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On Laplacian energy of non-commuting graphs of finite groups

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Abstract. Let G be a finite non-abelian group with center $Z(G)$. The non-commuting graph of G is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy \neq yx$. In this paper, we compute Laplacian energy of the non-commuting graphs of some classes of finite non-abelian groups.

Keywords: Non-commuting graph, L-spectrum, Laplacian energy, finite group.

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1. Introduction

Let $\mathcal{G}$ be a graph. Let $A(\mathcal{G})$ and $D(\mathcal{G})$ denote the adjacency matrix and degree matrix of $\mathcal{G}$ respectively. Then the Laplacian matrix of $\mathcal{G}$ is given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$. Let $\beta_1, \beta_2, \ldots, \beta_m$ be the eigenvalues of $L(\mathcal{G})$ with multiplicities $b_1, b_2, \ldots, b_m$. Then the Laplacian spectrum of $\mathcal{G}$, denoted by Ł-spec$(\mathcal{G})$, is the set $\{\beta_1^{b_1}, \beta_2^{b_2}, \ldots, \beta_m^{b_m}\}$. The Laplacian energy of $\mathcal{G}$, denoted by $LE(\mathcal{G})$, is given by

$$LE(\mathcal{G}) = \sum_{\mu \in \text{Ł-spec}(\mathcal{G})} \left| \mu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|} \right| \tag{1}$$

where $v(\mathcal{G})$ and $e(\mathcal{G})$ are the sets of vertices and edges of $\mathcal{G}$ respectively. It is worth mentioning that the notion of Laplacian energy of a graph was introduced by Gutman

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and Zhou [19]. A graph $\mathcal{G}$ is called L-integral if Ł-spec($\mathcal{G}$) contains only integers. Various properties of L-integral graphs and $LE(\mathcal{G})$ are studied in [2, 21, 23, 31, 32].

Let G be a finite non-abelian group with center $Z(G)$. The non-commuting graph of G, denoted by $\mathcal{A}_G$, is a simple undirected graph such that $v(\mathcal{A}_G) = G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy \neq yx$. Various aspects of non-commuting graphs of different families of finite non-abelian groups are studied in [1, 3, 9, 17, 30]. Note that the complement of $\mathcal{A}_G$ is the commuting graph of G denoted by $\overline{\mathcal{A}}_G$. Commuting graphs of finite groups are studied extensively in [4, 12–14, 20, 24, 27, 28]. In [11], Dutta et al. have computed the Laplacian spectrum of the non-commuting graphs of several well-known families of finite non-abelian groups. In this paper we compute the Laplacian energy of those classes of finite groups. It is worth mentioning that Ghorbani and Gharavi-Alkhansari [18] have computed the energy of non-commuting graphs of the projective special linear group $PSL(2, 2^k)$, where $k \geqslant 2$, the general linear group $GL(2, q)$, where $q = p^n$ (p is a prime and $n \geqslant 4$) and the quasi-dihedral group QD_{2n} recently.

2. Groups with known central factors

In this section, we compute Laplacian energy of some families of finite groups whose central factors are well-known.

Theorem 2.1 Let G be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle$. Then

$$LE(\mathcal{A}_G) = \left(\frac{120}{19}|Z(G)| + 30\right)|Z(G)|.$$

Proof. It is clear that $|v(\mathcal{A}_G)| = 19|Z(G)|$. Since $\frac{G}{Z(G)} \cong Sz(2)$, we have

$$\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^5Z(G) = b^4Z(G) = Z(G), b^{-1}abZ(G) = a^2Z(G)\rangle.$$

Note that for any $z \in Z(G)$,

$$\begin{array}{l} C_G(a) = C_G(az) = Z(G) \sqcup aZ(G) \sqcup a^2Z(G) \sqcup a^3Z(G) \sqcup a^4Z(G), \\ C_G(ab) = C_G(abz) = Z(G) \sqcup abZ(G) \sqcup a^4b^2Z(G) \sqcup a^3b^3Z(G), \\ C_G(a^2b) = C_G(a^2bz) = Z(G) \sqcup a^2bZ(G) \sqcup a^3b^2Z(G) \sqcup ab^3Z(G), \\ C_G(a^2b^3) = C_G(a^2b^3z) = Z(G) \sqcup a^2b^3Z(G) \sqcup ab^2Z(G) \sqcup a^4bZ(G), \\ C_G(b) = C_G(bz) = Z(G) \sqcup bZ(G) \sqcup b^2Z(G) \sqcup b^3Z(G) \quad \text{and} \\ C_G(a^3b) = C_G(a^3bz) = Z(G) \sqcup a^3bZ(G) \sqcup a^2b^2Z(G) \sqcup a^4b^3Z(G) \end{array}$$

