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# Conservation laws and invariant solutions of time-dependent Calogero-Bogoyavlenskii-Schiff equation

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**Abstract.** This paper uses the classical Lie method to determine symmetry reductions and exact solutions of the time-dependent Calogero-Bogoyavlenskii-Schiff equation (vCBS). This classical method generates some exact arbitrary solutions and exhibits various qualitative behaviors. Here, we derived the infinitesimal symmetries and six basic combinations of vector fields in the linear forms that can be utilized to transform the given equation into the PDEs with their variables. Further, we obtain comprehensive invariant solutions of the vCBS equation. Next, we apply a direct method to explore conservation laws. Finally, we determine the conservation laws of the vCBS equation via the Bluman-Anco homotopy formula.

**Keywords:** Lie algebras, vCBS equation, reduction equations, conservation laws.

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## 1. Introduction

In 2020, Wazwaz introduced three new completely integrable Calogero Bogoyavlenskii Schiff (CBS) equations:

- Extended CBS (eCBS);
- Time-dependent CBS (vCBS);
- Time-dependent negative-order CBS (vnCBS),

which appear in propagation of waves [15]. Several integrable models have been generated and proposed in the context of (2+1)-dimensional equations. Multidimensional integrable systems have been primarily used for solving the problems in integrable systems. It is

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essential to keep the symmetry for solving the equations, as it is the key for solving non-linear differential equations. Moreover, it is vital for understanding many physical phenomena in various fields, such as physics science and dynamics of fluids problems [2, 3, 6]. Such phenomena can be described as a system of differential equations and, in potential form, a fourth-order partial differential equation. The classical and non-classical methods generate some exact arbitrary function and thus exhibit various solutions. The symmetry group of equations is known as the most fundamental transformation local group, acting on the dependent and independent variables in the system [9–11].

Moreover, the symmetry group of differential equations enables transforming the solutions into other valuable solutions in the system. Non-linear evolution equations have been derived and broadly used to interpret the fundamental characteristics of non-linearity and to obtain the peculiar nature of non-linearity in science and engineering phenomena [5, 7]. In this study, we explore the time-dependent Calogero-Bogoyavlenskij-Schiff (vCBS) equation written as follows:

$$f(t)u_{xt} + \alpha u_{xxxy} + 2\beta u_x u_{xy} + \beta u_{xx} u_y = 0. \tag{1}$$

In order to calculate the solutions for partial differential equations (PDEs) in non-linear forms, we analyzed their corresponding symmetry groups, as they are the most common and powerful techniques for solving such equations [4, 8]. In this approach, first, the symmetry groups for the non-linear PDEs are recognized. Next, the identified symmetry groups are used to construct the certain solutions and eventually transform them into other solutions. In the analysis of PDEs, the conservation laws play a crucial role in analyzing the fundamental properties of the solutions and in particular, the study of their stability, existence, and uniqueness [1, 14]. The current work will be described with the following structure: Section 2 classifies a group of vCBS equations. In the next section (Section 3), we obtain the optimal one-dimensional subalgebra for the problem. In Section 4, we earn the reductions for Eq. (1) and present definite solutions. In the last section (Section 5), we obtain the associated conservation laws for Eq. (1) using direct methods and provide concluding remarks.

## 2. Symmetry classification of the vCBS equation

We first consider a system of p-th order differential equations in the form of

$$\Delta_\alpha(X, U^{(p)}) = 0, \qquad \alpha = 1, ..., t, \tag{2}$$

is a system of PDE of order $p$th, where $X = (x^1, ..., x^m)$ and $U = (u^1, ..., u^n)$ are $m$ independent and $n$ dependent variables respectively, and $U^{(i)}$ is the $i-$ order derivative of $U$ with respect to $x$, $0 \leqslant i \leqslant p$. Infinitesimal transformations Lie group acts on both $X$ and $U$ are

$$\hat{x}^i = x^i + \delta\varphi^i(X, U) + o\left(\delta^2\right), \qquad i = 1, \ldots, m,$$
$$\hat{u}_j = u^j + \delta\phi_j(X, U) + o\left(\delta^2\right), \qquad j = 1, \ldots, n.$$

In equations mentioned above, $\varphi^i$ and $\phi_j$ are the infinitesimal transformations for $\left\{x^1, \ldots x^m\right\}$ and $\left\{u^1, \ldots, u^n\right\}$ respectively. We can define an arbitrary generator based on the transformation group as follows:

$$V = \sum_{i=1}^{p} \varphi^i(X, U)\partial x^i + \sum_{j=1}^{q} \phi_j(X, U)\partial u^j. \tag{3}$$

