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Common fixed points for a pair of mappings in b-Metric spaces via digraphs and altering distance functions

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Abstract. In this paper, we discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings satisfying some generalized contractive type conditions in b-metric spaces endowed with graphs and altering distance functions. Finally, some examples are provided to justify the validity of our results.

Keywords: b-metric, digraph, altering distance function, common fixed point.

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1. Introduction

It is well known that Banach contraction principle [6] in metric spaces is one of the pivotal result in fixed point theory and nonlinear analysis. It remains a source of inspiration for researchers of this domain. Because of its simplicity and usefulness, it becomes an important tool in various research activities. In 1989, Bakhtin [8] introduced the concept of b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces. Recently, the study of fixed point theory with a graph takes a prominent place in many aspects. Many important results of [2–4, 10–12, 16, 17, 20, 25] have become the source of motivation for many researchers that do research in fixed point theory. Motivated by the work in [5, 14, 24], we will prove some common fixed point results for a pair of self-mappings satisfying some

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new contractive conditions in the setting of b-metric spaces. More precisely, we discuss the existence and uniqueness of points of coincidence and common fixed points for mappings satisfying a generalized contractive condition in b-metric spaces endowed with a graph and altering distance functions. Our results extend and unify several existing results in the literature. Finally, we give some examples to justify the validity of our results.

2. Some Basic Concepts

In 1984, Khan et. al. [21] introduced the concept of altering distance functions as follows.

Definition 2.1 The function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties hold:

(i) φ is continuous and non-decreasing.
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Now, let us recall some basic notations and definitions in b-metric spaces.

Definition 2.2 [13] Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leqslant s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is worth mentioning that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces.

***Example* 2.3** [24] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

***Example* 2.4** [23] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that $d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0)$. It is easy to verify that $s = \frac{3}{2}$.

Definition 2.5 [9] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 1 *[9] In a b-metric space (X, d), the following assertions hold:*

(i) A convergent sequence has a unique limit.
(ii) Each convergent sequence is Cauchy.
(iii) In general, a b-metric is not continuous.

Lemma 2.6 [3] Let (X, d) be a b-metric space with $s \geqslant 1$ and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have

$$\frac{1}{s^2}d(x,y) \leqslant \liminf_{n\to\infty} d(x_n,y_n) \leqslant \limsup_{n\to\infty} d(x_n,y_n) \leqslant s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, we have

$$\frac{1}{s}d(x,z) \leqslant \liminf_{n\to\infty} d(x_n,z) \leqslant \limsup_{n\to\infty} d(x_n,z) \leqslant sd(x,z)$$

for each $z \in X$.

Definition 2.7 [1] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.8 [19] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, $T(Sx) = S(Tx)$ whenever $Sx = Tx$.

Proposition 2.9 [1] Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.

Lemma 2.10 [22] Every sequence (x_n) of elements from a b-metric space (X, d), having the property that there exists $\gamma \in [0, 1)$ such that $d(x_{n+1}, x_n) \leqslant \gamma\, d(x_n, x_{n-1})$ for every $n \in \mathbb{N}$, is Cauchy.

We next review some basic notions in graph theory. Let (X, d) be a b-metric space. We consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We also assume that G has no parallel edges. So we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G i.e., the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G}) = E(G) \cup E(G^{-1})$.

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 15, 18]. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

3. Main results

Suppose that (X, d) is a b-metric space with the coefficient $s \geqslant 1$ and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges. Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $fx_0 = gx_1$. Continuing in this way, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$.

Definition 3.1 [23] Let (X, d) be a b-metric space endowed with a graph G and $f, g : X \to X$ be such that $f(X) \subseteq g(X)$. We define C_{gf} the set of all elements x_0 of X such that $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and for every sequence (gx_n) such that $gx_n = fx_{n-1}$.

Taking $g = I$, the identity map on X, C_{gf} becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.

Theorem 3.2 Let (X, d) be a b-metric space endowed with a graph G and let f, g : $X \to X$ be mappings. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(sd(fx, fy)) \leqslant \psi(M_s(gx, gy)) - \varphi(M_s(gx, gy)) \tag{1}$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, where

$$M_s(gx, gy) = max\{d(gx, gy), d(gy, fy), \frac{d(gx, fy)}{2s}\}.$$

Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the following property:

$(*)$ If (gx_n) is a sequence in X such that $gx_n \to x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geqslant 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, x) \in E(\tilde{G})$ for all $i \geqslant 1$.

Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the following property:

$(**)$ If x, y are points of coincidence of f and g in X, then $(x, y) \in E(\tilde{G})$.

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$.

We assume that $gx_n \neq gx_{n-1}$ for every $n \in \mathbb{N}$. If $gx_n = gx_{n-1}$ for some $n \in \mathbb{N}$, then $gx_n = fx_{n-1} = gx_{n-1}$ and hence gx_n is a point of coincidence of f and g. We now show that $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0$. Since $\frac{d(gx_{n-1},gx_{n+1})}{2s} \leqslant \frac{d(gx_{n-1},gx_n)+d(gx_n,gx_{n+1})}{2}$, it follows that

$$\begin{aligned} M_s(gx_{n-1}, gx_n) &= max\{d(gx_{n-1}, gx_n), d(gx_n, fx_n), \frac{d(gx_{n-1}, fx_n)}{2s}\} \\ &= max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{d(gx_{n-1}, gx_{n+1})}{2s}\} \\ &= max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}. \end{aligned}$$