are the only centralizers of non-central elements of G. Since all these distinct centralizers are abelian, we have

$$\overline{\mathcal{A}}_G = K_{4|Z(G)|} \sqcup 5K_{3|Z(G)|}$$

and hence $|e(\mathcal{A}_G)| = 150|Z(G)|^2$. By Theorem 3.1 of [11], we have

$$\text{Ł-spec}(\mathcal{A}_G) = \{0, (15|Z(G)|)^{4|Z(G)|-1}, (16|Z(G)|)^{15|Z(G)|-5}, (19|Z(G)|)^5\}.$$

So, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{300}{19}|Z(G)|$, $\left|15|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{15}{19}|Z(G)|$, $\left|16|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{4}{19}|Z(G)|$ and $\left|19|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{61}{19}|Z(G)|$. By (1), we have

$$\begin{aligned} LE(\mathcal{A}_G) = &\frac{300}{19}|Z(G)| + (4|Z(G)| - 1)\left(\frac{15}{19}|Z(G)|\right) \\ &+ (15|Z(G)| - 5)\left(\frac{4}{19}|Z(G)|\right) + 5\left(\frac{61}{19}|Z(G)|\right). \end{aligned}$$

Hence, the result follows. ■

Theorem 2.2 Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then

$$LE(\mathcal{A}_G) = 2p(p-1)|Z(G)|.$$

Proof. It is clear that $|v(\mathcal{A}_G)| = (p^2 - 1)|Z(G)|$. Since $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, we have $\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^p, aba^{-1}b^{-1} \in Z(G)\rangle$, where a, $b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$,

$$\begin{aligned} C_G(a) = C_G(a^i z) = Z(G) \sqcup aZ(G) \sqcup \cdots \sqcup a^{p-1}Z(G) \text{ for } 1 \leqslant i \leqslant p-1 \quad \text{and} \\ C_G(a^j b) = C_G(a^j bz) = Z(G) \sqcup a^j bZ(G) \sqcup \cdots \sqcup a^j b^{p-1}Z(G) \text{ for } 1 \leqslant j \leqslant p \end{aligned}$$

are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Therefore

$$\overline{\mathcal{A}}_G = K_{|C_G(a)\setminus Z(G)|} \sqcup (\bigsqcup_{j=1}^{p} K_{|C_G(a)\setminus Z(G)|}).$$

Since, $|C_G(a)| = |C_G(a^j b)| = p|Z(G)|$ for $1 \leqslant j \leqslant p$, we have $\overline{\mathcal{A}}_G = (p+1)K_{(p-1)|Z(G)|}$ and hence $|e(\mathcal{A}_G)| = \frac{p(p+1)(p-1)^2}{2}|Z(G)|^2$. By Theorem 3.2 of [11], we have

$$\text{Ł-spec}(\mathcal{A}_G) = \{0, ((p^2 - p)|Z(G)|)^{(p^2-1)|Z(G)|-p-1}, ((p^2-1)|Z(G)|)^p\}.$$

Therefore, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = p(p-1)|Z(G)|$, $\left|(p^2 - p)|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = 0$ and $\left|(p^2 - 1)|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = (p-1)|Z(G)|$. By (1), we have

$$LE(\mathcal{A}_G) = p(p-1)|Z(G)| + ((p^2-1)|Z(G)| - p - 1)0 + p((p-1)|Z(G)|).$$

Hence the result follows. ■

Corollary 2.3 Let G be a non-abelian group of order p^3, for any prime p. Then

$$LE(\mathcal{A}_G) = 2p^2(p-1).$$

Proof. Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.2. ■

Theorem 2.4 Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geqslant 2$. Then

$$LE(\mathcal{A}_G) = \frac{(2m^2-3m)(m-1)|Z(G)|^2 + m(4m-3)|Z(G)|}{2m-1}.$$

Proof. Clearly, $|v(\mathcal{A}_G)| = (2m-1)|Z(G)|$. Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G)\rangle$, where $x, y \in G$ with $xy \neq yx$. It is easy to see that for any $z \in Z(G)$,

