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Smooth biproximity spaces and P-smooth quasi-proximity spaces

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Abstract. The notion of smooth biproximity space (X, δ_1, δ_2) where δ_1, δ_2 are gradation proximities defined by Ghanim et al. [10]. In this paper, we show every smooth biproximity space (X, δ_1, δ_2) induces a supra smooth proximity space δ_{12} finer than δ_1 and δ_2. We study the relationship between (X, δ_{12}) and the FP^*-separation axioms which had been introduced by Ramadan et al. [23]. Furthermore, we show for each smooth bitopological space which is FP^*T_4, the associated supra smooth topological space is a smooth supra proximal. The notion of FP-(resp. FP^*-) proximity map are also introduced. In addition, we introduce the concept of P-smooth quasi-proximity spaces and prove that the associated smooth bitopological space $(X, \tau_\delta, \tau_{\delta^{-1}})$ satisfies FP-separation axioms in sense of Ramadan et al. [10].

Keywords: Smooth bitopological space, supra smooth proximity, smooth quasi-proximity, FP-separation axioms, compatibility, FP-proximity map.

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1. Introduction

Šostak [26], introduced the fundamental concept of a 'fuzzy topological structure', as an extension of both crisp topology and Chang's fuzzy topology [5], indicating that not only

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the object were fuzzified, but also the axiomatic. Subsequently, Badard [4], introduced the concept of 'smooth topological space'. Chattopadhyay et al. [6] and Chattopadhyay and Samanta [7] re-introduced the same concept, calling it 'gradation of openess'. Ramadan [22] and his colleagues introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have re-introduced and further studied smooth topological space (cf. [6, 7, 9, 27]). Thus, the terms 'fuzzy topology', in Šostak's sense, 'gradation of openness' and 'smooth topology' are essentially referring to the same concept. In our paper, we adopt the term smooth topology. Lee et al. [20] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil's fuzzy bitopological spaces [12].

The so-called supra topology was established by Mashhour et al. [21] (recall that a supra topology on a set X is a collection of subsets of X, which is closed under arbitrary unions). Abd El-Monsef and Ramadan [2] introduced the concept supra fuzzy topology, followed by Ghanim et al. [11] who introduced the supra fuzzy topology in Šostak sense. Abbas [1] generated the supra fuzzy topology (X, τ_{12}) from fuzzy bitopological space (X, τ_1, τ_2) in Šostak sense as an extension of supra fuzzy topology due to Kandil et al. [13].

The concept of proximity space was first described by Frigyes Riesz (1909) but ignored at the time. It was rediscovered and axiomatized by Efremovic under the name of infinitesimal space [8]. Katsaras [15, 16] introduced and studied fuzzy proximity spaces. Samanta [25] introduced the concept of gradations of fuzzy proximity. It was shown that this fuzzy proximity is more general than that of Artico and Moresco [3]. On the other hand, Ghanim et al. [10] introduced gradation proximity spaces with somewhat different definition of Samanta [25]. Kandil et al. [14] introduced the concept of supra fuzzy proximity and fuzzy biproximity spaces. Ghanim et al. [11] introduced the concept of gradation of supra proximity. Kim and Park [18] introduced the concept of fuzzy quasi-proximity spaces in view of definition of Ghanim et al. [10].

In this paper, we consider the gradation proximity in the sense of Ghanim et al. [10]. In Section 2, we give an alternative description of the fuzzy closure operator C_δ, that introduced in [25], by using fuzzy points and the concept of q-coincidence and we study some properties of fuzzy closure operator C_δ. In Section 3, we introduce the notion of smooth biproximity space (X, δ_1, δ_2) and we generate a supra smooth proximity (X, δ_{12}) from (X, δ_1, δ_2). We discuss the supra smooth topological structure $(X, \tau_{\delta_{12}})$ based on this supra smooth proximity. It will be shown that the induced supra smooth topology $(X, \tau_{\delta_{12}})$ are FP^*R_i-space, $i = 0, 1, 2, 3$ and FP^*T_i-space, $i = 0, 1, 2, 3, 4$. Moreover, for each smooth bitopological space which is FP^*T_4, the induced supra smooth topological space is a supra smooth proximal. The notion of FP-(resp. FP^*-) proximity map are also introduced. Finally, in Section 4, we introduce the concept of P-smooth quasi-proximity δ and study its basic properties. We show the associated smooth bts $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPR_i-space $i = 0, 1, 2, 3$ and FPT_i-space $i = 0, 1, 2, 3, 4$ in the sense of Ramadan et al. [23] when δ is separated.

2. Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$, $I_1 = [0, 1)$ and I^X be the family of all fuzzy sets on X. For any $\mu_1, \mu_2 \in I^X$,

$$(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x) : x \in X\},$$

$$(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x) : x \in X\}.$$

For $\lambda \in I^X$, $\bar{1} - \lambda$ denotes the complement of λ. For fixed $\alpha \in I$, $\bar{\alpha}(x) = \alpha\ \forall x \in X$ is a fuzzy constant set on X. By $\bar{0}$ and $\bar{1}$, we denote constant fuzzy sets on X with value 0 and 1, respectively. For $x \in X$ and $t \in I_0$, a fuzzy point $x_t(y)$, , denoted by x_t, which takes t if $x = y$ and 0 otherwise, for all $y \in X$. Let $Pt(X)$ be a family of all fuzzy points in X. The fuzzy point x_t is said to be contained in a fuzzy set λ iff $\lambda(x) \geqslant t$. A fuzzy point x_t is said to be quasi-coincident with a fuzzy set λ, denoted by $x_t\ q\ \lambda$ if and only if $\lambda(x) + t > 1$. In general, for $\mu, \lambda \in I^X$, μ is called quasi-coincident with λ, denoted by $\mu\ q\ \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu\ \bar{q}\ \lambda$. Equivalently, $\mu\ q\ \lambda$ if and only if $\exists\ x_t \in Pt(X);\ x_t \in \mu$ and $x_t\ q\ \lambda$. For $\lambda_1, \lambda_2 \in I^X$, $\lambda_1 \leqslant \lambda_2$ if and only if $\lambda_1\ \bar{q}\ \bar{1} - \lambda_2$. Also, $\lambda_1 \leqslant \lambda_2$ if and only if $(\forall x_t \in Pt(X))\ (x_t\ q\ \lambda_1 \Longrightarrow x_t\ q\ \lambda_2\ )$. FP (resp. FP^*) stand for fuzzy pairwise (resp. fuzzyP^*). The indices are $i, j \in \{1, 2\}$ and $i \neq j$.

Definition 2.1 [4, 6, 22, 26] A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,
(2) $\tau(\mu_1 \wedge \mu_2) \geqslant \tau(\mu_1) \wedge \tau(\mu_2)$, $\forall\ \mu_1, \mu_2 \in I^X$,
(3) $\tau(\bigvee_{i \in J} \mu_i) \geqslant \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-open fuzzy set of X if $\tau(\mu) \geqslant r$, and μ is an r-closed fuzzy set of X if $\tau(\bar{1} - \mu) \geqslant r$. Note, Šostak [26] used the term 'fuzzy topology' and Chattopadhyay [6], the term 'gradation of openness' for a smooth topology τ.

If τ satisfies conditions (1) and (3), then τ is said to be a supra smooth topology and (X, τ) is said to be supra smooth topological space [11].