For any natural number n, we have by using (1) that

$$\begin{aligned} \psi(d(gx_n, gx_{n+1})) &\leqslant \psi(sd(gx_n, gx_{n+1})) \\ &= \psi(sd(fx_{n-1}, fx_n)) \\ &\leqslant \psi(M_s(gx_{n-1}, gx_n)) - \varphi(M_s(gx_{n-1}, gx_n)) \\ &= \psi(max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}) \\ &\quad - \varphi(max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}) \\ &< \psi(max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\}). \end{aligned} \tag{2}$$

If $max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\} = d(gx_n, gx_{n+1})$, then it follows from (2) that

$$\psi(d(gx_n, gx_{n+1})) \leqslant \psi(d(gx_n, gx_{n+1})) - \varphi(d(gx_n, gx_{n+1})) < \psi(d(gx_n, gx_{n+1})),$$

which is a contradiction. Therefore, $max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\} = d(gx_{n-1}, gx_n)$. Thus, (2) reduces to

$$\psi(d(gx_n, gx_{n+1})) \leqslant \psi(d(gx_{n-1}, gx_n)) - \varphi(d(gx_{n-1}, gx_n)) < \psi(d(gx_{n-1}, gx_n)). \tag{3}$$

Since ψ is non-decreasing, $(d(gx_n, gx_{n+1}))_{n\in\mathbb{N}\cup\{0\}}$ is a non-increasing sequence of positive numbers. Hence, there exists $r \geqslant 0$ such that $\lim\limits_{n\to\infty} d(gx_n, gx_{n+1}) = r$. Taking limit as $n \to \infty$ in (3), we have $\psi(r) \leqslant \psi(r) - \varphi(r) \leqslant \psi(r)$, which implies that $\varphi(r) = 0$ and so, $r = 0$. Therefore,

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{4}$$

We now show that (gx_n) is Cauchy in $g(X)$. If possible, suppose (gx_n) is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences (gx_{m_i}) and (gx_{n_i}) of (gx_n) such that n_i is the smallest positive integer satisfying

$$d(gx_{m_i}, gx_{n_i}) \geqslant \epsilon \; for \; n_i > m_i > i. \tag{5}$$

So, it must be the case that

$$d(gx_{m_i}, gx_{n_i-1}) < \epsilon. \tag{6}$$

By using conditions (5) and (6), we obtain

$$\begin{aligned}
\epsilon &\leqslant d(gx_{m_i}, gx_{n_i}) \\
&\leqslant sd(gx_{m_i}, gx_{m_i-1}) + sd(gx_{m_i-1}, gx_{n_i}) \\
&\leqslant sd(gx_{m_i}, gx_{m_i-1}) + s^2 d(gx_{m_i-1}, gx_{n_i-1}) + s^2 d(gx_{n_i-1}, gx_{n_i}).
\end{aligned}$$

Taking the upper limit as $i \to \infty$ and using (4), we get $\frac{\epsilon}{s^2} \leqslant \limsup\limits_{i\to\infty} d(gx_{m_i-1}, gx_{n_i-1})$. Again,

$$d(gx_{m_i-1}, gx_{n_i-1}) \leqslant sd(gx_{m_i-1}, gx_{m_i}) + sd(gx_{m_i}, gx_{n_i-1}).$$

Taking the upper limit as $i \to \infty$ and using (4) and (6), then $\limsup\limits_{i\to\infty} d(gx_{m_i-1}, gx_{n_i-1}) \leqslant \epsilon s$. Therefore, we obtain

$$\frac{\epsilon}{s^2} \leqslant \limsup_{i\to\infty} d(gx_{m_i-1}, gx_{n_i-1}) \leqslant \epsilon s. \tag{7}$$

From condition (5), we have

$$\begin{aligned}
\epsilon &\leqslant d(gx_{m_i}, gx_{n_i}) \\
&\leqslant sd(gx_{m_i}, gx_{m_i-1}) + sd(gx_{m_i-1}, gx_{n_i}) \\
&\leqslant sd(gx_{m_i}, gx_{m_i-1}) + s^2 d(gx_{m_i-1}, gx_{n_i-1}) + s^2 d(gx_{n_i-1}, gx_{n_i}).
\end{aligned}$$

Taking the upper limit as $i \to \infty$ and using condition (7), it follows that

$$\frac{\epsilon}{s} \leqslant \limsup_{i\to\infty} d(gx_{m_i-1}, gx_{n_i}) \leqslant \epsilon s^2. \tag{8}$$

By using (1), we get

$$\begin{aligned} \psi(sd(gx_{m_i}, gx_{n_i})) &= \psi(sd(fx_{m_i-1}, fx_{n_i-1})) \\ &\leqslant \psi(M_s(gx_{m_i-1}, gx_{n_i-1})) - \varphi(M_s(gx_{m_i-1}, gx_{n_i-1})), \end{aligned} \tag{9}$$

where

$$\begin{aligned} M_s(gx_{m_i-1}, gx_{n_i-1}) &= max \begin{Bmatrix} d(gx_{m_i-1}, gx_{n_i-1}), d(gx_{n_i-1}, fx_{n_i-1}), \\ \frac{d(gx_{m_i-1}, fx_{n_i-1})}{2s} \end{Bmatrix} \\ &= max\{d(gx_{m_i-1}, gx_{n_i-1}), d(gx_{n_i-1}, gx_{n_i}), \frac{d(gx_{m_i-1}, gx_{n_i})}{2s}\}. \end{aligned}$$

Taking the upper limit as $i \to \infty$ and using conditions (4), (7) and (8), we have

$$\begin{aligned} \limsup_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1}) &= max \begin{Bmatrix} \limsup_{i\to\infty} d(gx_{m_i-1}, gx_{n_i-1}), 0, \\ \frac{\limsup_{i\to\infty} d(gx_{m_i-1}, gx_{n_i})}{2s} \end{Bmatrix} \\ &\leqslant max\{\epsilon s, \frac{\epsilon s^2}{2s}\} = \epsilon s. \end{aligned} \tag{10}$$