$$\begin{aligned} &C_G(xy^j) = C_G(xy^jz) = Z(G) \sqcup xy^jZ(G), 1 \leqslant j \leqslant m \quad \text{and} \\ &C_G(y) = C_G(y^iz) = Z(G) \sqcup yZ(G) \sqcup \cdots \sqcup y^{m-1}Z(G), 1 \leqslant i \leqslant m-1 \end{aligned}$$

are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G and $|C_G(xy^j)| = 2|Z(G)|$ for $1 \leqslant j \leqslant m$ and $|C_G(y)| = m|Z(G)|$. Hence

$$\overline{\mathcal{A}}_G = K_{(m-1)|Z(G)|} \sqcup mK_{|Z(G)|}$$

and $|e(\mathcal{A}_G)| = \frac{3m(m-1)|Z(G)|^2}{2}$. By Theorem 3.4 of [11], we have

$$\text{L-spec}(\mathcal{A}_G) = \{0, (m|Z(G)|)^{(m-1)|Z(G)|-1}, (2(m-1)|Z(G)|)^{m|Z(G)|-m}, ((2m-1)|Z(G)|)^m\}.$$

Therefore,

$$\begin{aligned} \left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| &= \frac{3m(m-1)|Z(G)|}{2m-1}, \\ \left|m|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| &= \frac{m(m-1)|Z(G)|}{2m-1}, \\ \left|2(m-1)|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| &= \frac{(m-1)(m-2)|Z(G)|}{2m-1}, \\ \left|(2m-1)|Z(G)| - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| &= \frac{(m^2-m+1)|Z(G)|}{2m-1}. \end{aligned}$$

By (1), we have

$$\begin{aligned} LE(\mathcal{A}_G) =& \frac{3m(m-1)|Z(G)|}{2m-1} + ((m-1)|Z(G)|-1)\left(\frac{m(m-1)|Z(G)|}{2m-1}\right) \\ &+ (m|Z(G)|-m)\left(\frac{(m-1)(m-2)|Z(G)|}{2m-1}\right) + m\left(\frac{(m^2-m+1)|Z(G)|}{2m-1}\right) \end{aligned}$$

and hence, the result follows. ■

Using Theorem 2.4, we now compute the Laplacian energy of the non-commuting graphs of the groups M_{2mn}, D_{2m} and Q_{4n} respectively.

Corollary 2.5 Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1} \rangle$ be a metacyclic group, where $m > 2$. Then

$$LE(\mathcal{A}_{M_{2mn}}) = \begin{cases} \frac{m(2m-3)(m-1)n^2+m(4m-3)n}{2m-1}, & \text{if m is odd} \\ \frac{m(m-2)(m-3)n^2+m(2m-3)n}{m-1}, & \text{if m is even.} \end{cases}$$

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Proof. Observe that $Z(M_{2mn}) = \langle b^2 \rangle$ or $\langle b^2 \rangle \cup a^{\frac{m}{2}} \langle b^2 \rangle$ according as m is odd or even. Also, it is easy to see that $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m according as m is odd or even. Hence, the result follows from Theorem 2.4 ■

Corollary 2.6 Let $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order $2m$, where $m > 2$. Then

$$LE(\mathcal{A}_{D_{2m}}) = \begin{cases} m^2, & \text{if m is odd} \\ \frac{m(m^2-3m+3)}{m-1}, & \text{if m is even.} \end{cases}$$

Corollary 2.7 Let $Q_{4m} = \langle x, y : y^{2m} = 1, x^2 = y^m, yxy^{-1} = y^{-1} \rangle$, where $m \geqslant 2$, be the generalized quaternion group of order $4m$. Then

$$LE(\mathcal{A}_{Q_{4m}}) = \frac{2m(4m^2 - 6m + 3)}{2m - 1}.$$

Proof. The result follows from Theorem 2.4 noting that $Z(Q_{4m}) = \{1, a^m\}$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. ■

3. Some well-known groups

In this section, we compute Laplacian energy of the non-commuting graphs of some well-known families of finite non-abelian groups.

