

New three-step iteration process and fixed point approximation in Banach spaces

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Abstract. In this paper we propose a new iteration process, called the K^* iteration process, for approximation of fixed points. We show that our iteration process is faster than the existing well-known iteration processes using numerical examples. Stability of the K^* iteration process is also discussed. Finally we prove some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces. Our results are the extension, improvement and generalization of many well-known results in the literature of iterations in fixed point theory.

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1. Introduction

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. However, once the existence of a fixed point of some mapping is established, then to find the value of that fixed point is not an easy task, that is why we use iterative processes for computing them. Many iterative processes have been developed and it is impossible to cover them all. The well-known Banach contraction theorem uses the Picard iteration process for approximation of fixed points. Some of the

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other well-known iterative processes are those of Mann [12], Ishikawa [8], Agarwal [2], Noor [13], Abbas [1], SP [16], S* [9], CR [4], Normal-S [19], Picard Mann [11], Picard-S [6], Thakur New [24] and so on.

Two qualities “Fastness” and “Stability” play important role for an iteration process to be preferred on another iteration process. In [17], Rhoades mentioned that the Mann iteration process for decreasing function converge faster than the Ishikawa iteration process and for increasing function the Ishikawa iteration process is better than the Mann process. Also the Mann iteration process appears to be independent of the initial guess (see also [18]). In [2], the authors claimed that Agarwal iteration process converge at a rate same as that of the Picard iteration process and faster than the Mann iteration process for contraction mappings. In [1], the authors claimed that Abbas iteration process converge faster than Agarwal iteration process. In [4], the authors claimed that the CR iteration process is equivalent to and faster than those of Picard, Mann, Ishikawa, Agarwal, Noor and SP iterative processes for quasi-contractive operators in Banach spaces. Also in [10] the authors proved that the CR iterative process converge faster than the S* iterative process for the class of contraction mappings. In [6], authors claimed that the Picard-S iteration process converge faster than all of Picard, Mann, Ishikawa, Noor, SP, CR, Agarwal, S*, Abbas and Normal-S for contraction mappings. In [24], the authors proved with the help of numerical example that the Thakur New iteration process converge faster than those of Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration processes for the class of Suzuki generalized nonexpansive mappings.

Motivated by above, in this paper, we introduce a new iteration process namely the K^* iteration process and then prove analytically that our process is stable. An example of Suzuki generalized nonexpansive mapping is given which is not a nonexpansive mapping. Numerically we compare the speed of convergence of our new iteration process with the most leading two-step Agarwal iteration process and most leading three-step Picard-S iteration process. Finally we prove some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings, which is the generalization of nonexpansive as well as contraction mappings, in the setting of uniformly convex Banach spaces.

2. Preliminaries

First we recall some definitions, propositions and lemmas to be used in the next two sections.

A Banach space X is called uniformly convex [5] if for each $\varepsilon \in (0, 2]$ there is a $\delta > 0$ such that for $x, y \in X$,

$$\left. \begin{array}{l} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x - y\| > \varepsilon \end{array} \right\} \implies \left\| \frac{x + y}{2} \right\| \leq \delta.$$

A Banach space X is said to satisfy the Opial property [14] if for each sequence $\{x_n\}$ in X , converging weakly to $x \in X$, we have

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in X$ such that $y \neq x$.

A point p is called a fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of the mapping T . Let C be a nonempty subset of a Banach space X . A mapping $T : C \rightarrow C$ is called a contraction if there exists $\theta \in (0, 1)$ such that $\|Tx - Ty\| \leq \theta \|x - y\|$, for all $x, y \in C$. A mapping $T : C \rightarrow C$ is called a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have $\|Tx - p\| \leq \|x - p\|$. In 2008, Suzuki [23] introduced the concept of generalized nonexpansive mappings which is a condition on mapping, called condition (C). A mapping $T : C \rightarrow C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|.$$

Suzuki [23] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The mapping satisfying condition (C) is called Suzuki generalized nonexpansive mapping. Also, he obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mapping. In 2011, Phuengrattana [15] proved convergence theorems for Suzuki generalized nonexpansive mappings using the Ishikawa iteration in uniformly convex Banach spaces and $CAT(0)$ spaces. Recently, fixed point theorems for Suzuki generalized nonexpansive mapping have been studied by a number of authors see e.g. [24] and references therein.