From conditions (7) and (10), it follows that

$$\frac{\epsilon}{s^2} \leqslant \limsup_{i\to\infty} d(gx_{m_i-1}, gx_{n_i-1} \leqslant \limsup_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1}) \leqslant \epsilon s.$$

Therefore,

$$\frac{\epsilon}{s^2} \leqslant \limsup_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1}) \leqslant \epsilon s. \tag{11}$$

Similarly,

$$\frac{\epsilon}{s^2} \leqslant \liminf_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1}) \leqslant \epsilon s. \tag{12}$$

Taking the upper limit as $i \to \infty$ in (9) and using conditions (5), (11), it follows that

$$\begin{aligned} \psi(\epsilon s) &\leqslant \psi(s \limsup_{i\to\infty} d(gx_{m_i}, gx_{n_i})) \\ &\leqslant \psi(\limsup_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1})) - \liminf_{i\to\infty} \varphi(M_s(gx_{m_i-1}, gx_{n_i-1})) \\ &\leqslant \psi(\epsilon s) - \varphi(\liminf_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1})), \end{aligned}$$

which gives that $\varphi(\liminf_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1})) = 0$, i.e., $\liminf_{i\to\infty} M_s(gx_{m_i-1}, gx_{n_i-1}) = 0$. This contradicts the condition (12). Thus, (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists an $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$. As $x_0 \in C_{gf}$, it follows that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geqslant 0$ and so by property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$ for all $i \geqslant 1$.

Again, using condition (1), we have

$$\psi(sd(fv, fx_{n_i})) \leqslant \psi(M_s(gv, gx_{n_i})) - \varphi(M_s(gv, gx_{n_i})), \tag{13}$$

where

$$\begin{aligned} M_s(gv, gx_{n_i}) &= max\{d(gv, gx_{n_i}), d(gx_{n_i}, fx_{n_i}), \frac{d(gv, fx_{n_i})}{2s}\} \\ &= max\{d(gv, gx_{n_i}), d(gx_{n_i}, gx_{n_i+1}), \frac{d(gv, gx_{n_i+1})}{2s}\} \to 0 \ as\ i \to \infty. \end{aligned}$$

Taking the upper limit as $i \to \infty$ in (13), it follows that

$$\psi(s \limsup_{i\to\infty} d(fv, fx_{n_i})) \leqslant \psi(0) - \varphi(0) = 0,$$

which implies that $\psi(s \limsup_{i\to\infty} d(fv, gx_{n_i+1})) = 0$. Using Lemma 2.6, we obtain that

$$\psi(d(fv, gv)) = \psi(s\,\frac{1}{s} d(fv, gv)) \leqslant \psi(s \limsup_{i\to\infty} d(fv, gx_{n_i+1})) = 0.$$

This gives that $d(fv, gv) = 0$ and hence $fv = gv = u$. Therefore, u is a point of coincidence of f and g.

For uniqueness, assume that there is another point of coincidence u^* in X such that $fx = gx = u^*$ for some $x \in X$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. Then,

$$\begin{aligned} M_s(u, u^*) &= M_s(gv, gx) \\ &= max\{d(gv, gx), d(gx, fx), \frac{d(gv, fx)}{2s}\} \\ &= max\{d(u, u^*), d(u^*, u^*), \frac{d(u, u^*)}{2s}\} \\ &= d(u, u^*). \end{aligned}$$

By using condition (1), we get

$$\begin{aligned} \psi(d(u, u^*)) &\leqslant \psi(sd(u, u^*)) = \psi(sd(fv, fx)) \\ &\leqslant \psi(M_s(gv, gx)) - \varphi(M_s(gv, gx)) = \psi(d(u, u^*)) - \varphi(d(u, u^*)), \end{aligned}$$

which gives that $\varphi(d(u, u^*)) = 0$ i.e., $d(u, u^*) = 0$ and hence, $u = u^*$. Therefore, f and g have a unique point of coincidence in X. If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. ■

Corollary 3.3 Let (X,d) be a b-metric space endowed with a graph G and let f, g : $X \to X$ be mappings. Suppose that there exists $k \in [0,1)$ such that

$$d(fx,fy) \leqslant \frac{k}{s}\, max\{d(gx,gy), d(gy,fy), \frac{d(gx,fy)}{2s}\}$$

for all $x,y \in X$ with $(gx,gy) \in E(\tilde{G})$. Suppose $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X with the property $(*)$. Then f and g have a point of coincidence in X if $C_{gf} \neq \emptyset$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.2 by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0, +\infty)$. ■

Corollary 3.4 Let (X,d) be a complete b-metric space endowed with a graph G and let $f : X \to X$ be a mapping. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(sd(fx,fy)) \leqslant \psi(M_s(x,y)) - \varphi(M_s(x,y))$$

for all $x,y \in X$ with $(x,y) \in E(\tilde{G})$, where $M_s(x,y) = max\{d(x,y), d(y,fy), \frac{d(x,fy)}{2s}\}$. Suppose the triple (X,d,G) has the following property:

$(\acute{*})$ If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geqslant 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geqslant 1$.
Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:

$(\acute{**})$ If x, y are fixed points of f in X, then $(x,y) \in E(\tilde{G})$.