Definition 2.2 [7] Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure of λ is a mapping $C_\tau : I^X \times I_0 \to I^X$ such that

$$C_\tau(\lambda, r) = \bigwedge\{\rho \in I^X |\ \rho \geqslant \lambda,\ \tau(\bar{1} - \rho) \geqslant r\}. \tag{1}$$

A fuzzy interior of λ is a mapping $I_\tau : I^X \times I_0 \to I^X$ defined as

$$I_\tau(\lambda, r) = \bigvee\{\rho \in I^X |\ \rho \leqslant \lambda,\ \tau(\rho) \geqslant r\}, \tag{2}$$

which satisfies

$$I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r). \tag{3}$$

Definition 2.3 [7] A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda, \mu \in I^X$ and $r, s \in I_0$, the mapping C satisfies the following conditions:

(C1) $C(\bar{0}, r) = \bar{0}$,
(C2) $\lambda \leqslant C(\lambda, r)$,
(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$,
(C4) $C(\lambda, r) \leqslant C(\lambda, s)$ if $r \leqslant s$,
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \longrightarrow I$ given by

$$\tau_C(\lambda) = \bigvee\{r \in I|\ C(\bar{1} - \lambda, r) = \bar{1} - \lambda\}. \tag{4}$$

If C satisfies conditions (C1),(C2),(C4),(C5) and the following inequality:
$(C3)^*$ $\quad C(\lambda, r) \vee C(\mu, r) \leqslant C(\lambda \vee \mu, r)$,
then C is called supra fuzzy closure operator on X [1] and it generates a supra smooth topology $\tau_C : I^X \longrightarrow I$ as (4)

In analogs to Definition 2.3, one can use the equality (3) to obtain the definitions of fuzzy interior operator and supra fuzzy interior operator are obtained.

Definition 2.4 [22] Let (X, τ) and (Y, τ^*) be smooth topological spaces. A mapping $f : (X, \tau) \longrightarrow (Y, \tau^*)$ is called fuzzy continuous if $\tau(f^{-1}(\mu)) \geqslant \tau^*(\mu)$ for all $\mu \in I^Y$.

Another characterization of fuzzy continuous map is given below in terms of a fuzzy closure of a fuzzy set μ

Theorem 2.5 [6] Let (X, τ) and (Y, τ^*) be smooth topological spaces. Then, a mapping $f : (X, \tau) \longrightarrow (Y, \tau^*)$ is fuzzy continuous map iff $f(C_\tau(\mu, r)) \leqslant C_{\tau^*}(f(\mu), r)$, for all $\mu \in I^X$, for all $r \in I_0$.

Definition 2.6 [20] A triple (X, τ_1, τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts, for short).

The following theorem shows how to generate a supra fuzzy closure (resp. interior) operator from a smooth bts (X, τ_1, τ_2).

Theorem 2.7 [1] Let (X, τ_1, τ_2) be a smooth bts.

(1) For each $\lambda \in I^X, r \in I_0$, the mapping $C_{12} : I^X \times I_0 \to I^X$ defined as

$$C_{12}(\lambda, r) = C_{\tau_1}(\lambda, r) \wedge C_{\tau_2}(\lambda, r) \tag{5}$$

is a supra fuzzy closure operator on X which generates a supra smooth topology $\tau_{12} : I^X \longrightarrow I$ defined as in (4).

(2) The mapping $I_{12} : I^X \times I_0 \to I^X$ defined as

$$I_{12}(\lambda, r) = I_{\tau_1}(\lambda, r) \vee I_{\tau_2}(\lambda, r) \tag{6}$$

is a supra fuzzy interior operator on X, satisfies $I_{12}(\bar{1} - \lambda, r) = \bar{1} - C_{12}(\lambda, r)$.

Definition 2.8 [23] A smooth bitopological space (X, τ_1, τ_2) is called:

(1) FPR_0 if and only if $x_t\ \bar{q}\ C_{\tau_i}(y_m, r)$ implies that $y_m\ \bar{q}\ C_{\tau_j}(x_t, r)$.
(2) FPR_1 if and only if $x_t\ \bar{q}\ C_{\tau_i}(y_m, r)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geqslant r$, $\tau_j(\mu) \geqslant r$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda\ \bar{q}\ \mu$.
(3) FPR_2 if and only if $x_t\ \bar{q}\ \rho = C_{\tau_i}(\rho, r)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geqslant r$, $\tau_j(\mu) \geqslant r$ such that $x_t \in \lambda$, $\rho \leqslant \mu$ and $\lambda\ \bar{q}\ \mu$.
(4) FPR_3 if and only if $\rho_1 = C_{\tau_i}(\rho_1, r)\ \bar{q}\ \rho_2 = C_{\tau_j}(\rho_2, r)$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geqslant r$, $\tau_j(\mu) \geqslant r$ such that $\rho_2 \leqslant \lambda$, $\rho_1 \leqslant \mu$ and $\lambda\ \bar{q}\ \mu$.
(5) FPT_0 if and only if $x_t\ \bar{q}\ y_m$ implies that there exist $\lambda \in I^X$ such that for $i = 1$ or 2 $\tau_i(\lambda) \geqslant r$ and $x_t \in \lambda$, $y_m\ \bar{q}\ \lambda$ or $y_m \in \lambda$, $x_t\ \bar{q}\ \lambda$.
(6) FPT_1 if and only if $x_t\ \bar{q}\ y_m$ implies that there exist $\lambda \in I^X$ such that for $i = 1$ or 2 $\tau_i(\lambda) \geqslant r$, $x_t \in \lambda$ and $y_m\ \bar{q}\ \lambda$.

(7) FPT_2 if and only if $x_t \bar{q} y_m$ implies that there exist $\lambda, \mu \in I^X$ with $\tau_i(\lambda) \geqslant r$, $\tau_j(\mu) \geqslant r$ such that $x_t \in \lambda$, $y_m \in \mu$ and $\lambda \bar{q} \mu$.
(8) FPT_3 if and only if it is FPR_2 and FPT_1.
(9) FPT_4 if and only if it is FPR_3 and FPT_1.

Definition 2.9 [23] A smooth bitopological space (X, τ_1, τ_2) is called:

(1) FP^*R_i if and only if its associated supra smooth topological space (X, τ_{12}) is FR_i, $i = 0, 1, 2$.
(2) FP^*T_i if and only if its associated supra smooth topological space (X, τ_{12}) is FT_i, $i = 0, 1, 2, 3, 4$.

Lemma 2.10 [23] Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$. Then $x_t \, q \, C_{12}(\lambda, r)$ if and only if $\lambda \, q \, \mu$ for all $\mu \in I^X$ with $\tau_{12}(\mu) \geqslant r$ and $x_t \in \mu$.

Definition 2.11 [10] A mapping $\delta : I^X \times I^X \longrightarrow I$ is said to be a gradation of proximity on X if it satisfies the following axioms:
(FP1) $\delta(\mu, \rho) = \delta(\rho, \mu)$.
(FP2) $\delta(\mu \vee \rho, \lambda) = \delta(\mu, \lambda) \vee \delta(\rho, \lambda)$.
(FP3) $\delta(\bar{1}, \bar{0}) = 0$.
(FP4) $\delta(\mu, \rho) < r$ implies there exists $\eta \in I^X$ such that $\delta(\mu, \eta) < r$ and $\delta(\bar{1} - \eta, \rho) < r$.
(FP5) $\delta(\mu, \rho) \neq 1$ implies $\mu \, \bar{q} \, \rho$.
The pair (X, δ) is called a fuzzy proximity space.
If δ satisfies conditions (FP1),(FP3),(FP4),(FP5) and the following axiom:
(FP2*) $\delta(\mu, \lambda) \vee \delta(\rho, \lambda) \leqslant \delta(\mu \vee \rho, \lambda)$,
then δ is called a gradation of supra proximity on X and the pair (X, δ) is called a supra fuzzy proximity space [11].

In this paper the gradation of proximity δ on X refers to as smooth proximity on X and the fuzzy proximity space (X, δ) refers to as smooth proximity space. Also, the gradation of supra proximity on X and the supra fuzzy proximity space (X, δ) refers to as supra smooth proximity on X and supra smooth proximity space, respectively.