We now list some properties of a Suzuki generalized nonexpansive mappings.

Proposition 2.1 Let C be a nonempty subset of a Banach space X and $T : C \rightarrow C$ be any mapping. Then

- (i) [23, Proposition 1] If T is nonexpansive then T is a Suzuki generalized nonexpansive mapping.
- (ii) [23, Proposition 2] If T is a Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.
- (iii) [23, Lemma 7] If T is a Suzuki generalized nonexpansive mapping, then $\|x - Ty\| \leq 3 \|Tx - x\| + \|x - y\|$ for all $x, y \in C$.

Lemma 2.2 [23, Proposition 3] Let T be a mapping on a subset C of a Banach space X with the Opial property. Assume that T is a Suzuki generalized nonexpansive mapping. If $\{x_n\}$ converges weakly to z and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.

Lemma 2.3 [23, Theorem 5] Let C be a weakly compact convex subset of a uniformly convex Banach space X . Let T be a mapping on C . Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.

Lemma 2.4 [20, Lemma 1.3] Suppose that X is a uniformly convex Banach space and $\{t_n\}$ be any real sequence such that $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\limsup_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Let C be a nonempty closed convex subset of a Banach space X , and let $\{x_n\}$ be a bounded sequence in X . For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} \|x_n - x\|$. The asymptotic radius of $\{x_n\}$ relative to C is given by $r(C, \{x_n\}) = \inf \{r(x, \{x_n\}) : x \in C\}$ and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(C, \{x_n\})$ consists of exactly one

point.

Definition 2.5 [7] Let $\{t_n\}_{n=0}^{\infty}$ be an arbitrary sequence in C . Then, an iteration procedure $x_{n+1} = f(T, x_n)$ converging to fixed point p , is said to be T -stable or stable with respect to T , if for $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$, $n = 0, 1, 2, 3, \dots$, we have

$$\lim_{n \rightarrow \infty} \epsilon_n = 0 \iff \lim_{n \rightarrow \infty} t_n = p.$$

Definition 2.6 [3] Let $T, \tilde{T} : X \rightarrow X$ be two operators. We say that \tilde{T} is an approximate operator for T if for some $\varepsilon > 0$ we have $\|Tx - \tilde{T}x\| \leq \varepsilon$ for all $x \in X$.

Lemma 2.7 [25] Let $\{\psi_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the inequality $\psi_{n+1} \leq (1 - \phi_n)\psi_n + \varphi_n$, where $\phi_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \phi_n = \infty$ and $\frac{\varphi_n}{\phi_n} \rightarrow 0$ as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \psi_n = 0$.

3. K^* iteration Process and its Convergence Analysis

Through out this section we have $n \geq 0$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ and C be a nonempty subset of Banach Space X .

Following is the one step Mann iteration process for approximating fixed points:

$$\begin{cases} u_0 \in C \\ u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n \end{cases} \quad (1)$$

Agarwal et al. [2] introduced new iteration process known as Agarwal iteration process or S iteration process which converges faster than (1), as:

$$\begin{cases} u_0 \in C \\ v_n = (1 - \beta_n)u_n + \beta_n T u_n \\ u_{n+1} = (1 - \alpha_n)T u_n + \alpha_n T v_n \end{cases} \quad (2)$$

Recently Gursoy and Karakaya in [6] introduced new iteration process called Picard-S iteration process, as follow:

$$\begin{cases} u_0 \in C \\ w_n = (1 - \beta_n)u_n + \beta_n T u_n \\ v_n = (1 - \alpha_n)T u_n + \alpha_n T w_n \\ u_{n+1} = T v_n \end{cases} \quad (3)$$