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$, the identity map on X. ■

Corollary 3.5 Let (X,d) be a b-metric space and let f, g : $X \to X$ be mappings. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(sd(fx,fy)) \leqslant \psi(M_s(gx,gy)) - \varphi(M_s(gx,gy))$$

for all $x,y \in X$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. ■

Corollary 3.6 Let (X,d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(sd(fx,fy)) \leqslant \psi(M_s(x,y)) - \varphi(M_s(x,y))$$

for all $x,y \in X$. Then f has a unique fixed point in X.

Proof. It follows from Theorem 3.2 by putting $G = G_0$ and $g = I$. ■

Corollary 3.7 Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$ and let $f : X \to X$ be a mapping. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(sd(fx, fy)) \leqslant \psi(M_s(x, y)) - \varphi(M_s(x, y))$$

for all $x, y \in X$ with $x \preceq y$ or, $y \preceq x$. Suppose the triple $(X, d, \preceq)$ has the following property:

(†) If (x_n) is a sequence in X such that $x_n \to x$ and x_n, x_{n+1} are comparable for all $n \geqslant 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geqslant 1$.

If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for m, $n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:

(††) If x, y are fixed points of f in X, then x, y are comparable.

Proof. The proof can be obtained from Theorem 3.2 by taking $g = I$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. ■

Before presenting our second main theorem, we need the following definition.

Definition 3.8 Let (X, d) be a b-metric space endowed with a graph G and $f, T : X \to X$ be two mappings. We define $C_{f\vee T}$ the set of all elements x_0 of X such that $(x_n, x_m) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$ and for every sequence (x_n) such that $x_n = fx_{n-1}$ if n is odd and $x_n = Tx_{n-1}$ if n is even.

Taking $T = f$, $C_{f\vee T}$ reduces to C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$.

Theorem 3.9 Let (X, d) be a complete b-metric space endowed with a graph G and let $f, T : X \to X$ be mappings. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(s^4 d(fx, Ty)) \leqslant \psi(N_s(x, y)) - \varphi(N_s(x, y)) \tag{14}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where

$$N_s(x, y) = max\left\{d(x, y), min\{d(x, fx), d(y, Ty)\}, \tfrac{1}{2s} min\{d(x, Ty), d(y, fx)\}\right\}.$$

Suppose the triple (X, d, G) has the property $(\acute{*})$. Then f and T have a common fixed point in X if $C_{f\vee T} \neq \emptyset$. Moreover, f and T have a unique common fixed point in X if the graph G has the following property:

(‡) If x, y are common fixed points of f and T in X, then $(x, y) \in E(\tilde{G})$.

Proof. We first prove that u is a fixed point of T if and only if u is a fixed point of f. Suppose that u is a fixed point of T i.e., $Tu = u$. Then, by using condition (14), we obtain

$$\psi(s^4 d(fu, u)) = \psi(s^4 d(fu, Tu)) \leqslant \psi(N_s(u, u)) - \varphi(N_s(u, u))$$

where

$$N_s(u, u) = max\left\{d(u, u), min\{d(u, fu), d(u, Tu)\}, \frac{1}{2s} min\{d(u, Tu), d(u, fu)\}\right\} = 0.$$

Therefore, $\psi(s^4 d(fu,u)) \leqslant \psi(0)-\varphi(0)=0$, which implies that $d(fu,u)=0$ i.e., $fu=u$. By an argument similar to that used above, we can show that if u is a fixed point of f, then u is also a fixed point of T.

Suppose that $C_{f\vee T} \neq \emptyset$. We choose an $x_0 \in C_{f\vee T}$ and keep it fixed. We can construct a sequence (x_n) in X such that $x_n = fx_{n-1}$ if n is odd and $x_n = Tx_{n-1}$ if n is even, and $(x_n, x_m) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$.

We assume that $x_n \neq x_{n-1}$ for every $n \in \mathbb{N}$. If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_{2n} = fx_{2n}$. So, x_{2n} is a fixed point of f. By our previous discussion, it follows that x_{2n} is also a fixed point of T. If $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_{2n+1} = Tx_{2n+1}$ which gives that, x_{2n+1} is a fixed point of T. Therefore, it follows from our previous observation that x_{2n+1} is a fixed point of f. We now show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. As $(x_{2n}, x_{2n+1}) \in E(\tilde{G})$, by using condition (14), we obtain

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &\leqslant \psi(s^4 d(x_{2n+1}, x_{2n+2})) \\
&= \psi(s^4 d(fx_{2n}, Tx_{2n+1})) \\
&\leqslant \psi(N_s(x_{2n}, x_{2n+1})) - \varphi(N_s(x_{2n}, x_{2n+1})),
\end{aligned}$$

where

$$\begin{aligned}
N_s(x_{2n}, x_{2n+1}) &= max \left\{ \begin{array}{l} d(x_{2n}, x_{2n+1}), min\{d(x_{2n}, fx_{2n}), d(x_{2n+1}, Tx_{2n+1})\}, \\ \frac{1}{2s} min\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, fx_{2n})\} \end{array} \right\} \\
&= max \left\{ \begin{array}{l} d(x_{2n}, x_{2n+1}), min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}, \\ \frac{1}{2s} min\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} \end{array} \right\} \\
&= d(x_{2n}, x_{2n+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &\leqslant \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})) \\
&< \psi(d(x_{2n}, x_{2n+1})). \qquad (15)
\end{aligned}$$

Similarly, since $N_s(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n-1})$, we have

$$\begin{aligned}
\psi(d(x_{2n}, x_{2n+1})) &\leqslant \psi(s^4 d(x_{2n}, x_{2n+1})) \\
&= \psi(s^4 d(fx_{2n}, Tx_{2n-1})) \\
&\leqslant \psi(N_s(x_{2n}, x_{2n-1})) - \varphi(N_s(x_{2n}, x_{2n-1})) \\
&= \psi(d(x_{2n}, x_{2n-1})) - \varphi(d(x_{2n}, x_{2n-1})) \\
&< \psi(d(x_{2n-1}, x_{2n})). \qquad (16)
\end{aligned}$$

Combining conditions (15) and (16), we get $\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n))$ for all $n \in \mathbb{N}$. Since ψ is non-decreasing, $(d(x_n, x_{n+1}))_{n\in\mathbb{N}}$ is a non-increasing sequence of positive numbers. Hence there exists $r \geqslant 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Taking limit as $n \to \infty$ in (15), we have $\psi(r) \leqslant \psi(r) - \varphi(r) \leqslant \psi(r)$, which implies that $\varphi(r) = 0$ and so, $r = 0$.