Lemma 2.12 [10] Let (X, δ) be a smooth proximity space, $\lambda, \mu \in I^X$ and $r \in I_0$. If $\delta(\mu, \lambda) \geqslant r$, $\mu \leqslant \mu_1$ and $\lambda \leqslant \lambda_1$, then $\delta(\mu_1, \lambda_1) \geqslant r$.

In this paper we adopt Samanta's definition for the fuzzy closure C_δ, because this definition satisfies condition $(C4)$ of the fuzzy closure operator in Definition 2.3.

Theorem 2.13 [25] Let (X, δ) be a smooth proximity space. Then the mapping $C_\delta(\lambda, r)$: $I^X \times I_1 \longrightarrow I^X$ given by

$$C_\delta(\lambda, r) = \bar{1} - \bigvee \{\rho \in I^X \mid \ \rho \leqslant \bar{1} - \lambda, \ \delta(\rho, \lambda) < 1 - r\}, \tag{7}$$

is a fuzzy closure operator. Furthermore, C_δ generates a smooth topology $\tau_\delta : I^X \longrightarrow I$ as in (4).

Next, we recall the definition of fuzzy quasi-proximity spaces.

Definition 2.14 [18] A mapping $\delta : I^X \times I^X \longrightarrow I$ is said to be a fuzzy quasi-proximity on X if it satisfies the following axioms:
(FQP1) $\delta(\bar{1}, \bar{0}) = 0$ and $\delta(\bar{0}, \bar{1}) = 0$.
(FQP2) $\delta(\mu \vee \rho, \lambda) = \delta(\mu, \lambda) \vee \delta(\rho, \lambda)$ and $\delta(\mu, \rho \vee \nu) = \delta(\mu, \rho) \vee \delta(\mu, \nu)$.
(FQP3) If $\delta(\mu, \rho) < r$, then there exists $\eta \in I^X$ such that $\delta(\mu, \eta) < r$ and $\delta(\bar{1} - \eta, \rho) < r$.
(FQP4) If $\delta(\mu, \rho) \neq 1$, then $\mu \, \bar{q} \, \rho$.

The pair (X, δ) is called a fuzzy quasi-proximity space.

In this paper the fuzzy quasi-proximity on X refers to as smooth quasi-proximity on X and the fuzzy quasi-proximity space (X, δ) refers to as smooth quasi-proximity space.

Remark 1 *[18]*

(1) A smooth quasi-proximity space (X, δ) is called a smooth proximity space if δ satisfies: (FP1) $\delta(\mu, \rho) = \delta(\rho, \mu)$ for any $\mu, \rho \in I^X$.

(2) If (X, δ) is a smooth quasi-proximity space and $\lambda \leqslant \mu$, then, by (FQP2), we have $\delta(\lambda, \nu) \leqslant \delta(\mu, \nu)$ and $\delta(\rho, \lambda) \leqslant \delta(\rho, \mu)$ for any $\nu, \rho \in I^X$.

(3) If (X, δ) is a smooth quasi-proximity space. Then, $\delta^{-1} : I^X \times I^X \longrightarrow I$ defined by $\delta^{-1}(\mu, \rho) = \delta(\rho, \mu)$ for any $\mu, \rho \in I^X$, is also a smooth quasi-proximity and its called the conjugate of δ.

Definition 2.15 [17] Let (X, δ_1) and (X, δ_2) be smooth (resp. quasi-) proximity spaces. We say that δ_2 is finer than δ_1 (or δ_1 is coarser than δ_2) if and only if for any $\mu, \rho \in I^X$, $\delta_2(\mu, \rho) \leqslant \delta_1(\mu, \rho)$.

3. An alternative description of C_δ

In this section we give an alternative description of the fuzzy closure operator C_δ defined in (7), by using fuzzy points and the concept of q-coincidence. This alternative description will be used it in the next parts of the paper.

Theorem 3.1 Let (X, δ) be a smooth proximity space, $x_t \in Pt(X)$, $\lambda \in I^X$ and $r \in I_1$. Then, an operator $C_\delta : I^X \times I_1 \longrightarrow I^X$ defined as

$$x_t \; q \; C_\delta(\lambda, r) \text{ if and only if } \delta(x_t, \lambda) \geqslant 1 - r. \tag{8}$$

is a fuzzy closure operator on X.

Proof. We apply Definition 2.3 as follows

(C1) Since $\delta(\bar{1}, \bar{0}) = 0$, then from Lemma 2.12 we get $\delta(x_t, \bar{0}) = 0$ for all $x_t \in Pt(X)$, it follows that $x_t \; \bar{q} \; C_\delta(\bar{0}, r)$ for all $x_t \in Pt(X)$ and for all $r \in I_1$, thus $C_\delta(\bar{0}, r) = \bar{0}$.

(C2) Let $x_t \; q \; \lambda$. Then by (FP5) axiom we get $\delta(x_t, \lambda) = 1 \geqslant 1 - r$ for all $r \in I_1$ this implies $x_t \; q \; C_\delta(\lambda, r)$. Hence $\lambda \leqslant C_\delta(\lambda, r)$.

(C3)

$$\begin{aligned} x_t \; q \; C_\delta(\lambda \vee \mu, r) &\Longleftrightarrow \delta(x_t, \lambda \vee \mu) \geqslant 1 - r \\ &\Longleftrightarrow \delta(x_t, \lambda) \geqslant 1 - r \quad \vee \quad \delta(x_t, \mu) \geqslant 1 - r \\ &\Longleftrightarrow x_t \; q \; C_\delta(\lambda, r) \quad \vee \quad x_t \; q \; C_\delta(\mu, r) \\ &\Longleftrightarrow x_t \; q \; [C_\delta(\lambda, r) \vee C_\delta(\mu, r)]. \end{aligned}$$

Hence, $C_\delta(\lambda \vee \mu, r) = C_\delta(\lambda, r) \vee C_\delta(\mu, r)$.

(C4) Let $r, s \in I_1$ such that $r \leqslant s$ and $x_t \; q \; C_\delta(\lambda, r)$. Then, $\delta(x_t, \lambda) \geqslant 1 - r \geqslant 1 - s$ and this means $x_t \; q \; C_\delta(\lambda, s)$. Hence, if $r \leqslant s$, we have $C_\delta(\lambda, r) \leqslant C_\delta(\lambda, s)$.

(C5) Let $x_t \; q \; C_\delta(C_\delta(\lambda, r), r)$ and $x_t \; \bar{q} \; C_\delta(\lambda, r)$. Then, $\delta(x_t, \lambda) < 1 - r$. By (FP4) axiom there exists $\eta \in I^X$ such that $\delta(x_t, \eta) < 1 - r$ and $\delta(\bar{1} - \eta, \lambda) < 1 - r$. Since $\delta(\bar{1} - \eta, \lambda) < 1 - r$, then from Lemma 2.12, $\delta(y_m, \lambda) < 1 - r$ for all $y_m \in \bar{1} - \eta$

and therefore $y_m \bar{q}\ C_\delta(\lambda, r)$ for all $y_m \in \bar{1} - \eta$, this means, $\bar{1} - \eta\ \bar{q}\ C_\delta(\lambda, r)$. So, $C_\delta(\lambda, r) \leqslant \eta$. Since $\delta(x_t, \lambda) < 1 - r$ and $C_\delta(\lambda, r) \leqslant \eta$, then from Lemma 2.12, we have $\delta(x_t, C_\delta(\lambda, r)) < 1 - r$. Hence, $x_t\ \bar{q}\ C_\delta(C_\delta(\lambda, r), r)$ which is a contradiction. The other inclusion follows directly from (C2). Hence $C_\delta(C_\delta(\lambda, r), r) = C_\delta(\lambda, r)$. Thus, using Definition 2.3, C_δ is a fuzzy closure operator.

■

Corollary 3.2 Let (X, δ) be a smooth proximity space, $\rho, \lambda \in I^X$ and $r \in I_1$. If $\rho\ q\ C_\delta(\lambda, r)$, then $\delta(\rho, \lambda) \geqslant 1 - r$.