They proved that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings. Also, by providing an example, it is shown that the Picard-S iteration process converge faster than all of Mann, Ishikawa, Noor, SP, CR, S, S^* , Abbas, Normal-S and Two-step Mann iteration process. In 2015, Thakur et al. [24]

used the following new iteration process, we will call it Thakur New iteration process,

$$\begin{cases} u_0 \in C \\ w_n = (1 - \beta_n)u_n + \beta_n T u_n \\ v_n = T((1 - \alpha_n)u_n + \alpha_n w_n) \\ u_{n+1} = T v_n. \end{cases} \quad (4)$$

With the help of numerical example they proved that (4) is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iteration process for Suzuki generalized nonexpansive mappings. We note that the speed of convergence of iteration process (3) and (4) are almost same.

Problem 3.1 Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes (2), (3) and (4)?

To answer this, we introduced a new iteration process known as “ K^* Iteration Process” defined by:

$$\begin{cases} x_0 \in C \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n \\ y_n = T((1 - \alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T y_n \end{cases} \quad (5)$$

First we will prove that our new iteration process (5) is stable and have a good speed of convergence comparatively to other iteration processes.

Theorem 3.2 Let C be a nonempty closed convex subset of a Banach space X and $T : C \rightarrow C$ be a contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (5) with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converge strongly to a unique fixed point of T .

Proof. The well-known Banach theorem guarantees the existence and uniqueness of fixed point p . We will show that $x_n \rightarrow p$ for $n \rightarrow \infty$. From (5) we have

$$\begin{aligned} \|z_n - p\| &= \|(1 - \beta_n)x_n + \beta_n T x_n - (1 - \beta_n + \beta_n)p\| \\ &\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|T x_n - T p\| \\ &\leq (1 - \beta_n) \|x_n - p\| + \beta_n \theta \|x_n - p\| \\ &= (1 - \beta_n(1 - \theta)) \|x_n - p\|. \end{aligned} \quad (6)$$

Similarly using (6), we have

$$\begin{aligned} \|y_n - p\| &= \|T((1 - \alpha_n)z_n + \alpha_n T z_n) - T p\| \\ &\leq \theta \|(1 - \alpha_n)z_n + \alpha_n T z_n - p\| \\ &\leq \theta[(1 - \alpha_n) \|z_n - p\| + \alpha_n \|T z_n - p\|] \\ &\leq \theta[(1 - \alpha_n) \|z_n - p\| + \alpha_n \theta \|z_n - p\|] \\ &\leq \theta(1 - \alpha_n(1 - \theta)) \|z_n - p\| \\ &\leq \theta(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta)) \|x_n - p\|. \end{aligned} \quad (7)$$

Using (7), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq \theta \|y_n - p\| \\ &\leq \theta^2(1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta)) \|x_n - p\| \\ &\leq \theta^2(1 - \alpha_n(1 - \theta)) \|x_n - p\|, \end{aligned} \quad (8)$$

by using the fact that $(1 - \beta_n(1 - \theta)) < 1$, for $\theta \in (0, 1)$ and $\{\beta_n\}_{n=0}^\infty$ in $[0, 1]$. From (8) we have the following inequalities

$$\left\{ \begin{array}{l} \|x_{n+1} - p\| \leq \theta^2(1 - \alpha_n(1 - \theta)) \|x_n - p\| \\ \|x_n - p\| \leq \theta^2(1 - \alpha_{n-1}(1 - \theta)) \|x_{n-1} - p\| \\ \|x_{n-1} - p\| \leq \theta^2(1 - \alpha_{n-2}(1 - \theta)) \|x_{n-2} - p\| \\ \vdots \\ \vdots \\ \|x_1 - p\| \leq \theta^2(1 - \alpha_0(1 - \theta)) \|x_0 - p\|. \end{array} \right. \quad (9)$$