Proposition 3.1 Let G be a non-abelian group of order pq, where p and q are primes with $p \mid (q-1)$. Then

$$LE(\mathcal{A}_G) = \frac{2q(p^2 - 1)(q - 1)}{pq - 1}.$$

Proof. It is clear that $|v(\mathcal{A}_G)| = pq - 1$. Note that $|Z(G)| = 1$ and the centralizers of non-central elements of G are precisely the Sylow subgroups of G. The number of Sylow q-subgroups and Sylow p-subgroups of G are one and q respectively. Therefore, we have

$$\overline{\mathcal{A}_G} = K_{q-1} \sqcup qK_{p-1}$$

and hence $|e(\mathcal{A}_G)| = \frac{q(p^2-1)(q-1)}{2}$. By Proposition 4.1 of [11], we have

$$\text{Ł-spec}(\mathcal{A}_G) = \{0, (pq - q)^{q-2}, (pq - p)^{pq-2q}, (pq - 1)^q\}.$$

So, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{p^2q^2-p^2q-q^2+q}{pq-1}$, $\left|pq - q - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{q(q-p)(p-1)}{pq-1}$, $\left|pq - p - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{(q-p)(q-1)}{pq-1}$ and $\left|pq - 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{p^2q+q^2-2pq-q+1}{pq-1}$. By (1), we have

$$LE(\mathcal{A}_G) = \frac{p^2q^2-p^2q-q^2+q}{pq-1} + (q-2)\left(\frac{q(q-p)(p-1)}{pq-1}\right) + (pq-2q)\left(\frac{(q-p)(q-1)}{pq-1}\right) + q\left(\frac{p^2q+q^2-2pq-q+1}{pq-1}\right)$$

and hence, the result follows. ■

Proposition 3.2 Let QD_{2^n} denotes the quasidihedral group $\langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1}\rangle$, where $n \geqslant 4$. Then

$$LE(\mathcal{A}_{QD_{2^n}}) = \frac{2^{3n-3} - 2^{2n} + 3.2^n}{2^{n-1} - 1}.$$

Proof. It is clear that $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$; and so $|v(\mathcal{A}_{QD_{2^n}})| = 2(2^{n-1} - 1)$. Note that

$$C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a\rangle \text{ for } 1 \leqslant i \leqslant 2^{n-1}-1, i \neq 2^{n-2} \quad \text{and}$$
$$C_{QD_{2^n}}(a^jb) = \{1, a^{2^{n-2}}, a^jb, a^{j+2^{n-2}}b\} \text{ for } 1 \leqslant j \leqslant 2^{n-2}$$

are the only centralizers of non-central elements of QD_{2^n}. Note that these centralizers are abelian subgroups of QD_{2^n}. Therefore, we have

$$\overline{\mathcal{A}}_{QD_{2^n}} = K_{|C_{QD_{2^n}}(a)\setminus Z(QD_{2^n})|} \sqcup \big(\bigsqcup_{j=1}^{2^{n-2}} K_{|C_{QD_{2^n}}(a^jb)\setminus Z(QD_{2^n})|}\big).$$

Since $|C_{QD_{2^n}}(a)| = 2^{n-1}$ and $|C_{QD_{2^n}}(a^jb)| = 4$ for $1 \leqslant j \leqslant 2^{n-2}$, we have $\overline{\mathcal{A}}_{QD_{2^n}} = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$. Hence

$$|e(\mathcal{A}_{QD_{2^n}})| = \frac{3.2^{2n-2} - 6.2^{n-1}}{2}.$$

By Proposition 4.2 of [11], we have

$$\text{Ł-spec}(\mathcal{A}_{QD_{2^n}}) = \{0, (2^{n-1})^{2^{n-1}-3}, (2^n-4)^{2^{n-2}}, (2^n-2)^{2^{n-2}}\}.$$

Therefore, $\left|0 - \frac{2|e(\mathcal{A}_{QD_{2^n}})|}{|v(\mathcal{A}_{QD_{2^n}})|}\right| = \frac{3.2^{n-1}(2^{n-1}-2)}{2.2^{n-1}-2}$, $\left|2^{n-1} - \frac{2|e(\mathcal{A}_{QD_{2^n}})|}{|v(\mathcal{A}_{QD_{2^n}})|}\right| = \frac{2^{2n-2}-4.2^{n-1}}{2.2^{n-1}-2}$, $\left|2^n - 4 - \frac{2|e(\mathcal{A}_{QD_{2^n}})|}{|v(\mathcal{A}_{QD_{2^n}})|}\right| = \frac{2^{2n-2}-6.2^{n-1}+8}{2.2^{n-1}-2}$ and $\left|2^n - 2 - \frac{2|e(\mathcal{A}_{QD_{2^n}})|}{|v(\mathcal{A}_{QD_{2^n}})|}\right| = \frac{2^{2n-2}-2.2^{n-1}+4}{2.2^{n-1}-2}$. By (1), we have