Proof. Let $\rho\ q\ C_\delta(\lambda, r)$. Then, there exists $x_t \in \rho$ such that $x_t\ q\ C_\delta(\lambda, r)$, this implies, $\delta(x_t, \lambda) \geqslant 1 - r$ and from Lemma 2.12 it follows $\delta(\rho, \lambda) \geqslant 1 - r$. ■

Corollary 3.3 Let (X, δ) be a smooth proximity space, for $\rho, \lambda \in I^X$ and $r \in I_1$. If $\delta(\bar{1} - \rho, \lambda) < 1 - r$, then $C_\delta(\lambda, r) \leqslant \rho$.

Corollary 3.4 Let (X, δ) be a smooth proximity space, $\rho, \lambda \in I^X$ and $r \in I_1$. Then

$$C_\delta(\lambda, r) = \bigwedge \{\rho \in I^X \mid \delta(\bar{1} - \rho, \lambda) < 1 - r\}. \tag{9}$$

Proof. From Corollary 3.2 we conclude that $\delta(\bar{1} - \rho, \lambda) < 1 - r$ implies $\bar{1} - \rho\ \bar{q}\ C_\delta(\lambda, r)$ which means $C_\delta(\lambda, r) \leqslant \rho$ and hence

$$C_\delta(\lambda, r) \leqslant \bigwedge \{\rho \in I^X \mid \delta(\bar{1} - \rho, \lambda) < 1 - r\}.$$

To show the reverse inclusion, suppose $x_t\ \bar{q}\ C_\delta(\lambda, r)$. Then $\delta(x_t, \lambda) = \delta(\bar{1} - (\bar{1} - x_t), \lambda) < 1 - r$. Since $x_t\ \bar{q}\ \bar{1} - x_t$, then $x_t\ \bar{q}\ \bigwedge \{\rho \in I^X \mid \delta(\bar{1} - \rho, \lambda) < 1 - r\}$. Hence

$$\bigwedge \{\rho \in I^X \mid \delta(\bar{1} - \rho, \lambda) < 1 - r\} \leqslant C_\delta(\lambda, r).$$

■

Clearly C_δ given in (9) is coincide with that introduced by Samanta (7).

Proposition 3.5 Let (X, δ) be a smooth proximity space, for $\lambda, \rho \in I^X$ and $r \in I_1$ we have the following:
(1) If $\lambda \leqslant \rho$, then $C_\delta(\lambda, r) \leqslant C_\delta(\rho, r)$.
(2) $\delta(\lambda, \rho) = \delta(C_\delta(\lambda, r), C_\delta(\rho, r))$.

Proof.

(1) Let $\lambda \leqslant \rho$ and $x_t\ q\ C_\delta(\lambda, r)$. Then $\delta(x_t, \lambda) \geqslant 1 - r$. Since $\lambda \leqslant \rho$, then from Lemma 2.12, we get $\delta(x_t, \rho) \geqslant 1 - r$ this implies $x_t\ q\ C_\delta(\rho, r)$. Hence $C_\delta(\lambda, r) \leqslant C_\delta(\rho, r)$.

(2) Necessity follows directly from Theorem 2.3(C2) and Lemma 2.12.
Conversely, let $\delta(C_\delta(\lambda, r), C_\delta(\rho, r)) \geqslant 1 - r$ and suppose that $\delta(\lambda, \rho) < 1 - r$. Then by (FP4) axiom, there exists $\eta \in I^X$ such that $\delta(\lambda, \eta) < 1 - r$ and $\delta(\bar{1} - \eta, \rho) < 1 - r$. Since $\delta(\bar{1} - \eta, \rho) < 1 - r$, then from Corollary 3.3, $C_\delta(\rho, r) \leqslant \eta$ and from Lemma 2.12, $\delta(\lambda, \eta) < 1 - r$ implies $\delta(\lambda, C_\delta(\rho, r)) < 1 - r$. It then follows by apply (FP1) and (FP4) axioms on $\delta(\lambda, C_\delta(\rho, r)) < 1 - r$, we have $\delta(C_\delta(\lambda, r), C_\delta(\rho, r)) < 1 - r$ which a contradicts. Hence $\delta(\lambda, \rho) = \delta(C_\delta(\lambda, r), C_\delta(\rho, r))$.

■

Theorem 3.6 Let (X, δ) be a smooth proximity space and $\lambda \in I^X$. Then

(1) λ is an r-open fuzzy set in (X, τ_δ) if and only if $\forall x_t \in \lambda$, $\delta(x_t, \bar{1} - \lambda) < 1 - r$.
(2) λ is an r-closed fuzzy set in (X, τ_δ) if and only if $\delta(x_t, \lambda) \geqslant 1 - r$ implies $x_t \ q \ \lambda$.

Proof.

(1) Necessity, let λ be an r-open fuzzy set in (X, τ_δ), (i.e $C_\delta(\bar{1} - \lambda, r) = \bar{1} - \lambda$) and $x_t \in \lambda$. Suppose $\delta(x_t, \bar{1} - \lambda) \geqslant 1 - r$. Then $x_t \ q \ C_\delta(\bar{1} - \lambda, r) = \bar{1} - \lambda$ and this means $\lambda(x) < t$ which is a contradiction. Hence, $\forall x_t \in \lambda$ we have $\delta(x_t, \bar{1} - \lambda) < 1 - r$.
   Conversely, to prove λ is an r-open fuzzy set in (X, τ_δ) we must show $C_\delta(\bar{1} - \lambda, r) = \bar{1} - \lambda$. Suppose that there exists $x_t \in Pt(X)$ such that $x_t \ q \ C_\delta(\bar{1} - \lambda, r)$ and $x_t \ \bar{q} \ \bar{1} - \lambda$. It follows $x_t \in \lambda$ and $\delta(x_t, \bar{1} - \lambda) < 1 - r$. From (8) we have, $x_t \ \bar{q} \ C_\delta(\bar{1} - \lambda, r)$ which is a contradiction. Thus, $x_t \ q \ \bar{1} - \lambda$ and consequently, $C_\delta(\bar{1} - \lambda, r) \leqslant \bar{1} - \lambda$. The other inclusion follows directly from Definition 2.3(C2). So, $C_\delta(\bar{1} - \lambda, r) = \bar{1} - \lambda$ and hence λ is an r-open fuzzy set in (X, τ_δ).
(2) Necessity follows directly from hypothesis and (8).
   Conversely, let $x_t \ q \ C_\delta(\lambda, r)$. Then $\delta(x_t, \lambda) \geqslant 1 - r$ implies $x_t \ q \ \lambda$. Thus $C_\delta(\lambda, r) \leqslant \lambda$. The other inclusion follows directly from Definition 2.3(C2). Hence the required result.

■

4. δ_{12}-supra smooth proximity spaces

In this section we introduced the definition of smooth biproximity space (X, δ_1, δ_2). We prove that for every given smooth biproximity space (X, δ_1, δ_2) there is associated supra smooth proximity space (X, δ_{12}). We also discuss the supra smooth topological structure based on this supra smooth proximity.

Definition 4.1 A triple (X, δ_1, δ_2) is called a smooth biproximity space, where δ_1 and δ_2 are smooth proximities on X.

Now we generate a supra smooth proximity from a smooth biproximity space.

Theorem 4.2 Let (X, δ_1, δ_2) be a smooth biproximity space, $\mu, \rho \in I^X$. Then, the mapping $\delta_{12} : I^X \times I^X \longrightarrow I$ defined as

$$\delta_{12}(\mu, \rho) = \delta_1(\mu, \rho) \wedge \delta_2(\mu, \rho). \tag{10}$$

defines a supra smooth proximity on X, the space (X, δ_{12}) is called the associated supra smooth proximity space.