From (9) we can easily derive

$$\|x_{n+1} - p\| \leq \|x_0 - p\| \theta^{2(n+1)} \prod_{k=0}^n (1 - \alpha_k(1 - \theta)), \quad (10)$$

where $1 - \alpha_k(1 - \theta) \in (0, 1)$ because $\theta \in (0, 1)$ and $\alpha_n \in [0, 1]$, for all $n \in \mathbb{N}$. Since $1 - x \leq e^{-x}$ for all $x \in [0, 1]$, from (10) we get

$$\|x_{n+1} - p\| \leq \frac{\|x_0 - p\| \theta^{2(n+1)}}{e^{(1-\theta) \sum_{k=0}^n \alpha_k}}. \quad (11)$$

Taking the limit of both sides of inequality (11) yields $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$, i.e. $x_n \rightarrow p$ for $n \rightarrow \infty$, as required. \blacksquare

Theorem 3.3 Let C be a nonempty closed convex subset of a Banach space X and $T : C \rightarrow C$ be a contraction mapping. Let $\{x_n\}_{n=0}^\infty$ be an iterative sequence generated by (5) with real sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in $[0, 1]$ satisfying $\sum_{n=0}^\infty \alpha_n = \infty$. Then the iterative process (5) is T -stable.

Proof. Let $\{t_n\}_{n=0}^\infty \subset X$ be any arbitrary sequence in C . Let the sequence generated by (5) is $x_{n+1} = f(T, x_n)$ converging to unique fixed point p (by Theorem 3.2) and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. We will prove that $\lim_{n \rightarrow \infty} \epsilon_n = 0 \iff \lim_{n \rightarrow \infty} t_n = p$. Let $\lim_{n \rightarrow \infty} \epsilon_n = 0$. By using (8) we get

$$\begin{aligned} \|t_{n+1} - p\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| \\ &= \epsilon_n + \left\| T(T((1 - \alpha_n)((1 - \beta_n)t_n + \beta_n T t_n)) \right. \\ &\quad \left. + \alpha_n T((1 - \beta_n)t_n + \beta_n T t_n)) - p \right\| \\ &\leq \theta^2(1 - \alpha_n(1 - \theta)) \|t_n - p\| + \epsilon_n. \end{aligned}$$

Define $\psi_n = \|t_n - p\|$, $\phi_n = \alpha_n(1 - \theta) \in (0, 1)$ and $\varphi_n = \epsilon_n$. Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$, which

implies that $\frac{\varphi_n}{\phi_n} \rightarrow 0$ as $n \rightarrow \infty$. Thus all conditions of Lemma 2.7 are fulfilled by above inequality. Hence by Lemma 2.7 we get $\lim_{n \rightarrow \infty} t_n = p$. Conversely, let $\lim_{n \rightarrow \infty} t_n = p$. Now, we have

$$\begin{aligned} \epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\ &\leq \|t_{n+1} - p\| + \|f(T, t_n) - p\| \\ &\leq \|t_{n+1} - p\| + \theta^2(1 - \alpha_n(1 - \theta)) \|t_n - p\|. \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Hence (5) is stable with respect to T . ■

For numerical interpretations first we construct an example of a Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 3.4 Define a mapping $T : [0, 1] \rightarrow [0, 1]$ by

$$Tx = \begin{cases} 1 - x & \text{if } x \in [0, \frac{1}{7}) \\ \frac{x+6}{7} & \text{if } x \in [\frac{1}{7}, 1]. \end{cases}$$

We need to prove that T is a Suzuki generalized nonexpansive mapping but not nonexpansive. If $x = \frac{7}{50}$, $y = \frac{1}{7}$ we see that

$$\|Tx - Ty\| = |Tx - Ty| = \left| 1 - \frac{7}{50} - \frac{43}{49} \right| = \frac{43}{2450} > \frac{1}{350} = \|x - y\|.$$