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{17}$$

We shall show that (x_n) is a Cauchy sequence in X. It is sufficient to show that (x_{2n}) is a Cauchy sequence. If possible, suppose that (x_{2n}) is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences (x_{2m_i}) and (x_{2n_i}) of (x_{2n}) such that n_i is the smallest positive integer for which

$$d(x_{2m_i}, x_{2n_i}) \geqslant \epsilon \; for \; n_i > m_i > i. \tag{18}$$

This implies that

$$d(x_{2m_i}, x_{2n_i-2}) < \epsilon. \tag{19}$$

By repeated use of the triangular inequality and by condition (19), we have

$$\begin{aligned} d(x_{2n_i+1}, x_{2m_i}) &\leqslant sd(x_{2n_i+1}, x_{2n_i}) + sd(x_{2n_i}, x_{2m_i}) \\ &\leqslant sd(x_{2n_i+1}, x_{2n_i}) + s^2 d(x_{2n_i}, x_{2n_i-1}) \\ &\quad + s^3 d(x_{2n_i-1}, x_{2n_i-2}) + s^3 d(x_{2n_i-2}, x_{2m_i}) \\ &< sd(x_{2n_i+1}, x_{2n_i}) + s^2 d(x_{2n_i}, x_{2n_i-1}) \\ &\quad + s^3 d(x_{2n_i-1}, x_{2n_i-2}) + \epsilon s^3. \end{aligned}$$

Taking the upper limit as $i \to \infty$, then $\limsup\limits_{i\to\infty} d(x_{2n_i+1}, x_{2m_i}) \leqslant \epsilon s^3$. From (18), we get

$$\epsilon \leqslant d(x_{2m_i}, x_{2n_i}) \leqslant sd(x_{2m_i}, x_{2n_i+1}) + sd(x_{2n_i+1}, x_{2n_i}).$$

Taking the upper limit as $i \to \infty$, we get $\frac{\epsilon}{s} \leqslant \limsup\limits_{i\to\infty} d(x_{2m_i}, x_{2n_i+1})$. Therefore,

$$\frac{\epsilon}{s} \leqslant \limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i+1}) \leqslant \epsilon s^3. \tag{20}$$

Again,

$$\begin{aligned} d(x_{2n_i}, x_{2m_i-1}) &\leqslant sd(x_{2n_i}, x_{2n_i-1}) + sd(x_{2n_i-1}, x_{2m_i-1}) \\ &\leqslant sd(x_{2n_i}, x_{2n_i-1}) + s^2 d(x_{2n_i-1}, x_{2n_i-2}) \\ &\quad + s^3 d(x_{2n_i-2}, x_{2m_i}) + s^3 d(x_{2m_i}, x_{2m_i-1}). \end{aligned}$$

Taking the upper limit as $i \to \infty$, we obtain $\limsup\limits_{n\to\infty} d(x_{2n_i}, x_{2m_i-1}) \leqslant \epsilon s^3$. Also,

$$\epsilon \leqslant d(x_{2n_i}, x_{2m_i}) \leqslant sd(x_{2n_i}, x_{2m_i-1}) + sd(x_{2m_i-1}, x_{2m_i}).$$

Taking the upper limit as $i \to \infty$, we get $\frac{\epsilon}{s} \leqslant \limsup_{i\to\infty} d(x_{2n_i}, x_{2m_i-1})$. Thus,

$$\frac{\epsilon}{s} \leqslant \limsup_{i\to\infty} d(x_{2n_i}, x_{2m_i-1}) \leqslant \epsilon s^3. \tag{21}$$

Similarly, we can obtain

$$\epsilon \leqslant \limsup_{i\to\infty} d(x_{2n_i}, x_{2m_i}) \leqslant \epsilon s^4 \tag{22}$$

and

$$\frac{\epsilon}{s^2} \leqslant \limsup_{i\to\infty} d(x_{2n_i+1}, x_{2m_i-1}) \leqslant \epsilon s^4. \tag{23}$$

As $(x_{2n_i}, x_{2m_i-1}) \in E(\tilde{G})$, by using condition (14), we obtain

$$\begin{aligned} \psi(s^4 d(x_{2n_i+1}, x_{2m_i})) &= \psi(s^4 d(fx_{2n_i}, Tx_{2m_i-1})) \\ &\leqslant \psi(N_s(x_{2n_i}, x_{2m_i-1})) - \varphi(N_s(x_{2n_i}, x_{2m_i-1})), \end{aligned} \tag{24}$$

where

$$\begin{aligned} N_s(x_{2n_i}, x_{2m_i-1}) &= max\left\{ \begin{array}{l} d(x_{2n_i}, x_{2m_i-1}), min\{d(x_{2n_i}, fx_{2n_i}), d(x_{2m_i-1}, Tx_{2m_i-1})\}, \\ \frac{1}{2s} min\{d(x_{2n_i}, Tx_{2m_i-1}), d(x_{2m_i-1}, fx_{2n_i})\} \end{array} \right\} \\ &= max\left\{ \begin{array}{l} d(x_{2n_i}, x_{2m_i-1}), min\{d(x_{2n_i}, x_{2n_i+1}), d(x_{2m_i-1}, x_{2m_i})\}, \\ \frac{1}{2s} min\{d(x_{2n_i}, x_{2m_i}), d(x_{2m_i-1}, x_{2n_i+1})\} \end{array} \right\}. \end{aligned} \tag{25}$$