Proof. To prove that δ_{12} is a supra smooth proximity on X, we must show δ_{12} satisfies (FP1), ($FP2^*$), (FP3), (FP4) and (FP5) axioms in Definition 2.11. Since δ_1 and δ_2 are smooth proximities on X, then from Definition 2.11 the proof comes directly. ■

Theorem 4.3 Let (X, δ_1, δ_2) be a smooth biproximity space and (X, δ_{12}) its associated supra smooth proximity space. Then

(1) The operator $C_{\delta_{12}} : I^X \times I_1 \longrightarrow I^X$ defined by

$$C_{\delta_{12}}(\lambda, r) = C_{\delta_1}(\lambda, r) \wedge C_{\delta_2}(\lambda, r) \tag{11}$$

for all $\lambda \in I^X$ and for all $r \in I_1$, is a supra fuzzy closure operator.

(2) The mapping $\tau_{\delta_{12}} : I^X \longrightarrow I$ defined by

$$\tau_{\delta_{12}}(\lambda) = \bigvee\{r \in I \mid \ C_{\delta_{12}}(\bar{1} - \lambda, r) = \bar{1} - \lambda\} \tag{12}$$

for all $\lambda \in I^X$, is a supra smooth topology on X such that $\tau_{\delta_i} \leqslant \tau_{\delta_{12}}$, $i = 1, 2$.

Proof.

(1) We only prove condition (C5), the other conditions are deduced from (11) and (5).

(C5) Suppose $C_{\delta_{12}}(C_{\delta_{12}}(\lambda, r), r) \nleqslant C_{\delta_{12}}(\lambda, r)$. Then, there exist $x \in X$ and $t \in I_0$ such that

$$C_{\delta_{12}}(C_{\delta_{12}}(\lambda, r), r)(x) > t > C_{\delta_{12}}(\lambda, r)(x). \tag{13}$$

Since $C_{\delta_{12}}(\lambda, r)(x) < t$, then by (11), $C_{\delta_1}(\lambda, r)(x) < t$ or $C_{\delta_2}(\lambda, r)(x) < t$. Suppose $C_{\delta_1}(\lambda, r)(x) < t$. Then, there exists $\rho \in I^X$ with $\delta_1(\bar{1} - \rho, \lambda) < 1 - r$ such that $\rho(x) < t$. Since $\delta_1(\bar{1} - \rho, \lambda) < 1 - r$, then by (FP4) axiom there exists $\eta \in I^X$ such that $\delta_1(\bar{1} - \rho, \eta) < 1 - r$ and $\delta_1(\bar{1} - \eta, \lambda) < 1 - r$. From Corollary 3.3, $\delta_1(\bar{1} - \eta, \lambda) < 1 - r$ which implies $C_{\delta_1}(\lambda, r) \leqslant \eta$. From Lemma 2.12, we have $\delta_1(\bar{1} - \rho, C_{\delta_1}(\lambda, r)) < 1 - r$ which means $C_{\delta_1}(C_{\delta_1}(\lambda, r), r) \leqslant \rho$. The latter inequality implies $C_{\delta_1}(C_{\delta_1}(\lambda, r), r)(x) \leqslant \rho(x) < t$. This contradicts (13). Thus, $C_{\delta_{12}}(C_{\delta_{12}}(\lambda, r), r) \leqslant C_{\delta_{12}}(\lambda, r)$. Similarly. if $C_{\delta_2}(\lambda, r)(x) < t$ we get $C_{\delta_{12}}(C_{\delta_{12}}(\lambda, r), r) \leqslant C_{\delta_{12}}(\lambda, r)$. The other inclusion follows directly from Definition 2.3(C2). Hence, $C_{\delta_{12}}(C_{\delta_{12}}(\lambda, r), r) = C_{\delta_{12}}(\lambda, r)$

(2) From (1) and Definition 2.3, it follows $\tau_{\delta_{12}}$ is a supra smooth topology on X. Clearly from (11) we have $C_{\delta_{12}}(\lambda, r) \leqslant C_{\delta_i}(\lambda, r)$ for all $\lambda \in I^X$, $r \in I_1$ and $i = 1, 2$, then it follows $\tau_{\delta_i}(\lambda) \leqslant \tau_{\delta_{12}}(\lambda)$.

■

Proposition 4.4 Let (X, δ_1, δ_2) be a smooth biproximity space, (X, δ_{12}) its associated supra smooth proximity space and $C_{\delta_{12}}$ be a supra fuzzy closure operator. Then

(1) $\delta_{12}(\bar{1} - \rho, \lambda) < 1 - r$ implies $C_{\delta_{12}}(\lambda, r) \leqslant \rho$.
(2) $x_t \; q \; C_{\delta_{12}}(\lambda, r)$ if and only if $\delta_{12}(x_t, \lambda) \geqslant 1 - r$.
(3) $\delta_{12}(\lambda, \rho) = \delta_{12}(C_{\delta_{12}}(\lambda, r), C_{\delta_{12}}(\rho, r))$.

Proof. The results follows directly from (10), (11), Corollary 3.3, Theorem 3.1, Lemma 2.12 and Proposition 3.5(2).

■

Definition 4.5 A smooth proximity δ is said to be separated if and only if

$$x_t \; \bar{q} \; y_m \; \text{ implies } \; \delta(x_t, y_m) \neq 1 \tag{14}$$

for all $x_t, y_m \in Pt(X)$.

Proposition 4.6 Let (X, δ_1, δ_2) be a smooth biproximity space and (X, δ_{12}) its associated supra smooth proximity space. Then

$$\delta_{12} \; \text{ is separated } \iff \delta_1 \text{ or } \delta_2 \text{ is separated.} \tag{15}$$

Proof. Straightforward. ■

Theorem 4.7 Let (X, δ_1, δ_2) be a smooth biproximity space and (X, δ_{12}) its associated supra smooth proximity space. Then, $(X, \tau_{\delta_{12}})$ is FP^*R_i-space, $i = 0, 1, 2, 3$.

Proof.

(0) Let $x_t \; \bar{q} \; C_{\delta_{12}}(y_m, r)$, $r \in I_1$. Then, $\delta_{12}(x_t, y_m) < 1 - r$ and from $(FP1)$ axiom we get, $\delta_{12}(y_m, x_t) < 1 - r$. This implies $y_m \; \bar{q} \; C_{\delta_{12}}(x_t, r)$ and hence $(X, \tau_{\delta_{12}})$ is FP^*R_0.

(1) Let $x_t \; \bar{q} \; C_{\delta_{12}}(y_m, r)$, $r \in I_1$. Then, $\delta_{12}(x_t, y_m) < 1 - r$. From $(FP4)$ axiom there exists $\rho \in I^X$ such that $\delta_{12}(x_t, \rho) < 1 - r$ and $\delta_{12}(\bar{1} - \rho, y_m) < 1 - r$. This implies $x_t \; \bar{q} \; C_{\delta_{12}}(\rho, r)$ and $y_m \; \bar{q} \; C_{\delta_{12}}(\bar{1} - \rho, r)$ which means $x_t \in \bar{1} - C_{\delta_{12}}(\rho, r) = I_{\delta_{12}}(\bar{1} - \rho, r)$, $y_m \in \bar{1} - C_{\delta_{12}}(\bar{1} - \rho, r) = I_{\delta_{12}}(\rho, r)$ and $I_{\delta_{12}}(\bar{1} - \rho, r) \; \bar{q} \; I_{\delta_{12}}(\rho, r)$. Hence $(X, \tau_{\delta_{12}})$ is FP^*R_1.