To use the Lie group procedure for Eq. (1) and obtain infinitesimal transformations, we examine a one-parameter Lie group: (we apply x, y, t in replacement of $x^1, x^2, x^3$ respectively). So $x^1 = x, x^2 = y, x^3 = t, u^1 = u$ and

$$\begin{aligned}
\tilde{x} &= x + \delta\varphi^1(x, y, t, u) + o\left(\delta^2\right), \\
\tilde{y} &= y + \delta\varphi^2(x, y, t, u) + o\left(\delta^2\right), \\
\tilde{t} &= t + \delta\varphi^3(x, y, t, u) + o\left(\delta^2\right), \\
\hat{u} &= u + \delta\phi_1(x, y, t, u) + o\left(\delta^2\right).
\end{aligned}$$

The obtained symmetry generator can be described in the following form:

$$V = \varphi^1(x, y, t, u)\partial x + \varphi^2(x, y, t, u)\partial y + \varphi^3(x, y, t, u)\partial t + \phi_1(x, y, t, u)\partial u. \tag{4}$$

The condition of being invariance for the equation is

$$pr^{(4)}v\left[f(t)u_{xt} + \alpha u_{xxxy} + 2\beta u_x u_{xy} + \beta u_{xx}u_y\right] = 0,$$

with

$$f(t)u_{xt} + \alpha u_{xxxy} + 2\beta u_x u_{xy} + \beta u_{xx}u_y = 0.$$

Assuming $\varphi^1, \varphi^2, \varphi^3$ and $\phi_1$ are the only dependents for $x, y, t$ and $u$, we adjust the individual coefficients to be zero. The total number of these equations is 99. We cannot reach the solution of the above PDE equations with the arbitrary function $f$. But if $f(t) = e^t$, by finding solutions for the above PDE equations, we obtain the results as follows:

**Theorem 2.1** The point symmetries Lie groups of Eq. (1) contain Lie algebra generated by Eq. (4). The obtained coefficients are the infinitesimals in the following forms:

$$\begin{aligned}
\phi_1 &= \frac{1}{2\beta}\left(u\beta c_2 e^{-t} + 2e^t(\partial t F_1(t))y + 2F_2(t)\beta + u(c_1 + c_6)\beta + (yc_2 - c_5)x\right), \\
\varphi^1 &= -\frac{1}{2}xc_2e^{-t} + F_1(t) + \frac{1}{2}(-c_1 - c_6)x, \\
\varphi^2 &= (-yc_2 + c_5)e^{-t} + yc_6 + c_4, \\
\varphi^3 &= c_1 + c_2e^{-t} + c_3e^t,
\end{aligned}$$

where $c_i \in \mathbb{R},\ i = 1, \ldots, 6$ and $F_1(t)$ and $F_2(t)$ are arbitrary function.

**Corollary 2.2** All one-parameter Lie groups of point symmetries for Eq. (1) have the following infinitesimal generators:

$$\nu_6 = -\frac{1}{2}x\partial_x + \partial_t + \frac{1}{2}u\partial_u,$$
$$\nu_5 = -\frac{1}{2}xe^{-t}\partial_x - ye^{-t}\partial_y + e^{-t}\partial_t + \frac{1}{2}\frac{(u\beta e^{-t} + yx)\partial_u}{\beta},$$
$$\nu_4 = e^t\partial_t,$$
$$\nu_3 = \partial_y,$$
$$\nu_2 = e^{-t}\partial_y - \frac{1}{2}\frac{x\partial_u}{\beta},$$
$$\nu_1 = -\frac{1}{2}x\partial_x + y\partial_y + \frac{1}{2}u\partial_u,$$
$$\nu_\mu = \mu(t)\partial_x + \frac{e^t(\partial_t\mu(t))y}{\beta}\partial_u, \qquad \nu_\theta = \theta(t)\partial_u,$$

where $\mu = F_1$ and $\theta = F_2$ are arbitrary function.