Taking the upper limit as $i \to \infty$ in (25) and using conditions (17), (21)-(23), we get

$$\begin{aligned} \frac{\epsilon}{2s^3} = min[\frac{\epsilon}{s}, \frac{\epsilon}{2s}, \frac{1}{2s}\frac{\epsilon}{s^2}] &\leqslant \limsup_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1}) \\ &= max\left\{ \begin{array}{l} \limsup_{i\to\infty} d(x_{2n_i}, x_{2m_i-1}), 0, \\ \\ \frac{1}{2s} min\{\limsup_{i\to\infty} d(x_{2n_i}, x_{2m_i}), \\ \\ \limsup_{i\to\infty} d(x_{2m_i-1}, x_{2n_i+1})\} \end{array} \right\} \\ &\leqslant max\{\epsilon s^3, \frac{1}{2s}\epsilon s^4\} = \epsilon s^3. \end{aligned}$$

Therefore,

$$\frac{\epsilon}{2s^3} \leqslant \limsup_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1}) \leqslant \epsilon s^3. \tag{26}$$

Similarly, we can obtain

$$\frac{\epsilon}{2s^3} \leqslant \liminf_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1}) \leqslant \epsilon s^3. \tag{27}$$

Taking the upper limit as $i \to \infty$ in (24) and using conditions (20), (26), (27), we get

$$\begin{aligned}\psi(\epsilon s^3) = \psi(s^4 \frac{\epsilon}{s}) &\leqslant \psi(s^4 \limsup_{i\to\infty} d(x_{2n_i+1}, x_{2m_i}))\\ &\leqslant \psi(\limsup_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1})) - \liminf_{i\to\infty} \varphi(N_s(x_{2n_i}, x_{2m_i-1}))\\ &\leqslant \psi(\epsilon s^3) - \varphi(\liminf_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1})),\end{aligned}$$

which implies that $\varphi(\liminf_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1})) = 0$. Consequently, it follows that $\liminf_{i\to\infty} N_s(x_{2n_i}, x_{2m_i-1}) = 0$, a contradiction to (27). Therefore, (x_n) is a Cauchy sequence in X. Since (X, d) is complete, there exists an $u \in X$ such that $x_n \to u$ as $n \to \infty$. As $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geqslant 0$, so by property $(*)'$, there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, u) \in E(\tilde{G})$ for all $i \geqslant 1$. If n_i is even, then $x_{n_i+1} = fx_{n_i}$. So, by using condition (14), we obtain

$$\psi(s^4 d(x_{n_i+1}, Tu)) = \psi(s^4 d(fx_{n_i}, Tu)) \leqslant \psi(N_s(x_{n_i}, u)) - \varphi(N_s(x_{n_i}, u)), \tag{28}$$

where

$$\begin{aligned}N_s(x_{n_i}, u) &= max \begin{Bmatrix} d(x_{n_i}, u), min\{d(x_{n_i}, fx_{n_i}), d(u, Tu)\}, \\ \frac{1}{2s} min\{d(x_{n_i}, Tu), d(u, fx_{n_i})\} \end{Bmatrix}\\ &\leqslant max \begin{Bmatrix} d(x_{n_i}, u), d(x_{n_i}, x_{n_i+1}), \\ \frac{1}{2s} d(u, x_{n_i+1}) \end{Bmatrix} \to 0 \ as \ i \to \infty.\end{aligned}$$

Therefore, taking the upper limit as $i \to \infty$ in (28), we have

$$\begin{aligned}\psi(s^4 \limsup_{i\to\infty} d(x_{n_i+1}, Tu)) &\leqslant \psi(\limsup_{i\to\infty} N_s(x_{n_i}, u)) - \varphi(\liminf_{i\to\infty} N_s(x_{n_i}, u))\\ &= \psi(0) - \varphi(0) = 0,\end{aligned}$$

which gives that $\psi(s^4 \limsup_{i\to\infty} d(x_{n_i+1}, Tu)) = 0$. By using Lemma 2.6, we have

$$\psi(s^3 d(u, Tu)) = \psi(s^4 \frac{1}{s} d(u, Tu)) \leqslant \psi(s^4 \limsup_{i\to\infty} d(x_{n_i+1}, Tu)) = 0,$$

which gives that $d(u, Tu) = 0$ i.e., $Tu = u$ and hence by our previous observation, u is also a fixed point of f. If n_i is odd, then proceeding as above, we can show that $fu = u$. Therefore, u is a common fixed point of f and T.