Hence T is not a nonexpansive mapping. To verify that T is a Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x \in [0, \frac{1}{7})$, then $\frac{1}{2} \|x - Tx\| = \frac{1-2x}{2} \in (\frac{5}{14}, \frac{1}{2}]$. For $\frac{1}{2} \|x - Tx\| \leq \|x - y\|$ we must have $\frac{1-2x}{2} \leq y - x$, i.e., $\frac{1}{2} \leq y$, hence $y \in [\frac{1}{2}, 1]$. We have

$$\|Tx - Ty\| = \left| \frac{y+6}{7} - (1-x) \right| = \left| \frac{y+7x-1}{7} \right| < \frac{1}{7}$$

and $\|x - y\| = |x - y| > |\frac{1}{7} - \frac{1}{2}| = \frac{5}{14}$. Hence $\frac{1}{2} \|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Case II: Let $x \in [\frac{1}{7}, 1]$, then $\frac{1}{2} \|x - Tx\| = \frac{1}{2} |\frac{x+6}{7} - x| = \frac{6-6x}{14} \in [0, \frac{18}{49}]$. For $\frac{1}{2} \|x - Tx\| \leq \|x - y\|$ we must have $\frac{6-6x}{14} \leq |y - x|$, which gives two possibilities:

(a). Let $x < y$, then $\frac{6-6x}{14} \leq y - x \implies y \geq \frac{6+8x}{14} \implies y \in [\frac{50}{98}, 1] \subset [\frac{1}{7}, 1]$. So

$$\|Tx - Ty\| = \left| \frac{x+6}{7} - \frac{y+6}{7} \right| = \frac{1}{7} \|x - y\| \leq \|x - y\|.$$

Hence $\frac{1}{2} \|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

(b). Let $x > y$, then $\frac{6-6x}{14} \leq x - y \implies y \leq x - \frac{6-6x}{14} = \frac{20x-6}{14} \implies y \in [-\frac{22}{98}, 1]$. Since $y \in [0, 1]$, $y \leq \frac{20x-6}{14} \implies x \in [\frac{3}{10}, 1]$. So the case is $x \in [\frac{3}{10}, 1]$ and $y \in [0, 1]$.

Now $x \in [\frac{3}{10}, 1]$ and $y \in [\frac{1}{7}, 1]$ is already included in (a). So let $x \in [\frac{3}{10}, 1]$ and $y \in [0, \frac{1}{7})$, then $\|Tx - Ty\| = |\frac{x+6}{7} - (1-y)| = |\frac{x+7y-1}{7}|$. For convenience, first we consider $x \in [\frac{3}{10}, \frac{1}{2}]$ and $y \in [0, \frac{1}{7})$, then $\|Tx - Ty\| \leq \frac{1}{14}$ and $\|x - y\| > \frac{11}{70}$. Hence $\|Tx - Ty\| \leq \|x - y\|$.

Table 1. Sequences generated by K^* , Picard-S and S iteration processes.

	K^*	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.998620346016926	0.997959183673469	0.985714285714286
x_2	0.999988271772595	0.999963995259702	0.998235767725379
x_3	0.999999947984460	0.999999410903228	0.999797939807358
x_4	0.99999999949083	0.99999990799730	0.999977910151591
x_5	1	0.99999999860798	0.999997660439039
x_6	1	0.9999999997942	0.99999757854153
x_7	1	0.999999999997	0.99999975372301
x_8	1	1.	0.999999997529402
x_9	1	1.	0.99999999754887
x_{10}	1	1.	0.99999999975903

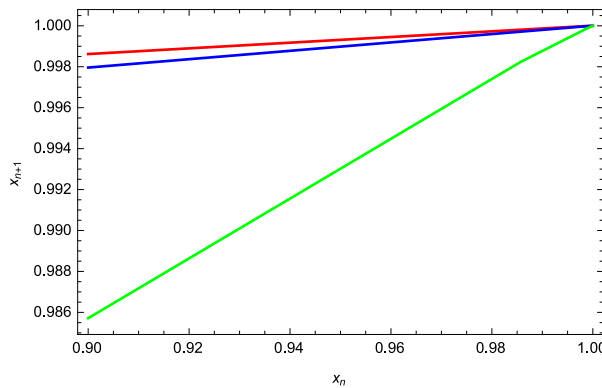


Figure 1. Convergence of iterative sequences generated by K^* , Picard-S and S iteration processes to the fixed point 1 of the mapping T defined in Example 3.4.