We provide Lie algebra for Eq. (1) by Table (1). The expression $[\nu_i, \nu_j] = \nu_i\nu_j - \nu_j\nu_i$ determines the entry in row $i^{\text{th}}$ and column $j^{\text{th}}$ , $i, j = 1, \ldots, 6$. The $G_i$ groups containing one parameter produced by the $\nu_i$ are described in the following expressions:

$$\exp(\varepsilon v_i)(x, y, t, u) = \left(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}\right):$$
$$G_1\colon (xe^{-\frac{1}{2}\varepsilon}, ye^4, t, ue^{\frac{1}{2}\varepsilon}),$$
$$G_2\colon (x, e^{-t}\varepsilon + y, t, -\frac{1}{2}\frac{x\varepsilon}{\beta} + u),$$
$$G_3\colon (x, \varepsilon + y, t, u),$$
$$G_4\colon \left(xe^{-\frac{1}{2}\varepsilon}, y, \varepsilon + t, ue^{\frac{1}{2}\varepsilon}\right).$$

Table 1. Lie algebra for Eq(1).

| [ , ] | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ |
|---|---|---|---|---|---|---|
| $\nu_1$ | 0 | $-\nu_2$ | $-\nu_3$ | 0 | 0 | 0 |
| $\nu_2$ | $\nu_2$ | 0 | 0 | $\nu_3$ | 0 | $\nu_2$ |
| $\nu_3$ | $\nu_3$ | 0 | 0 | 0 | $-\nu_2$ | 0 |
| $\nu_4$ | 0 | $-\nu_3$ | 0 | 0 | $\nu_1 - 2\nu_6$ | $-\nu_4$ |
| $\nu_5$ | 0 | 0 | $\nu_2$ | $-\nu_1 + 2\nu_6$ | 0 | $\nu_5$ |
| $\nu_6$ | 0 | $-\nu_2$ | 0 | $\nu_4$ | $-\nu_5$ | 0 |

Table 2. Adjoint representation of the Lie algebra.

| Ad | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ |
|---|---|---|---|---|---|---|
| $\nu_1$ | $\nu_1$ | $e^{-s_1}\nu_2$ | $e^{-s_1}\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ |
| $\nu_2$ | $\nu_1$ | $s_2\nu_1 + \nu_2 + s_2\nu_6$ | $\nu_3 + s_2\nu_4$ | $\nu_4$ | $\nu_5$ | $\nu_6$ |
| $\nu_3$ | $\nu_1$ | $\nu_2 - s_3\nu_5$ | $s_3\nu_1 + \nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ |
| $\nu_4$ | $\nu_1 + s_4\nu_5$ | $\nu_2$ | $-s_4\nu_2 + \nu_3$ | $\nu_4 + s_4^2\nu_5 - s_4\nu_6$ | $\nu_5$ | $-2s_4\nu_5 + \nu_6$ |
| $\nu_5$ | $\nu_1 - s_5\nu_4$ | $\nu_2 + s_5\nu_3$ | $\nu_3$ | $\nu_4$ | $s_5^2\nu_4 + \nu_5 + s_5\nu_6$ | $2s_5\nu_4 + \nu_6$ |
| $\nu_6$ | $\nu_1$ | $e^{-s_1}\nu_2$ | $e^{-s_1}\nu_3$ | $\nu_4$ | $\nu_5$ | $\nu_6$ |

## 3. Classification of one-Dimensional subalgebras

We can now employ the developed symmetry group and determine the optimal system with one parameter group for Eq. (1). It should be noted that obtaining the subgroups of the system is vital. These subgroups offer various types of solutions for the equation. Thus, it is essential to find solutions that remain unaffected. The proposed approach provides an expression for an optimal group of subalgebras [12, 13]. The classification procedure of subalgebras would be homogenous with the classification for representation orbits that are adjoined. The solution for a problem with an optimal group of subalgebras can be obtained by assigning one representative from every subalgebra group within the system. The adjoint representation of all $\nu_t,\ t = 1, \ldots, 6$ would be defined as

$$\mathrm{Ad}\left(\exp\left(s\nu_t\right)\nu_r\right) = \nu_r - s\left[\nu_t, \nu_r\right] + \frac{s^2}{2}\left[\nu_t, \left[\nu_t, \nu_r\right]\right] - \ldots.$$

In this representation, $s$ is a variable and $[\nu_t, \nu_r]$ is defined in Table 1 for $t, r = 1, \ldots, 6$. We assume $g$ to be the Lie algebra determined by the Corollary 2.2. Then we can obtain the adjoint action for the equation in Table 2. Further, we can set up an optimal system of the subalgebras for Eq. (1).