For uniqueness, let v be another common fixed point of f and T in X. By property

(‡), we have $(u,v) \in E(\tilde{G})$. Then

$$\begin{aligned} N_s(u,v) &= max \begin{Bmatrix} d(u,v), min\{d(u,fu), d(v,Tv)\}, \\ \frac{1}{2s} min\{d(u,Tv), d(v,fu)\} \end{Bmatrix} \\ &= max\{d(u,v), 0, \frac{1}{2s} d(u,v)\} \\ &= d(u,v). \end{aligned}$$

As $(u,v) \in E(\tilde{G})$, we have

$$\begin{aligned} \psi(d(u,v)) &\leqslant \psi(s^4 d(u,v)) = \psi(s^4 d(fu,Tv)) \\ &\leqslant \psi(N_s(u,v)) - \varphi(N_s(u,v)) = \psi(d(u,v)) - \varphi(d(u,v)), \end{aligned}$$

which implies that $\varphi(d(u,v)) = 0$ and hence $d(u,v) = 0$ i.e., $u = v$. Therefore, f and T have a unique common fixed point in X. ■

Corollary 3.10 Let (X,d) be a complete b-metric space endowed with a graph G and let $f, T : X \to X$ be mappings. Suppose that there exists $k \in [0,1)$ such that

$$d(fx,Ty) \leqslant \frac{k}{s^4} max\{d(x,y), min\{d(x,fx), d(y,Ty)\}, \frac{1}{2s} min\{d(x,Ty), d(y,fx)\}\}$$

for all $x, y \in X$ with $(x,y) \in E(\tilde{G})$. Suppose the triple (X,d,G) has the property $(*')$. Then f and T have a common fixed point in X if $C_{f \vee T} \neq \emptyset$. Moreover, f and T have a unique common fixed point in X if the graph G has the property (‡).

Proof. The proof follows from Theorem 3.9 by considering altering distance functions $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0,+\infty)$. ■

Corollary 3.11 Let (X,d) be a complete b-metric space endowed with a graph G and let $f : X \to X$ be a mapping. Suppose that there exists $k \in [0,1)$ such that

$$d(fx,fy) \leqslant \frac{k}{s^4} max\{d(x,y), min\{d(x,fx), d(y,fy)\}, \frac{1}{2s} min\{d(x,fy), d(y,fx)\}\}$$

for all $x, y \in X$ with $(x,y) \in E(\tilde{G})$. Suppose the triple (X,d,G) has the property $(*')$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:
$(\ddagger')$ If x, y are fixed points of f in X, then $(x,y) \in E(\tilde{G})$.

Proof. The proof follows from Theorem 3.9 by considering $T = f$ and altering distance functions $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0,+\infty)$. ■

Corollary 3.12 Let (X,d) be a complete b-metric space and let $f, T : X \to X$ be mappings. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(s^4 d(fx,Ty)) \leqslant \psi(N_s(x,y)) - \varphi(N_s(x,y))$$

for all $x, y \in X$. Then f and T have a unique common fixed point in X.

Proof. The proof can be obtained from Theorem 3.9 by putting $G = G_0$. ■

Corollary 3.13 Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$ and let $f : X \to X$ be a mapping. Suppose that there exist two altering distance functions ψ and φ such that

$$\psi(s^4 d(fx, fy)) \leqslant \psi(N_s^{'}(x, y)) - \varphi(N_s^{'}(x, y))$$

for all $x, y \in X$ with $x \preceq y$ or, $y \preceq x$, where

$$N_s^{'}(x, y) = max\{d(x, y), min\{d(x, fx), d(y, fy)\}, \frac{1}{2s} min\{d(x, fy), d(y, fx)\}\}.$$

Suppose the triple $(X, d, \preceq)$ has the property (†). If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m,\ n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the property (††) holds.

Proof. The proof can be obtained from Theorem 3.9 by taking $G = G_2$ and $T = f$. ■

Remark 2 *If we take $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ for all $t \in [0, +\infty)$, where $k \in [0, 1)$ is a constant, then conditions (1) and (14) reduce to*

$$d(fx, fy) \leqslant \frac{k}{s} M_s(gx, gy) \ \ and\, d(fx, Ty) \leqslant \frac{k}{s^4} N_s(x, y)$$

respectively. Corollaries 3.3 and 3.10 give a unique common fixed point of a pair of mappings satisfying above mentioned contractive type conditions. However, it is valuable to note that without altering distance functions, one can finds a unique common fixed point of a pair of mappings satisfying above types of contractive conditions by using Lemma 2.10.

We furnish some examples in favour of our results.

***Example* 3.14** Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) =\mid x - y \mid^2$ for all $x,\ y \in X$. Then (X, d) is a b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : x \leqslant y \ with\ x, y \in [0, \infty)\}$.

Let $f,\ g : X \to X$ be defined by $fx = \frac{x}{2}$ if $x \in 2\mathbb{Z}$ and otherwise, $fx = 0$, and $gx = 3x$ if $x \in [0, \infty)$ and otherwise, $gx = [x]$ if $x \in (-\infty, 0)$. Obviously, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also, define altering distance functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t,\ \varphi(t) = (1 - k)t$ where $k = \frac{1}{8}$. We now verify that condition (1) holds for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$. Let $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$.
Case-I: If $x, y \in X \setminus 2\mathbb{Z}$, then

$$\psi(sd(fx, fy)) = 0 \leqslant \psi(M_s(gx, gy)) - \varphi(M_s(gx, gy)).$$

Case-II: If $x, y \in 2\mathbb{Z}$, then

$$\begin{aligned}\psi(sd(fx, fy)) &= \psi\left(2d\left(\frac{x}{2}, \frac{y}{2}\right)\right) = 2\left(\frac{x}{2} - \frac{y}{2}\right)^2 \\ &= \frac{1}{2}(x - y)^2 \leqslant \frac{9}{8}(x - y)^2 \\ &= kd(gx, gy) \leqslant kM_s(gx, gy) \\ &= \psi(M_s(gx, gy)) - \varphi(M_s(gx, gy)).\end{aligned}$$

Thus,

$$\psi(sd(fx, fy)) \leqslant \psi(M_s(gx, gy)) - \varphi(M_s(gx, gy))$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$.