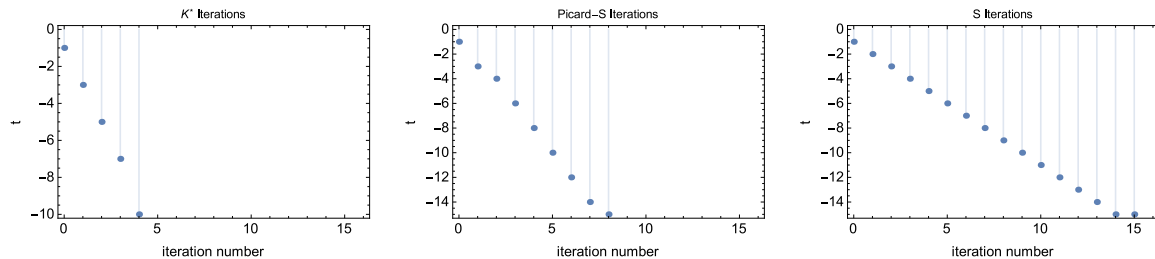


Figure 2. Graphs for K^* , Picard-S and S iteration processes where the value of t indicates that the value of the recursion after a certain number of steps is only 10^t units away from fixed point 1 of the mapping T defined in Example 3.4.

Next consider $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{7})$, then $\|Tx - Ty\| \leq \frac{1}{7}$ and $\|x - y\| > \frac{5}{14}$. Hence $\|Tx - Ty\| \leq \|x - y\|$. So $\frac{1}{2} \|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$. Hence T is a Suzuki generalized nonexpansive mapping.

In Table 1, we can see some of the first terms of a sequence generated by the K^* , Picard-S and S iteration processes for $\alpha_n = \frac{2n}{\sqrt{7n+9}}$, $\beta_n = \frac{1}{\sqrt{3n+7}}$, where initial value $x_0 = 0.9$ and operator T is that of Example 3.4. Set the stop parameter to $\|x_n - 1\| \leq 10^{-15}$, where 1 is the fixed point of T . A graphic representation is given in Fig. 1. Note that sequence generated by each iterative process is denoted by x_n . In Fig. 2 we can see the order of magnitude of the difference between the value of the recursion after a certain number of iterations and 1, the convergence value.

Similarly Table 2 and Fig. 3 use mapping T defined by $T(x) = \sqrt{x^2 - 8x + 40}$ with fixed point 5.

Table 2. Sequences generated by K^* , Picard-S and S iteration processes.

	K^*	Picard-S	S
x_0	10	10	10
x_1	5.192471731486675	5.66076955440972	6.83624962725991
x_2	5.001263310733600	5.01960812810809	5.37044124393507
x_3	5.000007765124472	5.00044149419855	5.04752702755717
x_4	5.000000047709050	5.00000985609151	5.00540334935426
x_5	5.00000000293130	5.00000021998893	5.00060429338923
x_6	5.00000000001800	5.0000000491015	5.00006745514064
x_7	5.	5.00000000010960	5.00000752819304
x_8	5.	5.00000000000245	5.00000084014883
x_9	5.	5.00000000000006	5.00000009376064
x_{10}	5.	5.	5.00000001046369