**Theorem 3.1** A one-dimensional optimal system of Eq. (1) is given by

(1) $\nu_6 + c_1\nu_1 + c_2\nu_2 + c_3\nu_3 + c_4\nu_4 + c_5\nu_5$,
(2) $\nu_5 + c_1\nu_3 + c_2\nu_4$,
(3) $\nu_4 + c_1\nu_2$,
(4) $\nu_3 + c_1\nu_1$,
(5) $\nu_2 + c_1\nu_1$,
(6) $\nu_1$.

**Proof.** Looking at Table 1, it is enough to determine the subalgebras of $\langle \nu_1, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6 \rangle$. Function $F_t^s : g \to g$ defined by $x \to Ad\left(\exp\left(s\nu_i\right)x\right)$ is a linear map, for $i = 1, \ldots, 6$. The six matrices $m_t^s$ of $F_i^s$, $i = 1, \ldots, 6$, with respect to basis, are the followings:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-s_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s_2 & 1 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 1 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -s_3 & 0 \\ s_3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & s_4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & s_4^2 & -s_4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2s_4 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 & -s_5 & 0 & 0 \\ 0 & 1 & s_5 & 0 & 0 & s_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s_5^2 & 1 & s_5 \\ 0 & 0 & 0 & 2s_5 & 0 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-s_6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{s_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-s_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Alternatively, by acting these matrices on $x = \sum_{i=1}^{6} c_i\nu_i$, $x$ is a vector field, $x$ can be simplified as follows:
By taking $c_6 \neq 0$, the vector $x$ can be transformed into the case (1).

For $c_6 = 0$ and $c_5 \neq 0$, the coefficients of $\nu_1$ and $\nu_2$ can be disappeared by setting $s_4 = \dfrac{-c_1}{c_5}$ and $s_3 = \dfrac{c_2}{c_5}$ respectively. If needed, by scaling $x$, we assume $c_5 = 1$. Thus, $x$ can be converted to the case (2).

For $c_6 = c_5 = 0$ and $c_4 \neq 0$, the coefficients of $\nu_1$ and $\nu_3$ can be disappeared by setting $s_2 = \frac{-c_3}{c_4}$ and $s_5 = \frac{c_1}{c_4}$ respectively. If needed, by scaling $x$, we assume $c_5 = 1$. Thus, $x$ can be converted to the case (3).

For $c_6 = c_5 = c_4 = 0$ and $c_3 \neq 0$ we can remove the coefficient $\nu_2$ by setting $s_5 = -\dfrac{c_2}{c_3}$. If needed by scaling $x$, we assume $c_3 = 1$. Therefore, $x$ can be converted to the case (4).

For $c_6 = c_5 = c_4 = c_3 = 0$ and $c_2 \neq 0$, if needed by scaling $x$, we assume $c_2 = 1$. Therefore, $x$ can be converted to the case (5).

For $c_6 = c_5 = c_4 = c_3 = c_2 = 0$ and $c_1 \neq 0$, $x$ transforms into the case (6). ■

Table 3. The parameters of Lie invariants and the similarity solutions.

| $i$ | $H_i$ | $\xi_i$ | $\eta_i$ | $w_i$ | $u_i$ |
|---|---|---|---|---|---|
| 1 | $\nu_1$ | $t$ | $\dfrac{y}{x^2}$ | $xu$ | $\dfrac{1}{x}h(\xi,\eta)$ |
| 2 | $\nu_2$ | $t$ | $x$ | $-\dfrac{u}{2} - \dfrac{1}{4\beta}yxe^t$ | $-\dfrac{1}{2\beta}yxe^t - 2h(\xi,\eta)$ |
| 3 | $\nu_3$ | $t$ | $x$ | $u$ | $h(\xi,\eta)$ |
| 4 | $\nu_4$ | $x$ | $y$ | $u$ | $h(\xi,\eta)$ |
| 5 | $\nu_5$ | $\dfrac{y}{x^2}$ | $2\ln(x) + t$ | $-xu - \dfrac{x^2e^t}{2\beta}$ | $-\dfrac{1}{2\beta}xe^ty - \dfrac{1}{x}h(\xi,\eta)$ |
| 6 | $\nu_6$ | $y$ | $2\ln(x) + t$ | $u$ | $h(\xi,\eta)$ |
| 7 | $\nu_3 + \nu_4$ | $x$ | $-(ye^t + 1)e^{-t}$ | $u$ | $h(\xi,\eta)$ |
| 8 | $\nu_4 + \nu_6$ | $y$ | $\dfrac{xe^{\frac{1}{2}t}}{\sqrt{e^t + 1}}$ | $u$ | $h(\xi,\eta)$ |
| 9 | $\nu_4 + \nu_6 + \nu_1$ | $x$ | $\dfrac{(e^t + 1)e^{-t}}{y}$ | $u$ | $h(\xi,\eta)$ |

