*Journal of Linear and Topological Algebra*
*Vol.* 11*, No.* 02*,* 2022*,* 123- 128
*DOR: 20.1001.1.22520201.2022.11.02.5.8*
*DOI: 10.30495/JLTA.2022.1953833.1477*

# Bornological linearly topologized modules over a discrete valuation ring

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Received 2 March 2022; Revised 16 June 2022, Accepted 20 June 2022.
Communicated by Hamidreza Rahimi

**Abstract.** In this work the notion of a bornological linearly topologized module over a discrete valuation ring is introduced and it is shown that certain semimetrizable linearly topologized modules are bornological. The main result is a characterization of bornological linearly topologized modules, from which the completeness and the quasi-completeness of certain linearly topologized modules of continuous linear mappings are derived.

**Keywords:** Discrete valuation rings, linearly topologized modules, continuity of linear mappings.

**2010 AMS Subject Classification**: 13F30, 46H25, 46A17.

## 1. Introduction

Bornological locally convex spaces were introduced by Mackey [5] and redefined by Bourbaki [1], a bornological space being characterized by the property according to which every linear mapping on it which transforms bounded sets into bounded sets is continuous.

In this paper the notion of a bornological linearly topologized module over an arbitrary discrete valuation ring is defined. It is shown that certain semimetrizable linearly topologized modules are bornological and an example of a metrizable linearly topologized module which is not bornological is given. The main result established here is a characterization of bornological linearly topologized modules, motivated by a classical result of the theory of locally convex spaces, from which we derive the completeness and the quasi-completeness of certain linearly topologized modules of continuous linear mappings (with respect to the topology of bounded convergence). It should also be mentioned that

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.ctb.iau.ir

a major difference between the classical case and the one discussed in the present work is that in the former invertible scalars (that is, non-zero scalars) may be taken "topologically close to zero", which is not possible in the context under consideration. Linearly topologized rings and linearly topologized modules have been considered, for example, in [2, 4, 7–9].

In this paper $R$ will denote an arbitrary discrete valuation ring [7, Chapter I], $M$ its valuation ring, $\pi$ a generator of $M$ and $R_0$ the $R$-module $K/R$ (where $K$ is the field of fractions of $R$); and "$R$-module" will always mean "unitary $R$-module". Since

$$\{\pi^n R;\ n = 1, 2, \dots\}$$

constitutes a fundamental system of neighborhoods of 0 in $R$ formed by ideals of $R$ such that $\bigcap_{n \geq 1} \pi^n R = \{0\}$, it follows that $R$ is a metrizable linearly topologized ring.

Throughout this work we shall say that $E$ is a linearly topologized $R$-module if $E$ is a topological $R$-module whose origin admits a fundamental system of neighborhoods consisting of submodules of $E$. It is easily seen that a subset $B$ of a linearly topologized $R$-module $E$ is bounded if and only if for each neighborhood $U$ of 0 in $E$ which is a submodule of $E$ there is an integer $k \geq 1$ so that $\pi^k B \subset U$. For each Hausdorff linearly topologized $R$-module $E$, $E^*$ denotes the $R$-module of all continuous $R$-linear mappings from $E$ into $R_0$ ($R_0$ endowed with the discrete topology, under which it is a linearly topologized $R$-module) [3, 6].

## 2. Bornological linearly topologized $R$-modules: basic properties

**Definition 2.1** Let $E$ be a linearly topologized $R$-module. A subset $U$ of $E$ is said to be a *bornivorous* in $E$ if $U$ is a submodule of $E$ and for each bounded subset $B$ of $E$ there is an integer $k \geq 1$ such that $\pi^k B \subset U$ (that is, $(\pi^k R)B \subset U$).

***Remark 1*** *It is easily seen that the notion of a bornivorous set does not depend on the choice of the generator $\pi$ of $M$.*

Every neighborhood of 0 in a linearly topologized $R$-module $E$ which is a submodule of $E$ is a bornivorous in $E$. The next example shows that the converse of this assertion is not always true.

***Example* 2.2** Let $E$ be the product $R$-module $R^{\mathbb{N}}$ endowed with the product topology, which makes $E$ into a metrizable linearly topologized $R$-module. We claim that the submodule $U = \pi E$ of $E$, which is not a neighborhood of 0 in $E$, is a bornivorous in $E$. In fact, let $B$ be an arbitrary bounded subset of $E$ and consider the neighborhood $W = (\pi R) \times R \times R \times \cdots \times R \times \dots$ of 0 in $E$. By the boundedness of $B$ there is an integer $\ell \geq 1$ so that $\pi^\ell B \subset W$, which implies $\pi^{\ell+1} B \subset U$. Hence $U$ is a bornivorous in $E$. We may also observe that

$$\pi W = (\pi^2 R) \times (\pi R) \times (\pi R) \times \cdots \times (\pi R) \times \dots$$

is not a neighborhood of 0 in $E$.