$$LE(\mathcal{A}_{QD_{2^n}}) = \frac{3.2^{n-1}(2^{n-1}-2)}{2.2^{n-1}-2} + (2^{n-1}-3)\left(\frac{2^{2n-2}-4.2^{n-1}}{2.2^{n-1}-2}\right) + 2^{n-2}\left(\frac{2^{2n-2}-6.2^{n-1}+8}{2.2^{n-1}-2}\right) + 2^{n-2}\left(\frac{2^{2n-2}-2.2^{n-1}+4}{2.2^{n-1}-2}\right)$$

and hence, the result follows. ■

Proposition 3.3 Let G denotes the projective special linear group $PSL(2, 2^k)$, where $k \geqslant 2$. Then

$$LE(\mathcal{A}_G) = \frac{3.2^{6k} - 2.2^{5k} - 7.2^{4k} + 2^{3k} + 4.2^{2k} + 2^k}{2^{3k} - 2^k - 1}.$$

Proof. We have $|v(\mathcal{A}_G)| = 2^{3k} - 2^k - 1$, since G is a non-abelian group of order $2^k(2^{2k}-1)$ with trivial center. By Proposition 3.21 of [1], the set of centralizers of non-trivial elements of G is given by

$$\{xPx^{-1}, xAx^{-1}, xBx^{-1} : x \in G\}$$

where P is an elementary abelian 2-subgroup and A, B are cyclic subgroups of G having order $2^k, 2^k - 1$ and $2^k + 1$ respectively. Also the number of conjugates of P, A and B in G are $2^k + 1, 2^{k-1}(2^k + 1)$ and $2^{k-1}(2^k - 1)$ respectively. Hence $\overline{\mathcal{A}_G}$ is given by

$$(2^k + 1)K_{|xPx^{-1}|-1} \sqcup 2^{k-1}(2^k + 1)K_{|xAx^{-1}|-1} \sqcup 2^{k-1}(2^k - 1)K_{|xBx^{-1}|-1}.$$

That is, $\overline{\mathcal{A}}_G = (2^k + 1)K_{2^k-1} \sqcup 2^{k-1}(2^k + 1)K_{2^k-2} \sqcup 2^{k-1}(2^k - 1)K_{2^k}$. Therefore,

$$|e(\mathcal{A}_G)| = \frac{2^{6k} - 3.2^{4k} - 2^{3k} + 2.2^{2k} + 2^k}{2}.$$

By Proposition 4.3 of [11], we have

$$\begin{aligned}\text{Ł-spec}(\mathcal{A}_G) = \{0,&(2^{3k} - 2^{k+1} - 1)^{2^{3k-1} - 2^{2k} + 2^{k-1}}, (2^{3k} - 2^{k+1})^{2^{2k} - 2^k - 2},\\ &(2^{3k} - 2^{k+1} + 1)^{2^{3k-1} - 2^{2k} - 3.2^{k-1}}, (2^{3k} - 2^k - 1)^{2^{2k} + 2^k}\}.\end{aligned}$$

Now, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{2^{6k} - 3.2^{4k} - 2^{3k} + 2.2^{2k} + 2^k}{2^{3k} - 2^k - 1}$, $\left|2^{3k} - 2^{k+1} - 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{2^{3k} - 2.2^k - 1}{2^{3k} - 2^k - 1}$, $\left|2^{3k} - 2^{k+1} - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{2^k}{2^{3k} - 2^k - 1}$, $\left|2^{3k} - 2^{k+1} + 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{2^{3k} - 1}{2^{3k} - 2^k - 1}$ and $\left|2^{3k} - 2^k - 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{2^{4k} - 2^{3k} - 2^{2k} + 2^k + 1}{2^{3k} - 2^k - 1}$. By (1), we have

$$\begin{aligned}LE(\mathcal{A}_G) =& \frac{2^{6k} - 3.2^{4k} - 2^{3k} + 2.2^{2k} + 2^k}{2^{3k} - 2^k - 1} + (2^{3k-1} - 2^{2k} + 2^{k-1})\left(\frac{2^{3k} - 2.2^k - 1}{2^{3k} - 2^k - 1}\right)\\ &+ (2^{2k} - 2^k - 2)\left(\frac{2^k}{2^{3k} - 2^k - 1}\right) + (2^{3k-1} - 2^{2k} - 3.2^{k-1})\left(\frac{2^{3k} - 1}{2^{3k} - 2^k - 1}\right)\\ &+ (2^{2k} + 2^k)\left(\frac{2^{4k} - 2^{3k} - 2^{2k} + 2^k + 1}{2^{3k} - 2^k - 1}\right)\end{aligned}$$

and hence, the result follows. ■

Proposition 3.4 Let G denotes the general linear group $GL(2,q)$, where $q = p^n > 2$ and p is a prime. Then

$$LE(\mathcal{A}_G) = \frac{q^9 - 2q^8 - q^7 + 2q^6 + 2q^5 + q^4 - 4q^3 + 2q^2 + q}{q^4 - q^3 - q^2 + 1}.$$