(2) Let $x_t \; \bar{q} \; \rho = C_{\delta_{12}}(\rho, r)$, $r \in I_1$. Then, $\delta_{12}(x_t, \rho) < 1 - r$ and from $(FP4)$ axiom there exists $\eta \in I^X$ such that $\delta_{12}(x_t, \eta) < 1 - r$ and $\delta_{12}(\bar{1} - \eta, \rho) < 1 - r$. This implies $x_t \; \bar{q} \; C_{\delta_{12}}(\eta, r)$ and $\rho \; \bar{q} \; C_{\delta_{12}}(\bar{1} - \eta, r)$ which means $x_t \in \bar{1} - C_{\delta_{12}}(\eta, r) = I_{\delta_{12}}(\bar{1} - \eta, r)$, $\rho \leqslant \bar{1} - C_{\delta_{12}}(\bar{1} - \eta, r) = I_{\delta_{12}}(\eta, r)$ and $I_{\delta_{12}}(\bar{1} - \eta, r) \; \bar{q} \; I_{\delta_{12}}(\eta, r)$. Hence $(X, \tau_{\delta_{12}})$ is FP^*R_2.

(3) The proof is similar to (2).

■

Theorem 4.8 Let (X, δ_1, δ_2) be a separated smooth biproximity space and (X, δ_{12}) its associated supra smooth proximity space. Then, $(X, \tau_{\delta_{12}})$ is FP^*T_i-space, $i = 0, 1, 2, 3, 4$.

Proof.

(0) Let $x_t \; \bar{q} \; y_m$. Since δ_{12} is separated, then $\delta_{12}(x_t, y_m) \neq 1$. Then, there exists $r \in I_1$ such that $\delta_{12}(x_t, y_m) < 1 - r$ and so it follows $y_m \; \bar{q} \; C_{\delta_{12}}(x_t, r)$. Since $x_t \in I_{\delta_{12}}(x_t, r) \leqslant C_{\delta_{12}}(x_t, r)$, then there exists $\lambda = I_{\delta_{12}}(x_t, r)$ such that $x_t \in \lambda$, $y_m \; \bar{q} \; \lambda$. Hence $(X, \tau_{\delta_{12}})$ is FP^*T_0.

(1) The proof is similar to part (0).

(2) Let $x_t \; \bar{q} \; y_m$. Since δ_{12} is separated, then $\delta_{12}(x_t, y_m) \neq 1$. This is implies that there exists $r \in I_1$ such that $\delta_{12}(x_t, y_m) < 1 - r$ from $(FP4)$ axiom there exits $\eta \in I^X$ such that $\delta_{12}(x_t, \eta) < 1 - r$ and $\delta_{12}(\bar{1} - \eta, y_m) < 1 - r$. This implies $x_t \; \bar{q} \; C_{\delta_{12}}(\eta, r)$ and $y_m \; \bar{q} \; C_{\delta_{12}}(\bar{1} - \eta, r)$ which yields, $x_t \in \bar{1} - C_{\delta_{12}}(\eta, r) = I_{\delta_{12}}(\bar{1} - \eta, r)$, $y_m \in \bar{1} - C_{\delta_{12}}(\bar{1} - \eta, r) = I_{\delta_{12}}(\eta, r)$ and $I_{\delta_{12}}(\bar{1} - \eta, r) \; \bar{q} \; I_{\delta_{12}}(\eta, r)$. Hence $(X, \tau_{\delta_{12}})$ is FP^*T_2.

(3) The results follows directly from Theorem 4.7(2) and part (1).

(4) The results follows directly from Theorem 4.7(3) and part (1).

■

In the following we shall give another description of the FP^*T_1 and FP^*R_3 spaces.

Theorem 4.9 A smooth bts (X, τ_1, τ_2) is a FP^*T_1 if and only $C_{12}(x_t, r) = x_t$, for each $x_t \in Pt(X)$, $r \in I_0$.

Proof. Necessity, let (X, τ_1, τ_2) be a FP^*T_1 to show that, $C_{12}(x_t, r) = x_t$, for each $x_t \in Pt(X)$, $r \in I_0$, let $y_m \; \bar{q} \; x_t$, then by FP^*T_1, there exists $\lambda \in I^X$, $\tau_{12}(\lambda) \geqslant r$ such that $y_m \in \lambda$ and $x_t \; \bar{q} \; \lambda$. From Lemma 2.10, $y_m \; \bar{q} \; C_{12}(x_t, r)$. Thus $C_{12}(x_t, r) \leqslant x_t$. On the other hand $x_t \leqslant C_{12}(x_t, r)$. Hence, $C_{12}(x_t, r) = x_t$.

Conversely, let $x_t \; \bar{q} \; y_m$. Then $x_t \; \bar{q} \; C_{12}(y_m, r)$ and by Lemma 2.10, there exists $\lambda \in I^X$, $\tau_{12}(\lambda) \geqslant r$ such that $x_t \in \lambda$ and $\lambda \; \bar{q} \; y_m$. Hence (X, τ_1, τ_2) is a FP^*T_1. ■

Theorem 4.10 A smooth bts (X, τ_1, τ_2) is FP^*R_3 if and only if for each $\rho, \lambda \in I^X$, $C_{12}(\rho, r) = \rho$ and $\tau_{12}(\lambda) \geqslant r$, $r \in I_0$ such that $\rho \leqslant \lambda$, there exists $\nu \in I^X$, $\tau_{12}(\nu) \geqslant r$ such that $\rho \leqslant \nu \leqslant C_{12}(\nu, r) \leqslant \lambda$.

Proof. First, suppose (X, τ_1, τ_2) is FP^*R_3 and let $\rho, \lambda \in I^X$ such that $C_{12}(\rho, r) = \rho$, $\tau_{12}(\lambda) \geqslant r$ and $\rho \leqslant \lambda$. Then, $\rho\ \bar{q}\ \bar{1} - \lambda$, so by FP^*R_3, there exists $\mu_1, \mu_2 \in I^X$ with $\tau_{12}(\mu_1) \geqslant r$ and $\tau_{12}(\mu_2) \geqslant r$ such that $\rho \leqslant \mu_1$, $\bar{1} - \lambda \leqslant \mu_2$ and $\mu_1\ \bar{q}\ \mu_2$. Since $\mu_1\ \bar{q}\ \mu_2$, then $C_{12}(\mu_1, r) \leqslant C_{12}(\bar{1} - \mu_2, r) = \bar{1} - \mu_2$. Therefore, $\rho \leqslant \mu_1$ implies $\rho \leqslant \mu_1 \leqslant C_{12}(\mu_1, r) \leqslant \bar{1} - \mu_2 \leqslant \lambda$. Put $\nu = \mu_1$, this means there exists $\nu \in I^X$, $\tau_{12}(\nu) \geqslant r$ such that $\rho \leqslant \nu \leqslant C_{12}(\nu, r) \leqslant \lambda$.

Conversely, suppose the condition holds. Let $\rho, \lambda \in I^X$ such that $C_{12}(\rho_1, r) = \rho_1\ \bar{q}\ \rho_2 = C_{12}(\rho_2, r)$. Then $\rho_1 \leqslant \bar{1} - \rho_2$. By hypothesis, there exists $\nu \in I^X$, $\tau_{12}(\nu) \geqslant r$ such that $\rho_1 \leqslant \nu \leqslant C_{12}(\nu, r) \leqslant \bar{1} - \rho_2$. This implies $\rho_1 \leqslant \nu$ and $\rho_2 \leqslant \bar{1} - C_{12}(\nu, r)$. Since $\nu\ \bar{q}\ \bar{1} - C_{12}(\nu, r)$, then (X, τ_1, τ_2) is FP^*R_3. ■

Definition 4.11 Let (X, τ) be a supra smooth topological space. If there exists a supra smooth proximity δ such that $\tau = \tau_\delta$, then τ and δ are said to compatible or the supra smooth topological space (X, τ) is a supra smooth proximal space.