**Definition 2.3** A linearly topologized $R$-module $E$ is said to be *bornological* if every bornivorous in $E$ is a neighborhood of 0 in $E$.

We have just seen that the metrizable linearly topologized $R$-module $E$, considered in Example 2.2, is not bornological.

**Proposition 2.4** Let $E$ be a semimetrizable linearly topologized $R$-module such that, for each integer $n \geq 1$ and for each neighborhood $U$ of 0 in $E$, $\pi^n U$ is a neighborhood of 0 in $E$. Then $E$ is bornological.

**Proof.** Let $U$ be a bornivorous in $E$ and let $(U_n)_{n\geq 1}$ be a decreasing fundamental system of neighborhoods of 0 in $E$. If $U$ is not a neighborhood of 0 in $E$, then $U$ does not contain $\pi^n U_n$ for every integer $n \geq 1$, because $\pi^n U_n$ is a neighborhood of 0 in $E$ for $n = 1, 2, \dots$ . Hence there exists a sequence $(x_n)_{n\geq 1}$ in $E$ such that $x_n \in U_n$ and $\pi^n x_n \notin U$ for $n = 1, 2, \dots$ . But, since $(x_n)_{n\geq 1}$ is a null sequence in $E$, the set $\{x_n; n = 1, 2, \dots\}$ is bounded in $E$. Thus the assumption that $U$ is a bornivorous in $E$ guarantees the existence of an integer $m \geq 1$ for which $\pi^m x_m \in U$, which does not occur. Therefore $U$ is a neighborhood of 0 in $E$, and $E$ is bornological. ■

Example 2.2 shows that the condition "$\pi^n U$ is a neighborhood of 0 in $E$ for each integer $n \geq 1$ and for each neighborhood $U$ of 0 in $E$" is essential for the validity of Proposition 2.4.

***Example* 2.5** By Proposition 2.4, every discrete linearly topologized $R$-module is bornological. In particular, $R_0$ is bornological.

**Corollary 2.6** If $E$ is a locally bounded linearly topologized $R$-module such that $\pi^n U$ is a neighborhood of 0 in $E$ for each integer $n \geq 1$ and for each neighborhood $U$ of 0 in $E$, then $E$ is bornological.

**Proof.** Since $E$ is locally bounded, there exists a bounded neighborhood $W$ of 0 in $E$. By hypothesis $\pi^n W$ is a neighborhood of 0 in $E$ for $n = 1, 2, \dots$, and it is easily seen that $(\pi^n W)_{n\geq 1}$ is a fundamental system of neighborhoods of 0 in $E$. Therefore $E$ is semimetrizable, and the result follows from Proposition 2.4. ■

**Proposition 2.7** Let $(E_\alpha)_{\alpha\in A}$ be a family of bornological linearly topologized $R$-modules, $E$ an $R$-module and, for each $\alpha \in A$, let $u_\alpha \colon E_\alpha \to E$ be an $R$-linear mapping. Assume that $E = \left[\bigcup_{\alpha\in A} \mathrm{Im}(u_\alpha)\right]$ and that $E$ is endowed with the final linear topology for the family $(E_\alpha, u_\alpha)_{\alpha\in A}$ [6]. Then the linearly topologized $R$-module $E$ is bornological.

**Proof.** Let $U$ be a bornivorous in $E$ and $\alpha \in A$ be arbitrary. We claim that the submodule $u_\alpha^{-1}(U)$ of $E_\alpha$ is a bornivorous in $E_\alpha$. In fact, if $B_\alpha$ is a bounded subset of $E_\alpha$, $u_\alpha(B_\alpha)$ is a bounded subset of $E$ (since $u_\alpha$ is continuous), and hence there is an integer $k \geq 1$ so that $\pi^k u_\alpha(B_\alpha) = u_\alpha(\pi^k B_\alpha) \subset U$. Consequently $\pi^k B_\alpha \subset u_\alpha^{-1}(U)$, proving that $u_\alpha^{-1}(U)$ is a bornivorous in $E_\alpha$, hence a neighborhood of 0 in $E_\alpha$ (since $E_\alpha$ is bornological). Therefore $U$ is a neighborhood of 0 in $E$, and $E$ is bornological. ■

An immediate consequence of Proposition 2.7 reads:

**Corollary 2.8** (a) A quotient of a bornological linearly topologized $R$-module, endowed with the quotient topology, is a bornological linearly topologized $R$-module.
(b) A topological direct sum of a family of bornological linearly topologized $R$-modules is a bornological linearly topologized $R$-module.