Proof. We have $|G| = (q^2-1)(q^2-q)$ and $|Z(G)| = q-1$. Therefore, $|v(\mathcal{A}_G)| = q^4 - q^3 - q^2 + 1$. By Proposition 3.26 of [1], the set of centralizers of non-central elements of $GL(2,q)$ is given by

$$\{xDx^{-1}, xIx^{-1}, xPZ(GL(2,q))x^{-1} : x \in GL(2,q)\}$$

where D is the subgroup of $GL(2,q)$ consisting of all diagonal matrices, I is a cyclic subgroup of $GL(2,q)$ having order q^2-1 and P is the Sylow p-subgroup of $GL(2,q)$ consisting of all upper triangular matrices with 1 in the diagonal. The orders of D and $PZ(GL(2,q))$ are $(q-1)^2$ and $q(q-1)$ respectively. Also the number of conjugates of D, I and $PZ(GL(2,q))$ in $GL(2,q)$ are $\frac{q(q+1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively. Hence the commuting graph of $GL(2,q)$ is given by

$$\frac{q(q+1)}{2}K_{|xDx^{-1}|-q+1} \sqcup \frac{q(q-1)}{2}K_{|xIx^{-1}|-q+1} \sqcup (q+1)K_{|xPZ(GL(2,q))x^{-1}|-q+1}.$$

Thus, $\overline{\mathcal{A}}_G = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}$. Hence, $|e(\mathcal{A}_G)| = \frac{q^8-2q^7-2q^6+5q^5+q^4-4q^3+q}{2}$. By Proposition 4.4 of [11], we have

$$\text{L-spec}(\mathcal{A}_G) = \{0, (q^4 - q^3 - 2q^2 + 2q)^{q^3-q^2-2q}, (q^4 - q^3 - 2q^2 + q + 1)^{\frac{q^4-2q^3+q}{2}}, (q^4 - q^3 - 2q^2 + 3q - 1)^{\frac{q^4-2q^3-2q^2+q}{2}}, (q^4 - q^3 - q^2 + 1)^{q^2+q}\}.$$

So, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{q^8-2q^7-2q^6+5q^5+q^4-4q^3+q}{q^4-q^3-q^2+1}$, $\left|q^4 - q^3 - 2q^2 + 2q - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{q^3-2q^2+q}{q^4-q^3-q^2+1}$, $\left|q^4 - q^3 - 2q^2 + q + 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{q^5-2q^4-q^3+3q^2-1}{q^4-q^3-q^2+1}$, $\left|q^4 - q^3 - 2q^2 + 3q - 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{q^5-2q^4+q^3-q^2+2q-1}{q^4-q^3-q^2+1}$ and $\left|q^4 - q^3 - q^2 + 1 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = \frac{q^6-3q^5+2q^4+2q^3-q^2-q+1}{q^4-q^3-q^2+1}$. By (1), we have

$$\begin{aligned} LE(\mathcal{A}_G) = &\frac{q^8 - 2q^7 - 2q^6 + 5q^5 + q^4 - 4q^3 + q}{q^4 - q^3 - q^2 + 1} + (q^3 - q^2 - 2q)\left(\frac{q^3 - 2q^2 + q}{q^4 - q^3 - q^2 + 1}\right) \\ &+ \left(\frac{q^4 - 2q^3 + q}{2}\right)\left(\frac{q^5 - 2q^4 - q^3 + 3q^2 - 1}{q^4 - q^3 - q^2 + 1}\right) \\ &+ \left(\frac{q^4 - 2q^3 - 2q^2 + q}{2}\right)\left(\frac{q^5 - 2q^4 + q^3 - q^2 + 2q - 1}{q^4 - q^3 - q^2 + 1}\right) \end{aligned}$$