We can verify that $0 \in C_{gf}$. In fact, $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$ gives that $gx_1 = f0 = 0 \Rightarrow x_1 = 0$ and so $gx_2 = fx_1 = 0 \Rightarrow x_2 = 0$. Proceeding in this way, we get $gx_n = 0$ for $n = 0, 1, 2, \cdots$ and hence $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$. Also, it is easy to verify that property $(*)$ holds. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.2 and 0 is the unique common fixed point of f and g in X.

The following example supports our Theorem 3.9.

***Example* 3.15** Let $X = [0, \infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = \mid x - y \mid^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : x \leqslant y \ and \ x, y \in \mathbb{N} \cup \{0\}\}$. Let $f, T : X \to X$ be defined by $fx = \ln(1 + \frac{x}{8})$ if $x \in \mathbb{N} \cup \{0\}$ and otherwise, $fx = 2$, and $Tx = \ln(1 + \frac{x}{6})$ if $x \in \mathbb{N} \cup \{0\}$ and otherwise, $Tx = 2$. Also, define altering distance functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = kt,\ \varphi(t) = (k-1)t$, where $1 < k \leqslant \frac{25}{16}$.

We now verify that condition (14) holds for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$. Let $x, y \in X$ with $(x, y) \in E(\tilde{G})$ and $y \leqslant x$.
Case-I: $(x, y) \in (\mathbb{N} \times \mathbb{N}) \cup \{(0, 0)\}$. If $\frac{y}{6} \leqslant \frac{x}{8}$, then

$$1 \leqslant \frac{1 + \frac{x}{8}}{1 + \frac{y}{6}} \leqslant \frac{1 + \frac{x}{6}}{1 + \frac{y}{6}}$$

and so

$$0 \leqslant \ln\left(\frac{1 + \frac{x}{8}}{1 + \frac{y}{6}}\right) \leqslant \ln\left(\frac{1 + \frac{x}{6}}{1 + \frac{y}{6}}\right).$$

By using the mean value theorem for the function $\ln(1 + t)$, for $t \in [\frac{y}{6}, \frac{x}{6}]$, we have

$$\begin{aligned}
\psi(s^4 d(fx, Ty)) &= 16kd(fx, Ty) = 16k\left(\ln\left(1 + \frac{x}{8}\right) - \ln\left(1 + \frac{y}{6}\right)\right)^2 \\
&= 16k\left(\ln\left(\frac{1 + \frac{x}{8}}{1 + \frac{y}{6}}\right)\right)^2 \leqslant 16k\left(\ln\left(\frac{1 + \frac{x}{6}}{1 + \frac{y}{6}}\right)\right)^2 \\
&= 16k\left(\ln\left(1 + \frac{x}{6}\right) - \ln\left(1 + \frac{y}{6}\right)\right)^2 \leqslant 16k\left(\frac{x}{6} - \frac{y}{6}\right)^2 \\
&\leqslant \frac{25}{36}(x - y)^2 \leqslant d(x, y) \\
&\leqslant N_s(x, y) = \psi(N_s(x, y)) - \varphi(N_s(x, y)).
\end{aligned}$$

If $\frac{x}{8} < \frac{y}{6}$, then $0 < \frac{y}{6} - \frac{x}{8} \leqslant \frac{y}{6} \Rightarrow \left(\frac{y}{6} - \frac{x}{8}\right)^2 \leqslant \frac{y^2}{36}$. By using the mean value theorem for the function $ln(1+t)$, for $t \in [\frac{x}{8}, \frac{y}{6}]$, we have

$$\begin{aligned}
\psi(s^4 d(fx, Ty)) = 16kd(fx, Ty) &= 16k\left(ln\left(1+\frac{x}{8}\right) - ln\left(1+\frac{y}{6}\right)\right)^2 \\
&\leqslant 16k\left(\frac{y}{6}-\frac{x}{8}\right)^2 \leqslant \frac{16}{36}ky^2 \\
&\leqslant \frac{25}{36}y^2 = \left(\frac{5y}{6}\right)^2 \\
&\leqslant \left(y - ln(1+\frac{y}{6})\right)^2 = d(y, Ty).
\end{aligned}$$

Again,

$$\begin{aligned}
\psi(s^4 d(fx, Ty)) &\leqslant 16k\left(\frac{y}{6}-\frac{x}{8}\right)^2 \leqslant 16k\left(\frac{x}{6}-\frac{x}{8}\right)^2 \\
&= 16k\frac{x^2}{24^2} \leqslant \frac{25}{24^2}x^2 \\
&\leqslant \frac{49}{64}x^2 \leqslant \left(x - ln(1+\frac{x}{8})\right)^2 \\
&= d(x, fx).
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi(s^4 d(fx, Ty)) &\leqslant min\{d(x, fx), d(y, Ty)\} \\
&\leqslant N_s(x, y) \\
&= \psi(N_s(x, y)) - \varphi(N_s(x, y)).
\end{aligned}$$

Case-II: $x = y$ *and* $x \notin \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
\psi(s^4 d(fx, Ty)) &\leqslant \psi(s^4 d(2, 2)) \\
&= 0 \\
&\leqslant N_s(x, y) \\
&= \psi(N_s(x, y)) - \varphi(N_s(x, y)).
\end{aligned}$$

The case $(x, y) \in E(\tilde{G})$ with $x \leqslant y$ may be treated similarly.

Thus, condition (14) is satisfied. It is easy to verify that property $(*)'$ holds and $0 \in C_{f \vee T}$ i.e., $C_{f \vee T} \neq \emptyset$. Thus, we have all the conditions of Theorem 3.9 and 0 is the unique common fixed point of f and T in X.

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