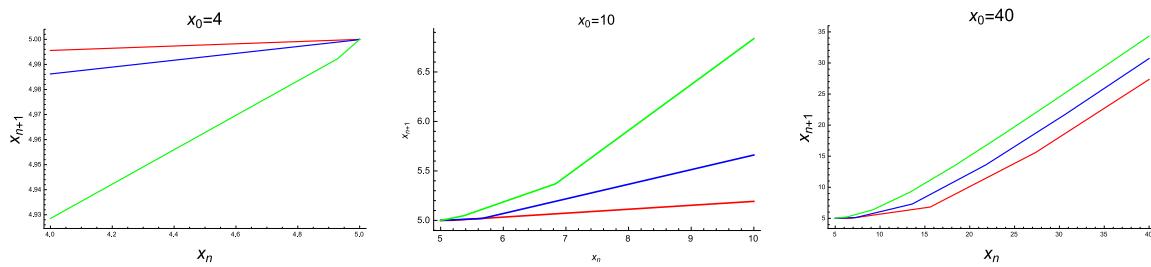


Figure 3. Convergence of iterative sequences generated by K^* , Picard-S and S iteration processes to the fixed point 5 of the mapping $T(x) = \sqrt{x^2 - 8x + 40}$ for different initial values.

4. Convergence results for Suzuki generalized nonexpansive mappings

In this section, we prove weak and strong convergence theorems of a sequence generated by a K^* iteration process for Suzuki generalized nonexpansive mappings in the setting of uniformly convex Banach spaces.

Lemma 4.1 Let C be a nonempty closed convex subset of a Banach space X , and let $T : C \rightarrow C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrarily chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (5), then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for any $p \in F(T)$.

Proof. Let $p \in F(T)$ and $z \in C$. Since T Suzuki generalized nonexpansive mapping, so

$$\frac{1}{2} \|p - Tp\| = 0 \leq \|p - z\| \text{ implies that } \|Tp - Tz\| \leq \|p - z\|.$$

So by Proposition 2.1(ii), we have

$$\begin{aligned} \|z_n - p\| &= \|(1 - \beta_n)x_n + \beta_nTx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{12}$$

Using (12), we get

$$\begin{aligned}
 \|y_n - p\| &= \|T((1 - \alpha_n)z_n + \alpha_n Tz_n) - p\| \\
 &\leq \|(1 - \alpha_n)z_n + \alpha_n Tz_n - p\| \\
 &\leq (1 - \alpha_n) \|z_n - p\| + \alpha_n \|Tz_n - p\| \\
 &\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \|z_n - p\| \\
 &\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \|x_n - p\| \\
 &= \|x_n - p\|.
 \end{aligned} \tag{13}$$

Similarly, by using (13), we have

$$\|x_{n+1} - p\| = \|Ty_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|. \tag{14}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, as required. \blacksquare

Theorem 4.2 Let C be a nonempty closed convex subset of a uniformly convex Banach space X , and let $T : C \rightarrow C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (5) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$.

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 4.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n \rightarrow \infty} \|x_n - p\| = r. \tag{15}$$

From (12) and (15), we have

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = r. \tag{16}$$

By Proposition 2.1(ii), we have

$$\limsup_{n \rightarrow \infty} \|Tx_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = r. \tag{17}$$

On the other hand by using (12), we have

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|Ty_n - p\| \\
 &\leq \|y_n - p\| \\
 &= \|T((1 - \alpha_n)z_n + \alpha_n Tz_n) - p\| \\
 &\leq \|(1 - \alpha_n)z_n + \alpha_n Tz_n - p\| \\
 &\leq (1 - \alpha_n) \|z_n - p\| + \alpha_n \|Tz_n - p\| \\
 &\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \|z_n - p\| \\
 &= \|x_n - p\| - \alpha_n \|x_n - p\| + \alpha_n \|z_n - p\|.
 \end{aligned}$$

This implies that

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n} \leq \|z_n - p\| - \|x_n - p\|.$$

So

$$\|x_{n+1} - p\| - \|x_n - p\| \leq \frac{\|x_{n+1} - p\| - \|x_n - p\|}{\alpha_n} \leq \|z_n - p\| - \|x_n - p\|$$

implies that $\|x_{n+1} - p\| \leq \|z_n - p\|$. Therefore,

$$r \leq \liminf_{n \rightarrow \infty} \|z_n - p\|. \tag{18}$$

By (16) and (18) we get

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} \|z_n - p\| \\ &= \lim_{n \rightarrow \infty} \|(1 - \beta_n)x_n + \beta_nTx_n - p\| \\ &= \lim_{n \rightarrow \infty} \|\beta_n(Tx_n - p) + (1 - \beta_n)(x_n - p)\|. \end{aligned} \tag{19}$$