Table 4. Reduced equations.

| | |
|---|---|
| 1 | $2f(\xi)\eta h_{\xi\eta} - f(\xi)h_\xi + 8\alpha\eta^3 h_{\eta\eta\eta} + 24\alpha\eta h_{\eta\eta\eta}$ $+6\alpha\eta h_{\eta\eta} + 12\beta\eta h_\eta h_{\eta\eta} + 2\beta\eta h_eta - 4\beta\eta h h_{\eta\eta} = 0,$ |
| 2 | $2\beta f(\xi)h_{\xi\eta} - 2\beta e^\xi h_\eta - \xi e^\xi \beta h_{\eta\eta} = 0,$ |
| 3 | $f(\xi)h_{\xi\eta} = 0,$ |
| 4 | $\alpha h_{\xi\xi\xi\eta} + 2\beta h_\xi h_{\xi\eta} + \beta h_\xi h_\eta = 0,$ |
| 5 | $-4\xi\beta h_{\xi\eta} + 4\beta h_{\eta\eta} - 2\beta h_\eta = 0,$ |
| 6 | $8\alpha h_{\xi\eta\eta\eta} - 24\alpha h_{\xi\eta\eta} + 22\alpha h_{\xi\eta} - 6\alpha h_\xi + 8\beta h_\eta h_{\xi\eta}$ $-10\beta h_\eta h_\xi - 4\beta h h_{\xi\eta} + 4\beta h h_\xi + 4\beta h_\xi h_{\eta\eta} = 0,$ |
| 7 | $-\alpha h_{\xi\xi\xi\eta} - 2\beta h_\xi h_{\xi\eta} - \beta h_\xi h_\eta h_{\xi\eta} = 0,$ |
| 8 | $\eta h_{\eta\eta} + 2h_\eta 2\alpha h_{\xi\eta\eta\eta} + 4\beta h_\eta h_{\xi\eta} + 2\beta h_{\eta\eta} h_\xi = 0,$ |
| 9 | $-h_{\xi\eta} - \alpha\eta h_{\xi\eta\eta\eta} - 2\beta\eta h_\xi h_{\xi\eta} - \beta\eta h_{\xi\xi} h_\eta = 0.$ |

## 4. Similarity reduction of Eq. (1)

Now, we perform the classification of symmetry reductions of Eq. (1) in the following section regarding subalgebras. Therefore, we must find a new expression for Eq. (1) in particular coordinates and reduce the equation. This specific coordinate can be determined by calculating $\xi, \eta$, (independent invariant), and $h$ regarding the infinitesimal generator. The similarity variables $\xi_i, \eta_i, h_i$ for Eq. (1) are represented in Table 3.

For example, we calculated the invariants related to subalgebra $H_6 = \nu_6 = -\frac{1}{2}\partial x + \partial t + \frac{1}{2}\partial u$, by solving the characteristic equation as follows:

$$-\frac{2\partial x}{x} = \frac{\partial y}{1} = \frac{\partial t}{1} = \frac{2\partial u}{u}.$$

Thus, the new variables are

$$\xi_i = y, \qquad \eta_i = 2\ln(x) + t, \qquad w_i = 6,$$

where $u = h(\xi, \eta)$ meets a reduced PDE with two variables as:

$$\begin{aligned} &8\alpha h_{\xi\eta\eta\eta} - 24\alpha h_{\xi\eta\eta} + 22\alpha h_{\xi\eta} - 6\alpha h_{\xi} \\ &+ 8\beta h_{\eta} h_{\xi\eta} - 10\beta h_{\eta} h_{\xi} - 4\beta h h_{\xi\eta} + 4\beta h h_{\xi} + 4\beta h_{\xi} h_{\eta\eta} = 0. \end{aligned} \tag{5}$$

Subalgebra $\nu_6$ and the reduced Eq. (5) are presented in Tables 3 and 4 by case (6). Equivalent solution of Eq. (5) becomes $u(x, y, t) = e^{\frac{1}{2}t} F(y)$, where $F$ is a function of $\xi = y$. Indeed, by Table 3, $h(\xi, \eta) = u(x, y, t)$. This solution does not depend on $x$. Using a similar argument for $H_7 = \nu_3 + \nu_4$, the Eq. (1) is reduced as:

$$-\alpha h_{\xi\xi\xi\eta} - 2\beta h_{\xi} h_{\xi\eta} - \beta h_{\xi} h_{\eta} h_{\xi\eta} = 0, \tag{6}$$