***Example* 2.9** For each integer $n \geqslant 1$, the product $R$-module $R^n$ endowed with the product topology is a metrizable linearly topologized $R$-module. Moreover, since

$$\pi^m((\pi^{m_1}R) \times \cdots \times (\pi^{m_n}R)) = (\pi^{m+m_1}R) \times \cdots \times (\pi^{m+m_n}R)$$

for arbitrary integers $m, m_1, \ldots, m_n \geqslant 1$, Proposition 2.4 ensures that $R^n$ is bornological. Consequently, in view of Corollary 2.8(b), the topological direct sum $R^{(\mathbb{N})}$ is bornological.

## 3. A characterization of bornological linearly topologized $R$-modules and some consequences

**Theorem 3.1** For a linearly topologized $R$-module $E$, the following conditions are equivalent:

(a) $E$ is bornological;
(b) for each linearly topologized $R$-module $F$, we have that each set of $R$-linear mappings from $E$ and $F$ which transforms bounded subsets of $E$ into bounded subsets of $F$ is equicontinuous;
(c) for each linearly topologized $R$-module $F$, we have that each $R$-linear mapping from $E$ into $F$ which transforms null sequences in $E$ into bounded subsets of $F$ is continuous;
(d) for each linearly topologized $R$-module $F$, we have that each $R$-linear mapping from $E$ into $F$ which transforms bounded subsets of $E$ into bounded subsets of $F$ is continuous.

**Proof.** (a) $\Rightarrow$ (b): Let $F$ be an arbitrary linearly topologized $R$-module and $\mathfrak{X}$ a set of $R$-linear mappings from $E$ into $F$ such that $\mathfrak{X}(B)$ is bounded in $F$ for every bounded subset $B$ of $E$. We claim that $\mathfrak{X}$ is equicontinuous. Indeed, let $V$ be a neighborhood of 0 in $F$ which is a submodule of $F$ and let us show that the submodule

$$U = \bigcap_{u \in \mathfrak{X}} u^{-1}(V)$$

of $E$ is a bornivorous in $E$. For this purpose, let $B$ be a bounded subset of $E$. Since, by hypothesis, $\mathfrak{X}(B)$ is bounded, there is an integer $\ell \geq 1$ so that $\pi^\ell \mathfrak{X}(B) \subset V$. Consequently

$$\pi^\ell B \subset U,$$

showing that $U$ is a bornivorous in $E$. Therefore, by (a), $U$ is a neighborhood of 0 in $E$, and $\mathfrak{X}$ is equicontinuous.

(b) $\Rightarrow$ (c): Let $F$ be a linearly topologized $R$-module and let $u$ be an $R$-linear mapping from $E$ into $F$ which transforms null sequences in $E$ into bounded subsets of $F$. Let $B$ be an arbitrary bounded subset of $E$. If $u(B)$ is not bounded, there exists a neighborhood $V$ of 0 in $F$ which is a submodule of $F$ such that

$$\pi^{2n}u(B) \not\subset V$$

for $n = 1, 2, \ldots$ . Then for each $n = 1, 2, \ldots$ there is an $x_n \in B$ so that

$$\pi^{2n}u(x_n) = \pi^n u(\pi^n x_n) \notin V.$$

Put $y_n = \pi^n x_n$ for $n = 1, 2, \ldots$; $(y_n)_{n\geq 1}$ is a null sequence in $E$, because $(\pi^n)_{n\geq 1}$ is a null sequence in $R$ and $(x_n)_{n\geq 1}$ is a bounded sequence in $E$. But the sequence $\big(u(y_n)\big)_{n\geq 1}$

is not bounded in $F$. For, if it were, there would exist an integer $\ell \geq 1$ for which $\pi^\ell u(y_\ell) = \pi^\ell u(\pi^\ell x_\ell) \in V$, which does not occur. Therefore $u(B)$ is bounded, and (b) ensures the continuity of $u$.

(d) $\Rightarrow$ (a): Let $\mathcal{V}$ be the filter base on $E$ formed by all bornivorous in $E$, which clearly satisfies conditions (ATG 1), (ATG 2), (TMN 1) and (TMN 3) of Theorem 12.3 of [9]. Let $x \in E$ and $V \in \mathcal{V}$ be given. By the boundedness of the set $\{x\}$, there is an integer $m \geq 1$ so that $(\pi^m R)x \subset V$, and condition (TMN 2) of the above-mentioned theorem is also fulfilled. Consequently one may guarantee the existence of a unique $R$-module topology on $E$ for which $\mathcal{V}$ is a fundamental system of neighborhoods of 0 (by construction, this topology is linear). Let us represent by $F$ the $R$-module $E$ endowed with this topology. Then the identity mapping $u \colon E \to F$ transforms bounded subsets of $E$ into bounded subsets of $F$, and condition (d) implies its continuity. Therefore every element of $\mathcal{V}$ is a neighborhood of 0 in $E$, proving that $E$ is bornological.