and hence, the result follows. ■

Proposition 3.5 Let $F = GF(2^n), n \geqslant 2$ and ϑ be the Frobenius automorphism of F, i.e., $\vartheta(x) = x^2$ for all $x \in F$. If G denotes the group

$$\left\{ U(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}$$

under matrix multiplication given by $U(a,b)U(a',b') = U(a+a', b+b'+a'\vartheta(a))$, then

$$LE(\mathcal{A}_G) = 2^{2n+1} - 2^{n+2}.$$

Proof. Note that $Z(G) = \{U(0,b) : b \in F\}$ and so $|Z(G)| = 2^n$. Therefore, $|v(\mathcal{A}_G)| = 2^n(2^n - 1)$. Let $U(a,b)$ be a non-central element of G. The centralizer of $U(a,b)$ in G is $Z(G) \sqcup U(a,0)Z(G)$. Hence $\overline{\mathcal{A}}_G = (2^n - 1)K_{2^n}$ and $|e(\mathcal{A}_G)| = \frac{2^{4n} - 3.2^{3n} + 2.2^{2n}}{2}$. By Proposition 4.5 of [11], we have

$$\text{Ł-spec}(\mathcal{A}_G) = \{0, (2^{2n} - 2^{n+1})^{(2^n-1)^2}, (2^{2n} - 2^n)^{2^n - 2}\}.$$

Thus, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = 2^{2n} - 2.2^n$, $\left|2^{2n} - 2^{n+1} - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = 0$ and $\left|2^{2n} - 2^n - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = 2^n$. By (1), we have

$$LE(\mathcal{A}_G) = 2^{2n} - 2.2^n + ((2^n - 1)^2)0 + (2^n - 2)2^n$$

and hence, the result follows. ■

Proposition 3.6 Let $F = GF(p^n)$ where p is a prime. If G denotes the group

$$\left\{ V(a,b,c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$, then

$$LE(\mathcal{A}_G) = 2(p^{3n} - p^{2n}).$$

Proof. We have $Z(G) = \{V(0,b,0) : b \in F\}$ and so $|Z(G)| = p^n$. Therefore, $|v(\mathcal{A}_G)| = p^n(p^{2n} - 1)$. The centralizers of non-central elements of $A(n,p)$ are given by

(1) If $b, c \in F$ and $c \neq 0$ then the centralizer of $V(0,b,c)$ in G is $\{V(0,b',c') : b', c' \in F\}$ having order p^{2n}.
(2) If $a, b \in F$ and $a \neq 0$ then the centralizer of $V(a,b,0)$ in G is $\{V(a',b',0) : a', b' \in F\}$ having order p^{2n}.
(3) If $a, b, c \in F$ and $a \neq 0, c \neq 0$ then the centralizer of $V(a,b,c)$ in G is $\{V(a',b',ca'a^{-1}) : a', b' \in F\}$ having order p^{2n}.

It can be seen that all the centralizers of non-central elements of $A(n,p)$ are abelian. Hence,

$$\overline{\mathcal{A}}_G = K_{p^{2n}-p^n} \sqcup K_{p^{2n}-p^n} \sqcup (p^n - 1)K_{p^{2n}-p^n} = (p^n + 1)K_{p^{2n}-p^n}$$

and $|e(\mathcal{A}_G)| = \frac{p^{6n}-p^{5n}-p^{4n}+p^{3n}}{2}$. By Proposition 4.6 of [11], we have

$$\text{Ł-spec}(\mathcal{A}_{A(n,p)}) = \{0, (p^{3n} - p^{2n})^{p^{3n}-2p^n-1}, (p^{3n} - p^n)^{p^n}\}.$$

So, $\left|0 - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = p^{3n} - p^{2n}$, $\left|p^{3n} - p^{2n} - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = 0$ and $\left|p^{3n} - p^n - \frac{2|e(\mathcal{A}_G)|}{|v(\mathcal{A}_G)|}\right| = p^{2n} - p^n$. By (1), we have

$$LE(\mathcal{A}_G) = p^{3n} - p^{2n} + (p^{3n} - 2p^n - 1)0 + p^n(p^{2n} - p^n)$$

and hence, the result follows. ■

4. Some consequences

In this section, we derive some consequences of the results obtained in Section 2 and Section 3. For a finite group G, the set $C_G(x) = \{y \in G : xy = yx\}$ is called the centralizer of an element $x \in G$. Let $|Cent(G)| = |\{C_G(x) : x \in G\}|$, that is the number of distinct centralizers in G. A group G is called an n-centralizer group if $|Cent(G)| = n$. The study of these groups was initiated by Belcastro and Sherman [6] in the year 1994. The readers may conf. [10] for various results on these groups. We begin with computing Laplacian energy of non-commuting graphs of finite n-centralizer groups for some positive integer n. It may be mentioned here that various energies of commuting graphs of finite n-centralizer groups have been computed in [15].