Equivalent solution of Eq. (6) becomes

$$h(\xi, \eta) = u(x, y, t) = c_1 x + F(-(ye^t + 1)e^{-t}),$$

where $F$ is a function of $\eta = -(ye^t + 1)e^{-t}$ Again using a similar argument for $H_8 = \nu_4 + \nu_6$, the reduced equation and equivalent solution of Eq(1) become

$$\eta h_{\eta\eta} + 2h_{\eta} 2\alpha h_{\xi\eta\eta\eta} + 4\beta h_{\eta} h_{\xi\eta} + 2\beta h_{\eta\eta} h_{\xi} = 0, \tag{7}$$

$$h(\xi, \eta) = u(x, y, t) = \frac{c_1 x e^{\frac{1}{2}t} + c_2\sqrt{e^t + 1}}{\sqrt{e^t + 1}x}. \tag{8}$$

For $H_9 = \nu_4 + \nu_6 + \nu_1$ the reduced equation and equivalent solution of Eq(1) become:

$$-h_{\xi\eta} - \alpha\eta h_{\xi\eta\eta\eta} - 2\beta\eta h_{\xi} h_{\xi\eta} - \beta\eta h_{\xi\xi} h_{\eta} = 0,$$

$$h(\xi, \eta) = u(x, y, t) = c_1 x + F\left(\frac{(e^t + 1)e^{-t}}{y}\right),$$

where $F$ is a function of $\eta = \dfrac{(e^t + 1)e^{-t}}{y}$.

## 5. Conservation laws

Many techniques, such as Noether and direct methods, etc., are used to explore conservation laws. In the current section, we utilize the direct method to analyze the conservation laws. Assume a differential equation in the form of $\rho\{x,u\}$ with $k$ order and independent variables of n, where x can be described as x $= \left(x^1,\ldots,x^n\right)$ and $U$ is a dependent variable, represented by $\rho[u] = \rho\left(x,U,\partial U,\ldots,\partial^k U\right) = 0$. Assuming a multiplier with $\Lambda\left(x,U,\partial U,\ldots,\partial^t U\right)$ can derived a conservation law in the form of $\Lambda[U]\rho[U] = D_i\phi^i[U] = 0$ for the equation $\rho\{x,u\}$ contingent upon

$$E_U\left(\Lambda\left(x,U,\partial U,\ldots,\partial^t U\right)\rho\left(x,U,\partial U,\ldots,\partial^k U\right)\right)\equiv 0.$$

By taking $U(x)$ as an expression for arbitrary functions into account, the expression $E_U$, which is the Euler operator acting on $U$, is described below:

$$E_U = \partial U - D_i\partial U + \cdots + (-1)^s D_{i_1}\ldots D_{i_s}\partial U_{i_1}\cdots_{i_s}$$

Since the CBS equation is concerned about $u,x,y,t$, it results in multipliers that further provide locally configured conservation laws for Eq. (1) in the format of $\Lambda = \Lambda\left(x,y,t,U,U_xU_y,U_t\right)$. We can now determine all nontrivial local conservation laws connected to the equation from multipliers. Subsequently, the expression $\Lambda = \Lambda\left(x,y,t,U,\partial_x U,\partial_y U,\partial_t U\right)$ is a conservation law multiplier concerning the Eq. (1) based on

$$E_U[\Lambda(x,y,t,U,\partial_x U,\partial_y U,\partial_t U)(f(t)u_{xt} + \alpha u_{\text{xxx y}} + 2r\beta u_x u_{xy} + \beta u_{xx}u_y)\equiv 0$$

for $U(x,y,t)$ in the form of an arbitrary function. We then identify all potential multipliers in the format of $\Lambda = \Lambda\left(x,y,t,u,\partial U_x,\partial U_y,\partial U_t\right)$ for the Eq. (1). Therefore, the Euler operator is determined to be as follows:

$$E_U = \partial U - D_i\partial U_i + \cdots + (-1)^3 D_{i_1}\ldots D_{i_3}\partial U_{i_1}\cdots_{i_3}$$