Using (15), (17), (19) and Lemma 2.4, we have $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Let $p \in A(C, \{x_n\})$. By Proposition 2.1(iii), we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n \rightarrow \infty} \|x_n - Tp\| \\ &\leq \limsup_{n \rightarrow \infty} (3\|Tx_n - x_n\| + \|x_n - p\|) \\ &\leq \limsup_{n \rightarrow \infty} \|x_n - p\| \\ &= r(p, \{x_n\}). \end{aligned}$$

This implies that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex, $A(C, \{x_n\})$ is singleton. Hence, we have $Tp = p$. Hence $F(T) \neq \emptyset$. ■

Now, we are in the position to prove weak convergence theorem.

Theorem 4.3 Let C be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property, and let $T : C \rightarrow C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (5) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. Then $\{x_n\}$ converges weakly to a fixed point of T .

Proof. Since $F(T) \neq \emptyset$, so by Theorem 4.2 we have that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex hence reflexive, so by Eberlin's theorem there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $q_1 \in X$. Since C is closed and convex, by Mazur's theorem $q_1 \in C$. By Lemma 2.2, $q_1 \in F(T)$. Now, we show that $\{x_n\}$ converges weakly to q_1 . In fact, if this is not true, so there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_2 \in C$ and

$q_2 \neq q_1$. By Lemma 2.2, $q_2 \in F(T)$. Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T)$. By Theorem 4.2 and Opial's property, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - q_1\| &= \lim_{j \rightarrow \infty} \|x_{n_j} - q_1\| \\ &< \lim_{j \rightarrow \infty} \|x_{n_j} - q_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - q_2\| \\ &= \lim_{k \rightarrow \infty} \|x_{n_k} - q_2\| \\ &< \lim_{k \rightarrow \infty} \|x_{n_k} - q_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - q_1\|, \end{aligned}$$

which is contradiction. So $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to a fixed point of T . ■

Next, we prove the strong convergence theorem.

Theorem 4.4 Let C be a nonempty compact convex subset of a uniformly convex Banach space X , and let $T : C \rightarrow C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (5) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. By Lemma 2.3, we have $F(T) \neq \emptyset$ so by Theorem 4.2 we have $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Since C is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.1(iii), we have

$$\|x_{n_k} - Tp\| \leq 3 \|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|, \text{ for all } n \geq 1.$$

Letting $k \rightarrow \infty$, we get $Tp = p$, i.e., $p \in F(T)$. Since, by Lemma 4.1, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for every $p \in F(T)$, so x_n converge strongly to p . ■

Senter and Dotson [21] introduced the notion of a mappings satisfying condition (I): A mapping $T : C \rightarrow C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p \in F(T)} \|x - p\|$.

Now, we prove the strong convergence theorem using condition (I).

Theorem 4.5 Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \rightarrow C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (5) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. By Lemma 4.1, we see that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F(T)$ and so $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists. Assume that $\lim_{n \rightarrow \infty} \|x_n - p\| = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Suppose $r > 0$, from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\|. \quad (20)$$

Since $F(T) \neq \emptyset$, by Theorem 4.3, we have $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. So (20) implies that

$$\lim_{n \rightarrow \infty} f(d(x_n, F(T))) = 0. \quad (21)$$

Since f is nondecreasing function, so from (21) we have $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$. Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$\|x_{n_k} - y_k\| < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

Using (14), we $\|x_{n_{k+1}} - y_k\| \leq \|x_{n_k} - y_k\| < \frac{1}{2^k}$. Hence,

$$\begin{aligned} \|y_{k+1} - y_k\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - y_k\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p . Since $F(T)$ is closed, therefore $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p . Since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, we have $x_n \rightarrow p \in F(T)$. Hence proved. \blacksquare

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