Theorem 4.12 Let (X, τ_1, τ_2) be a FP^*T_4 and (X, τ_{12}) its induced supra smooth topological space. Then the mapping $\delta : I^X \times I^X \longrightarrow I$ defined by

$$\delta(\mu, \rho) = \bigvee \{1 - r \in I |\ C_{12}(\mu, r)\ q\ C_{12}(\rho, r)\} \tag{16}$$

is a compatible separated supra smooth proximity on X.

Proof. First, we show that δ defines a supra smooth proximity.

(FP1) obvious.

(FP2*) Let $\delta(\mu, \lambda) \vee \delta(\rho, \lambda) = 1 - r$. Then $C_{12}(\mu, r)\ q\ C_{12}(\lambda, r)$ or $C_{12}(\rho, r)\ q\ C_{12}(\lambda, r)$, which implies $C_{12}(\lambda, r)\ q\ C_{12}(\mu, r) \vee C_{12}(\rho, r)$. Since $C_{12}(\mu, r) \vee C_{12}(\rho, r) \leqslant C_{12}(\mu \vee \rho, r)$, then $C_{12}(\lambda, r)\ q\ C_{12}(\mu \vee \rho, r)$. So $\delta(\mu \vee \rho, \lambda) = 1 - r$.

(FP3) Since $C_{12}(\bar{1}, r) = \bar{1}\ \bar{q}\ \bar{0} = C_{12}(\bar{0}, r)$ for all $r \in I_0$, then $\delta(\bar{1}, \bar{0}) = 0$.

(FP4) Let $\delta(\mu, \rho) < 1 - r$. Then $C_{12}(\mu, r)\ \bar{q}\ C_{12}(\rho, r)$, this implies, $C_{12}(\mu, r) \leqslant \bar{1} - C_{12}(\rho, r)$. Since (X, τ_1, τ_2) is a FP^*T_4, then by Theorem 4.10, there exists $\lambda \in I^X$ with $\tau_{12}(\lambda) \geqslant r$ such that

$$C_{12}(\mu, r) \leqslant \lambda \leqslant C_{12}(\lambda, r) \leqslant \bar{1} - C_{12}(\rho, r).$$

put $\nu = \bar{1} - \lambda$. Then $C_{12}(\nu, r) = C_{12}(\bar{1} - \lambda, r) = \bar{1} - \lambda$, and so we have

$$C_{12}(\mu, r) \leqslant \bar{1} - C_{12}(\nu, r) \leqslant C_{12}(\bar{1} - \nu, r) \leqslant \bar{1} - C_{12}(\rho, r),$$

implies

$$C_{12}(\mu, r) \leqslant \bar{1} - C_{12}(\nu, r) \text{ and } C_{12}(\bar{1} - \nu, r) \leqslant \bar{1} - C_{12}(\rho, r).$$

Then

$$C_{12}(\mu, r)\ \bar{q}\ C_{12}(\nu, r) \text{ and } C_{12}(\bar{1} - \nu, r)\ \bar{q}\ C_{12}(\rho, r).$$

Thus,

$$\delta(\mu,\nu) < 1-r \text{ and } \delta(\bar{1}-\nu,\rho) < 1-r.$$

Hence the result.

(FP5) Let $\mu\ q\ \rho$, then $C_{12}(\mu,r)\ q\ C_{12}(\rho,r)$ for all $r \in I_0$. It then follows from (16), $\delta(\mu,\rho) = 1$. Hence δ is a supra smooth proximity on X.

To show δ is separated, suppose $\delta(x_t, y_m) = 1$. Then $C_{12}(x_t,r)\ q\ C_{12}(y_m,r)$ for all $r \in I_0$. Since (X,τ_1,τ_2) is FP^*T_1, then by Theorem 4.9, $x_t\ q\ y_m$. Hence δ is separated.

To show $\tau_{12} = \tau_\delta$, first let μ be an r-closed fuzzy set in (X,τ_{12}), i.e. $C_{12}(\mu,r) = \mu$ and let $x_t\ q\ C_\delta(\mu,r)$. Then by (8), $\delta(x_t,\mu) \geqslant 1-r$ and from (16), $C_{12}(x_t,r)\ q\ C_{12}(\mu,r)$. Since (X,τ_1,τ_2) is FP^*T_1 and $C_{12}(\mu,r) = \mu$, then we have $x_t\ q\ \mu$. Thus $C_\delta(\mu,r) \leqslant \mu$, clearly $\mu \leqslant C_\delta(\mu,r)$. So $C_\delta(\mu,r) = \mu$ and hence μ is an r-closed fuzzy set in (X,τ_δ).

Conversely, let μ be an r- closed fuzzy set in (X,τ_δ) and $x_t\ q\ C_{12}(\mu,r)$. This implies $C_{12}(x_t,r)\ q\ C_{12}(\mu,r)$ and from (16), $\delta(x_t,\mu) \geqslant 1-r$, means that $x_t\ q\ C_\delta(\mu,r) = \mu$. Then $C_{12}(\mu,r) \leqslant \mu$, but $\mu \leqslant C_{12}(\mu,r)$. So $C_{12}(\mu,r) = \mu$ and hence μ is an r-closed fuzzy set in (X,τ_{12}). ■

Theorem 4.13 If a smooth bts (X,τ_1,τ_2) is a FP^*T_1 and the induced supra smooth topological space (X,τ_{12}) has a compatible supra smooth proximity and $\delta : I^X \times I^X \longrightarrow I$ defined by

$$\delta(\mu,\rho) = \bigvee\{1-r \in I|\ C_{12}(\mu,r)\ q\ C_{12}(\rho,r)\}. \tag{17}$$

Then (X,τ_δ) is a FT_4-space.

Proof. Let $\rho_1,\rho_2 \in I^X$ with $C_\delta(\rho_1,r) = \rho_1$, $C_\delta(\rho_2,r) = \rho_2$ and $\rho_1\ \bar{q}\ \rho_2$. This implies, $C_\delta(\rho_1,r)\ \bar{q}\ C_\delta(\rho_2,r)$ and so $C_{12}(\rho_1,r)\ \bar{q}\ C_{12}(\rho_2,r)$, by (17), we have $\delta(\rho_1,\rho_2) < 1-r$. Then by (FP4) axiom, there exists $\eta \in I^X$ such that $\delta(\rho_1,\eta) < 1-r$ and $\delta(\bar{1}-\eta,\rho_2) < 1-r$. By using Proposition 3.5(2), we have

$$\delta(\rho_1, C_\delta(\eta,r)) < 1-r \ \text{ and } \ \delta(C_\delta(\bar{1}-\eta,r),\rho_2) < 1-r.$$

Therefore, $\rho_1\ \bar{q}\ C_\delta(\eta,r)$ and $\rho_2\ \bar{q}\ C_\delta(\bar{1}-\eta,r)$. So, $\rho_1 \leqslant \bar{1}-C_\delta(\eta,r)$ and $\rho_2 \leqslant \bar{1}-C_\delta(\bar{1}-\eta,r)$. Since $\bar{1}-C_\delta(\eta,r)$, $\bar{1}-C_\delta(\bar{1}-\eta,r)$ are r-open fuzzy sets in (X,τ_δ) and $\bar{1}-C_\delta(\eta,r)\ \bar{q}\ \bar{1}-C_\delta(\bar{1}-\eta,r)$, then (X,τ_δ) is a FR_3. Since (X,τ_{12}) is a FT_1 and compatible with δ, then (X,τ_δ) is a FT_1. Hence, (X,τ_δ) is a FT_4. ■

Definition 4.14 Let (X,δ_1,δ_2) and $(Y,\delta_1^*,\delta_2^*)$ be two smooth biproximity spaces. Then a mapping $f : (X,\delta_1,\delta_2) \longrightarrow (X,\delta_1^*,\delta_2^*)$ is called:

(1) FP-proximity map if it satisfies, $\delta_i(\mu,\rho) \leqslant \delta_i^*(f(\mu),f(\rho))$ for every $\mu,\rho \in I^X$ and $i = 1,2$.