and the determining equation becomes

$$E_U[\Lambda(x,y,t,U,\partial U_x,\partial U_y,\partial U_t)(f(t)u_{xt} + \alpha u_{\text{xxx y}} + 2r\beta u_x u_{xy} + \beta u_{xx}u_y)\equiv 0,$$

where $U(x,y,t)$ is an arbitrary function. The above equation can be separated concerning $U_x,U_y,U_t$ to derive the over-determined equations:

$$\Lambda_{U,t,t} = \frac{-3f\Lambda_{U,t}f_t - g\Lambda_U f_{t,t} - \Lambda_U f_t^2}{f^2},\quad \Lambda_{x,x} = 0,\quad \Lambda_{x,y} = \frac{-\Lambda_{U,t}f - \Lambda_U f_t}{\beta},$$

$$\Lambda_{t,x} = \frac{U_x\Lambda_{U,t}f + U_x\Lambda_U f_t - \Lambda_x f_t}{f},\quad \Lambda_{U,x} = 0,\quad \Lambda_{U_x,x} = \Lambda_U,\quad \Lambda_U\Lambda_{U_t,y} = 0,$$

$$\Lambda_{U_y,x} = 0,\quad \Lambda_{U_t,x} = 0,\quad \Lambda_{y,y} = 0,\quad \Lambda_{U,y} = 0,\quad \Lambda_{U_x,y} = 0,\quad \Lambda_{U_y,y} = 2,$$

$$\Lambda_{U_x,t} = \frac{2\Lambda_U\beta U_y - \Lambda_{U_x}f_t - \Lambda_y\beta}{f},\quad \Lambda_{U_y,t} = \frac{2\Lambda_U\beta U_x - 2\Lambda_x\beta - \Lambda_{U_y}f_t}{f},$$

$$\Lambda_{U_t,t} = 4\Lambda_U,\quad \Lambda_{U,U} = 0,\quad \Lambda_{U,U_x} = 0,\quad \Lambda_{U,U_y} = 0,\quad \Lambda_{U,U_t} = 0,\quad \Lambda_{U_x,U_x} = 0,$$

$$\Lambda_{U_x,U_y} = 0,\quad \Lambda_{U_tU_x} = 0,\quad \Lambda_{U_yU_y} = 0,\quad \Lambda_{U_t,U_y} = 0,\quad \Lambda_{U_tU_t} = 0,$$

Solving the set of above equations, we find the infinite set of local multipliers:

$$\Lambda l(x, y, t, U, Ux, Uy, Ut) = C_1 U_t + F(t).$$

We then use homotopy formula (see [4]) and determine the conserved elements $\phi^t, \phi^x$ and $\phi^y$ corresponding to $\Lambda$ in the following format:

$$\begin{aligned}
\phi^t &= \frac{1}{3}U(2\beta U_x U_{xy} + \beta U_{xx} U_y)C_1 + \frac{1}{2}U_x C_1 U_t f + \frac{1}{2}U(fU_{tx} + \alpha U_{xxxy})C_1 + U_x F(t) f,\\
\phi^x &= \frac{2}{3}U_1 C_1 U_t \beta U_{xy} - \frac{1}{3}U\left(C_1 \beta U_y U_{tx} + C_1 U_t \beta U_{xy}\right) + UxC_1 U_t \beta U_y\\
&+ \frac{1}{2}UF(t)\beta U_{xy} - \frac{1}{2}U_{xy} C_1 \alpha U_{tx} - \frac{1}{2}U\left(C_1 U_t f_t + C_1 f U_{tt}\right) + \frac{1}{2}U_{xxy} C_1 U_t \alpha\\
&+ \frac{1}{2}U_y C_1 \alpha U_{txx} + \frac{3}{2}UxF(t)\beta U_y - U\left(F(t) f_t + F_2(t) f_t\right) + U_{xxy} F(t)\alpha,\\
\phi^y &= \frac{1}{3}UC_1 U_t \beta U_{xx} - \frac{1}{3}U\left(2C_1 \beta U_x U_{tx} + 2C_1 U_t \beta U_{xx}\right) - \frac{1}{2}UF(t)\beta U_{xx} - \frac{1}{2}UC_1 \alpha U_{txxx}.
\end{aligned}$$