This completes the proof of the theorem. ■

***Remark 2*** *We shall use Theorem 3.1 to give another explanation for the fact that the linearly topologized $R$-module $E$, considered in Example 2.2, is not bornological. In fact, let $F$ be the product $R$-module $R^{\mathbb{N}}$ endowed with the linear $R$-module topology for which the submodules $\pi^n R^{\mathbb{N}}$ $(n = 1, 2, \dots)$ of $R^{\mathbb{N}}$ constitute a fundamental system of neighborhoods of* 0*. Since $F$ is bounded, the identity mapping $u \colon E \to F$ transforms bounded subsets of $E$ into bounded subset of $F$. But $u$ is not continuous, because $\pi R^{\mathbb{N}}$ is a neighborhood of* 0 *in $F$ which is not a neighborhood of* 0 *in $E$. Thus, by Theorem 3.1, $E$ is not bornological.*

An immediate consequence of Theorem 3.1 reads:

**Corollary 3.2** Let $E$ be a bornological linearly topologized $R$-module and $F$ an arbitrary linearly topologized $R$-module. If $\tau_b$ is the linear $R$-module topology of bounded convergence on the $R$-module $\mathcal{L}(E; F)$ of all continuous $R$-linear mappings from $E$ into $F$, then every $\tau_b$-bounded subset of $\mathcal{L}(E; F)$ is equicontinuous. In particular, if $E$ is a Hausdorff space, every bounded subset of $E^*$ with respect to the strong topology $\beta(E^*, E)$ is equicontinuous.

**Corollary 3.3** Let $E$ be a bornological linearly topological $R$-module and $F$ a complete Hausdorff linearly topologized $R$-module. Then $(\mathcal{L}(E; F), \tau_b)$ is complete. In particular, if $E$ is a Hausdorff space, $(E^*, \beta(E^*, E))$ is complete.

**Proof.** Let $(u_i)_{i \in I}$ be a $\tau_b$-Cauchy net in $\mathcal{L}(E; F)$. It is easily seen that, for each $x \in E$, $(u_i(x))_{i \in I}$ is a Cauchy net in $F$; thus $(u_i(x))_{i \in I}$ converges to an element $u(x) \in F$, because $F$ is complete. The mapping $u \colon E \to F$ so defined is $R$-linear. We claim that $u \in \mathcal{L}(E; F)$ and that $(u_i)_{i \in I}$ converges to $u$ with respect to $\tau_b$. Indeed, let $B$ a bounded subset of $E$ and $V$ a neighborhood of 0 in $F$ which is a submodule of $F$. Then there exists an $i_0 \in I$ such that $u_i(x) - u_{i_0}(x) \in V$ for $x \in B$ and $i \in I$, $i \geq i_0$, which implies $u(x) - u_{i_0}(x) \in V$ for $x \in B$ (recall that $V$ is closed in $F$). On the other hand, there exists an integer $s \geq 1$ so that $\pi^s u_{i_0}(B) \subset V$, and hence

$$\pi^s u(B) = \pi^s(u(B) - u_{i_0}(B)) + \pi^s u_{i_0}(B) \subset \pi^s V + V \subset V + V = V.$$

Therefore $u(B)$ is bounded in $F$, and Theorem 3.1 furnishes the continuity of $u$. Finally it is clear that $(u_i)_{i \in I}$ converges to $u$ for $\tau_b$, thereby concluding the proof. ■

**Corollary 3.4** Let $E$ be a bornological linearly topologized $R$-module and $F$ a quasi-complete Hausdorff linearly topologized $R$-module. Then $(\mathcal{L}(E; F), \tau_b)$ is quasi-complete.

**Proof.** Let $\mathfrak{X}$ be a $\tau_b$-bounded and $\tau_b$-closed subset of $\mathcal{L}(E;F)$. By Corollary 3.2, $\mathfrak{X}$ is equicontinuous and, by Proposition 3.6 of [6], $\mathfrak{X}$ is $\tau_b$-complete. Thus, by definition, $(\mathcal{L}(E;F),\tau_b)$ is quasi-complete. ■

***Example* 3.5** Let $E$ be an arbitrary $R$-module endowed with the unique linear $R$-module topology for which $\{\pi^n E;\ n = 1, 2, \ldots\}$ constitutes a fundamental system of neighborhoods of 0. Then $E$ is semimetrizable and $\pi^n U$ is a neighborhood of 0 in $E$ for each integer $n \geqslant 1$ and for each neighborhood $U$ of 0 in $E$. Therefore, by Proposition 2.4, $E$ is bornological, and hence the first assertions of Corollaries 3.2 and 3.3, as well as Corollary 3.4, are valid for $E$.

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