Proposition 4.1 If G is a finite 4-centralizer group, then $LE(\mathcal{A}_G) = 4|Z(G)|$.

Proof. Let G be a finite 4-centralizer group. Then, by Theorem 2 of [6], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.2, the result follows. ■

Further, we have the following result.

Corollary 4.2 If G is a finite $(p+2)$-centralizer p-group for any prime p, then

$$LE(\mathcal{A}_G) = 2p(p-1)|Z(G)|.$$

Proof. Let G be a finite $(p+2)$-centralizer p-group. Then, by Lemma 2.7 of [5], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, by Theorem 2.2, the result follows. ■

Proposition 4.3 If G is a finite 5-centralizer group, then $LE(\mathcal{A}_G) = 12|Z(G)|$ or $\frac{18|Z(G)|^2+27|Z(G)|}{5}$.

Proof. Let G be a finite 5-centralizer group. Then by Theorem 4 of [6], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6. Now, if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then by Theorem 2.2, we have $LE(\mathcal{A}_G) = 12|Z(G)|$. If $\frac{G}{Z(G)} \cong D_6$, then by Theorem 2.4 we have $LE(\mathcal{A}_G) = \frac{18|Z(G)|^2+27|Z(G)|}{5}$. This completes the proof. ■

Let G be a finite group. The commutativity degree of G is given by the ratio

$$\Pr(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

The origin of commutativity degree of a finite group lies in a paper of Erdös and Turán (see [16]). Readers may conf. [7, 8, 25] for various results on $\Pr(G)$. In the following few results we shall compute Laplacian energy of non-commuting graphs of finite non-abelian groups G such that $\Pr(G) = r$ for some rational number r.

Proposition 4.4 Let G be a finite group and p the smallest prime divisor of $|G|$. If $\Pr(G) = \frac{p^2+p-1}{p^3}$, then $LE(\mathcal{A}_G) = 2p(p-1)|Z(G)|$.

Proof. If $\Pr(G) = \frac{p^2+p-1}{p^3}$, then by Theorem 3 of [22], we have $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.2. ■

As a corollary we have the following result.

Corollary 4.5 Let G be a finite group such that $\Pr(G) = \frac{5}{8}$. Then $LE(\mathcal{A}_G) = 4|Z(G)|$.

Proposition 4.6 If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}\}$, then $LE(\mathcal{A}_G) = 9, \frac{28}{3}, 25$ or $\frac{126}{5}$.

Proof. If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}\}$, then as shown in [29, pp. 246] and [26, pp. 451], we have $\frac{G}{Z(G)}$ is isomorphic to one of the groups in $\{D_6, D_8, D_{10}D_{14}\}$. Hence the result follows from Corollary 2.6. ■

Proposition 4.7 Let G be a group isomorphic to any of the following groups

(1) $\mathbb{Z}_2 \times D_8$
(2) $\mathbb{Z}_2 \times Q_8$
(3) $M_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle$
(4) $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$
(5) $D_8 * \mathbb{Z}_4 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb \rangle$
(6) $SG(16, 3) = \langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$.

Then $LE(\mathcal{A}_G) = 16$.

Proof. If G is isomorphic to any of the above listed groups, then $|G| = 16$ and $|Z(G)| = 4$. Therefore, $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus the result follows from Theorem 2.2. ■

Recall that genus of a graph is the smallest non-negative integer n such that the graph can be embedded on the surface obtained by attaching n handles to a sphere. A graph is said to be planar if the genus of the graph is zero. We conclude this paper with the following result.

Theorem 4.8 If the non-commuting graph of a finite non-abelian group G is planar, then

$$LE(\mathcal{A}_G) = \frac{28}{3} \text{ or } 9.$$

Proof. By Theorem 3.1 of [3], we have $G \cong D_6, D_8$ or Q_8. If $G \cong D_8$ or Q_8 then by Corollary 2.6 and Corollary 2.7 it follows that $LE(\mathcal{A}_G) = \frac{28}{3}$. If $G \cong D_6$ then, by Corollary 2.6, $LE(\mathcal{A}_G) = 9$. This completes the proof. ■

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