For case $C_1$, we have

$$\begin{aligned}
&\Lambda(x, y, t, U, U_x, U_y, U_t) = U_t,\\
\phi^t &= \frac{2}{3}U\beta UxUxy + \frac{1}{3}U\beta U_{xx} U_y + \frac{1}{2}U_x U_t f + \frac{1}{2}U_f U_{tx} + \frac{1}{2}U\alpha U_{xxxy},\\
\phi^x &= \frac{1}{3}UU_t \beta U_{xy} - \frac{1}{3}U\beta U_y U_{tx} + U_x U_t \beta U_y - \frac{1}{2}U_{xy}\alpha U_{tx} - \frac{1}{2}UU_t f_t - \frac{1}{2}UfU_{tt}\\
&+ \frac{1}{2}U_{xxy} U_t \alpha + \frac{1}{2}U_y \alpha U_{txx},\\
\phi^y &= -\frac{1}{3}UU_t \beta U_{xx} - \frac{2}{3}U\beta U_x U_{tx} - \frac{1}{2}U_\alpha U_{txxx}.
\end{aligned}$$

So, in this case, we find the following conservation law of Eq. (1):

$$D_t \phi^t + D_x \phi^x + D_y \phi^y = 0.$$

For case F, we obtain the following:

$$\begin{aligned}
&\Lambda(x, y, t, U, U_x, U_y, U_t) = F(t),\\
\phi^t &= U_x F(t) f,\\
\phi^x &= \frac{1}{2}UF(t)\beta U_{xy} + \frac{3}{2}U_x F(t)\beta U_y - UF_t f - UF(t) f_t + U_{xxy} F(t)\alpha,\\
\phi^y &= -\frac{1}{2}UF(t)\beta U_{xx}.
\end{aligned}$$

**Theorem 5.1** We have the following conservation law for Eq. (1):

$$D_t \phi^t + D_x \phi^x + D_y \phi^y = 0.$$

## References


[1] S. C. Anco, M. L. Gandarias, E. Recio, Line-solitons, line-shocks, and conservation laws of a universal KP-like equation in 2+1 dimensions, J. Math. Anal. Appl. 504 (2021), 1:125319.

[2] Y. AryaNejad, Symmetry Analysis of wave equation on conformally Flat spaces, J. Geom. phys 161 (2021), 161:104029.

[3] Y. AryaNejad, Exact solutions of Diffusion Equation on sphere, Comput. Methods Differ. Equ. 10(3) (2022), 789-798.

[4] G.W. Bluman, S.C Anco, Symmetry and integration methods for differential equations, Springer, 2002.

[5] M. S. Bruzon, M. l. Gandarias, C. Muriel, J. Ramierez, G. S. Saez, F. R. Romero, The Calogero-Bogoyavlenskii-Schiff equation in (z+1) dimensions, Theor. Math. phys. 137 (1) (2003), 1367-1377.

[6] M. L.Gandarias, M. Torrisi, A. Valenti, Symmetry classification, and optimal systems of a non-linear wave equation, Internet. J. Nonlinear Mech. 39 (2004), 389-398.

[7] Y. N. Grigoriev, N. H. Ibrgimov, V. F. kovalev, S. V. Meleshko, Symmetries of Integro-Differential Equations: with Applications in Mechanics and Plasma Physics, Springer, 2010.

[8] M. Jafari, A. Tanhaeivash, Symmetry group analysis and similarity reductions of the thin film equation, J. Linear. Topological. Algebra. 10 (4) (2021), 287-293.

[9] AP. Màrquez, TM. Garrido, E. Recio, R. De la Rosa, Lie symmetries and exact solutions for a fourth-order nonlinear diffusion equation, Math. Meth. Appl. Sci. 45 (17) (2022), 10614-10627.

[10] A. Motamednezhad, F. Khajevand, Symmetry analysis and exact solutions of acoustic equation, Comput. Methods. Diff. Equ. 8 (3) (2020), 523-536.

[11] M. Nadjafikhah, Lie symmetries of inviscid burgers equations, Adv. Appl. Clifford Algebras. 19 (1) (2009), 701-712.

[12] L. V. Ovsiannikov, Group analysis of differential equations, Academic press, New York, 1982.

[13] P. J. Olver, P. Rosenau, Group-invariant solutions of differential equations, SIAM J. Appl. Math. 47 (2) (1987), 263-278.

[14] S. Rashidi, S. R. Hejazi, Self-adjointness, Conservation laws and invariant solutions of the Buckmaster equation, Comput. Methods Diff. Equ. 8 (1) (2020), 85-98.

[15] A. M. Wazwaz, A variety of completely integrable Calogero Bogoyavlenskii Schiff equation with time dependent coefficints, Inter. J. Numerical Methods. Heat. Fluid. Flow. (2020), online.