(U) \subset U$ and $pCl(V) \subset V$ and $X \backslash U \subset V$. Since pCl is pre-isotonic and U and V are preclosure-separated, then $pCl(X \setminus U) \subset pCl(V) \subset X \setminus U$. If we consider the space (Y, pCl) by $Y = \{0, 1\}$, $pCl(\phi) = \phi$, $pCl(\{0\}) = \{0\}$, $pCl(\{1\}) = \{1\}$ and $pCl(Y) = Y$, then the space (Y, pCl) is a T_1-pre-grounded pre-isotonic space. We define the function $f : X \to Y$ as $f(U) = \{0\}$ and $f(X \setminus U) = \{1\}$. Let $A \neq \phi$ and $A \subset Y$. If $A = Y$, then $f^{-1}(A) = X$ and hence $pCl(X) = pCl(f^{-1}(A)) \subset X = f^{-1}(A) = f^{-1}(pCl(A))$. If $A = \{0\}$, then $f^{-1}(A) = U$ and hence $pCl(U) = pCl(f^{-1}(A)) \subset U = f^{-1}(A) = f^{-1}(pCl(A))$. If $A = \{1\}$, then $f^{-1}(A) = X \setminus U$ and so $pCl(X \setminus U) = pCl(f^{-1}(A)) \subset X \setminus U = f^{-1}(A) = f^{-1}(pCl(A))$. Hence, f is precontinuous. Since f is not constant, this is a contradiction. ■

Theorem 3.5 Let $f : (X, pCl) \to (Y, pCl)$ and $g : (Y, pCl) \to (Z, pCl)$ be precontinuous functions. Then, $g \circ f : X \to Z$ is precontinuous.

Proof. Suppose that f and g are precontinuous. For all $A \subset Z$ we have $pCl(g \circ f)^{-1}(A) = pCl(f^{-1}(g^{-1}(A))) \subset f^{-1}(pCl(g^{-1}(A))) \subset f^{-1}(g^{-1}(pCl(A))) = (g \circ f)^{-1}(pCl(A))$. Hence, $g \circ f : X \to Z$ is precontinuous. ■

Theorem 3.6 Let (X, pCl) and (Y, pCl) be generalized preclosure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto

Y. If X is preconnected, then Y is preconnected.

Proof. Suppose that $\{0,1\}$ is a generalized preclosure space with pre-grounded pre-isotonic pCl and $g: Y \to \{0,1\}$ is a precontinuous function. Since f is precontinuous, by Theorem 3.5, $g \circ f : X \to \{0,1\}$ is precontinuous. Since X is preconnected, $g \circ f$ is constant and hence g is constant. By Theorem 3.4, Y is preconnected. ■

Definition 3.7 Let (Y, pCl) be a generalized preclosure space with pre-grounded pre-isotonic pCl and more than one element. A generalized preclosure space (X, pCl) with pre-grounded pre-isotonic pCl is called Y-preconnected if any precontinuous function $f: X \to Y$ is constant.

Theorem 3.8 Let (Y, pCl) be a generalized preclosure space with pre-grounded pre-isotonic pCl and more than one element. Then every Y-preconnected generalized preclosure space with pre-grounded pre-isotonic is preconnected.

Proof. Let (X, pCl) be a Y-preconnected generalized preclosure space with pre-grounded pre-isotonic pCl. Suppose that $f: X \to \{0,1\}$ is a precontinuous function, where $\{0,1\}$ is a T_1-pre-grounded pre-isotonic space. Since Y is a generalized pre-closure space with pre-grounded pre-isotonic pre-enlarging pCl and more than one element, then there exists a precontinuous injection $g: \{0,1\} \to Y$. By Theorem 3.5, $g \circ f : X \to Y$ is precontinuous. Since X is Y-preconnected, then $g \circ f$ is constant and hence, by Theorem 3.4, X is preconnected. ■

Theorem 3.9 Let (X, pCl) and (Y, pCl) be generalized preclosure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is Z-preconnected, then Y is Z-preconnected.

Proof. Suppose that $g: Y \to Z$ is a precontinuous function. Then $g \circ f : X \to Z$ is precontinuous. Since X is Z-preconnected, then $g \circ f$ is constant. This implies that g is constant. Thus, Y is Z-preconnected. ■

Definition 3.10 A generalized preclosure space (X, pCl) is strongly preconnected if there is no countable collection of pairwise preclosure-separated sets $\{A_n\}$ such that $X = \cup A_n$.

Theorem 3.11 Every strongly preconnected generalized preclosure space with pre-grounded pre-isotonic pCl is preconnected.

Theorem 3.12 Let (X, pCl) and (Y, pCl) be generalized preclosure spaces with pre-grounded pre-isotonic pCl and $f : (X, pCl) \to (Y, pCl)$ be a precontinuous function onto Y. If X is strongly preconnected, then Y is strongly preconnected.

Proof. Suppose that Y is not strongly preconnected. Then, there exists a countable collection of pairwise preclosure-separated sets $\{A_n\}$ such that $Y = \cup A_n$. Since $f^{-1}(A_n) \cap pCl(f^{-1}(A_m)) \subset f^{-1}(A_n) \cap f^{-1}(pCl(A_m)) = \phi$ for all $n \neq m$, then the collection $\{f^{-1}(A_n)\}$ is pairwise preclosure separated. This is a contradiction. Hence, Y is strongly preconnected. ■

Theorem 3.13 Let $(X, (pCl)_X)$ and $(Y, (pCl)_Y)$ be generalized preclosure spaces. Then the following are equivalent for a function $f: X \to Y$.
(1) f is precontinuous,
(2) $f^{-1}(pInt(B)) \subseteq pInt(f^{-1}(B))$ for each $B \subseteq Y$.

Theorem 3.14 Let (X, pCl) be a generalized preclosure space with pre-grounded pre-isotonic pCl. Then (X, pCl) is strongly preconnected if and only if (X, pCl) is Y-

preconnected for any countable T_1-pre-grounded pre-isotonic space (Y, pCl).

Proof. Let (X, pCl) be strongly preconnected. Suppose that (X, pCl) is not Y-preconnected for some countable T_1-pre-grounded pre-isotonic space (Y, pCl). There exists a precontinuous function $f : X \to Y$ which is not constant and hence $K = f(X)$ is a countable set with more than one element. For each $y_n \in K$, there exists $U_n \subset X$ such that $U_n = f^{-1}(\{y_n\})$ and hence $Y = \cup U_n$. Since f is precontinuous and Y is pre-grounded, then for each $n \neq m, U_n \cap pCl(U_m) = f^{-1}(\{y_n\}) \cap pCl(f^{-1}(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(pCl(\{y_m\})) \subset f^{-1}(\{y_n\}) \cap f^{-1}(\{y_m\}) = \phi$. This contradict with the strong preconnectedness of X. Thus, X is Y-preconnected. Conversely, let X be Y-preconnected for any countable T_1-pre-grounded pre-isotonic space (Y, pCl). Suppose that X is not strongly preconnected. There exists a countable collection of pairwise preclosure-separated sets $\{U_n\}$ such that $X = \cup U_n$. We take the space (Z, pCl), where Z is the set of integers and $pCl : P(Z) \to P(Z)$ is defined by $pCl(K) = K$ for each $K \subset Z$. Clearly (Z, pCl) is countable T_1-pre-grounded pre-isotonic space. Put $U_k \in \{U_n\}$. We define a function $f : X \to Z$ by $f(U_k) = \{x\}$ and $f(X \setminus U_k) = \{y\}$ where $x, y \in Z$ and $x \neq y$. Since $pCl(U_k) \cap U_n = \phi$ for all $n \neq k$, then $pCl(U_k) \cap \cup_{n \neq k} U_k = \phi$ and hence $pCl(U_k) \subset U_k$. Let $\phi \neq K \subset Z$. If $x, y \in K$ then $f^{-1}(K) = X$ and $pCl(f^{-1}(K)) = pCl(X) \subset X = f^{-1}(K) = f^{-1}(pCl(K))$. If $x \in K$ and $y \notin K$, then $f^{-1}(K) = U_k$ and $pCl(f^{-1}(K)) = pCl(U_k) \subset U_k = f^{-1}(K) = f^{-1}(pCl(K))$. If $y \in K$ and $x \notin K$, then $f^{-1}(K) = X \setminus U_k$. Since $pCl(K) = K$ for each $K \subset Z$, then $pInt(K) = K$ for each $K \subset Z$. Also, $X \setminus U_K \subset \cup_{n \neq k} U_n \subset X \setminus pCl(U_k) = pInt(X \setminus U_k)$. Therefore, $f^{-1}(pInt(K)) = X \setminus U_k = f^{-1}(K) \subset pInt(X \setminus U_k) = pInt(f^{-1}(K))$. Hence we obtain that f is precontinuous. Since f is not constant, this is a contradiction with the Z-preconnectedness of X. Hence, X is strongly preconnected. ■

4. Conclusion

Closure spaces in point-set topology will give some new topological properties (for example: separation axioms, compactness, connectedness, continuity) which have been found to be very useful in the study of certain objects of digital topology [9]. Thus we may stress once more the importance of preclosure operators as a branch of them and the possible application in computer graphics [5] and quantum physics [4].

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