(2) FP^*-proximity map if it satisfies, $\delta_{12}(\mu,\rho) \leqslant \delta_{12}^*(f(\mu),f(\rho))$ for every $\mu,\rho \in I^X$

Proposition 4.15 Let $f : (X,\delta_1,\delta_2) \longrightarrow (Y,\delta_1^*,\delta_2^*)$, if f is a FP-proximity map, then f is a FP^*-proximity map.

Proof. The proof follows from the definition of δ_{12} and FP-proximity of f. ■

Next we give the relationship between smooth bitopological spaces and smooth biproximity spaces.

Theorem 4.16 Let $f : (X, \delta_1, \delta_2) \longrightarrow (Y, \delta_1^*, \delta_2^*)$ be a FP^*-proximity map. Then:
(1) $f : (X, \tau_{\delta_{12}}) \longrightarrow (Y, \tau_{\delta_{12}^*})$ is a f-continuous map.
(2) $C_{\delta_{12}}(f^{-1}(\mu), r) \leqslant f^{-1}(C_{\delta_{12}^*}(\mu, r))$, for all $r \in I_1$, for all $\mu \in I^Y$.

Proof. (1) By Theorem **??**, we will show that $f(C_{\delta_{12}}(\mu, r)) \leqslant C_{\delta_{12}^*}(f(\mu), r)$, for all $r \in I_1$, for all $\mu \in I^X$.

Suppose that, there exist $y \in Y$, $t \in I_0$, such that

$$f(C_{\delta_{12}}(\mu, r))(y) > t > C_{\delta_{12}^*}(f(\mu), r)(y). \tag{18}$$

By the definition of $C_{\delta_{12}^*}(f(\mu), r)$, there exists $\lambda \in I^Y$ such that $\lambda \leqslant \bar{1} - f(\mu)$, $\delta_{12}^*(\lambda, f(\mu)) < 1 - r$ and $\bar{1} - \lambda(y) < t$.

On the other hand, since $f^{-1}(\bar{1} - f(\mu)) = \bar{1} - f^{-1}(f(\mu))$, we have $\lambda \leqslant \bar{1} - f(\mu)$ implies $f^{-1}(\lambda) \leqslant f^{-1}(\bar{1} - f(\mu)) = \bar{1} - f^{-1}(f(\mu)) \leqslant \bar{1} - \mu$. Since f is a FP^*-proximity map, and $f(f^{-1}(\lambda)) \leqslant \lambda$, we have,

$$\delta_{12}(\mu, f^{-1}(\lambda)) \leqslant \delta_{12}^*(f(\mu), f(f^{-1}(\lambda))) < 1 - r.$$

Therefore, $\delta_{12}(\mu, f^{-1}(\lambda)) \leqslant 1 - r$ implies
$C_{\delta_{12}}(\mu, r) \leqslant \bar{1} - f^{-1}(\lambda)$. So,

$$f(C_{\delta_{12}}(\mu, r))(y) \leqslant \bar{1} - f(f^{-1}(\lambda))(y) = f(f^{-1}(\bar{1} - \lambda))(y) < t.$$

Which is contradicts for (18).

(2) Suppose that $C_{\delta_{12}}(f^{-1}(\mu), r) \nleqslant f^{-1}(C_{\delta_{12}^*}(\mu, r))$. Then, there exist $x \in X$, $t \in I_0$ such that

$$C_{\delta_{12}}(f^{-1}(\mu), r)(x) > t > f^{-1}(C_{\delta_{12}^*}(\mu, r))(x). \tag{19}$$

By the definition of $C_{\delta_{12}^*}(\mu, r)$, there exists $\lambda \in I^Y$ such that $\delta_{12}^*(\lambda, \mu) < 1 - r$ and $f^{-1}(\bar{1} - \lambda)(x) < t$.

Since f is a FP^*-proximity map, then we have $\delta_{12}(f^{-1}(\mu), f^{-1}(\lambda)) \leqslant \delta_{12}^*(f(f^{-1}(\mu)), f(f^{-1}(\lambda))) < \delta_{12}^*(\mu, \lambda) < 1 - r$. So, it follows that $C_{\delta_{12}}(f^{-1}(\mu), r)(x) \leqslant \bar{1} - f^{-1}(\lambda)(x) \leqslant t$. This contradicts (19). Hence the result. ■

5. *P*-smooth quasi-proximity spaces

In this section we introduce the concept of P-smooth quasi-proximity δ and study it is basic properties, and we discuss the structure of smooth bts based on P-smooth quasi-proximity.

Definition 5.1 A mapping $\delta : I^X \times I^X \longrightarrow I$ satisfying (FQP1), (FQP2), (FQP4) and the following axioms:
(FQP3 *)

(i) If $\delta(\mu, \rho) < r$, then there exists $\eta \in I^X$ such that $\delta(\mu, \eta) < r$ and $\delta(\bar{1} - \eta, \rho) < r$.
(ii) If $\delta(\rho, \mu) < r$, then there exists $\eta \in I^X$ such that $\delta(\rho, \eta) < r$ and $\delta(\bar{1} - \eta, \mu) < r$.

is called P-smooth quasi-proximity on X and the pair (X, δ) is called P-smooth quasi-proximity space.

Lemma 5.2 Let (X,δ) be a P-smooth quasi-proximity space, $\mu,\lambda \in I^X$ and $r \in I_0$. If $\delta(\mu,\lambda) \geqslant r$, $\mu \leqslant \mu_1$ and $\lambda \leqslant \lambda_1$, then $\delta(\mu_1,\lambda_1) \geqslant r$.

Proof. The result follows from (FQP2) axiom. ■

Theorem 5.3 Let (X,δ) be a P-smooth quasi-proximity space. Then

(1) The mapping $\delta^{-1} : I^X \times I^X \longrightarrow I$ defined by

$$\delta^{-1}(\mu,\rho) = \delta(\rho,\mu) \tag{20}$$

for all $\mu,\rho \in I^X$, is also a P-smooth quasi-proximity on X.

(2) The mapping $C_\delta : I^X \times I_1 \longrightarrow I^X$ given by

$$C_\delta(\lambda,r) = \bigwedge\{\rho \in I^X \mid \delta(\bar{1}-\rho,\lambda) < 1-r\}, \tag{21}$$

is a fuzzy closure operator. Furthermore, C_δ generates a smooth topology $\tau_\delta : I^X \longrightarrow I$ as (4).

Proof.

(1) By the definition of δ^{-1} and Remark *1*(3), it suffices to show axiom (FQP3$^*(ii)$). Let $\delta^{-1}(\rho,\mu) < r$. Since $\delta^{-1}(\rho,\mu) = \delta(\mu,\rho) < r$. Then, by (FQP3$^*(i)$), there exists $\eta \in I^X$ such that $\delta(\mu,\bar{1}-\eta) < r$ and $\delta(\eta,\rho) < r$. It follows, $\delta^{-1}(\bar{1}-\eta,\mu) < r$ and $\delta^{-1}(\rho,\eta) < r$, which is the required result. Hence, δ^{-1} is a P-smooth quasi-proximity on X.

(2) To prove that C_δ is a fuzzy closure operator, we need to satisfy conditions $(C1)-(C5)$ in Definition 2.3. The proof of the parts $(C1)-(C3)$ follows directly from axiom (FQP1), Remark *1*(2), (21) and axiom (FQP4).

(C4) Let $r,s \in I_1$, $r \leqslant s$. Suppose $C_\delta(\lambda,r) \nleqslant C_\delta(\lambda,s)$. Then, there exist $x \in X$ and $t \in I_0$ such that

$$C_\delta(\lambda,r)(x) > t > C_\delta(\lambda,s)(x). \tag{22}$$

Since $C_\delta(\lambda,s)(x) < t$. Then, there exists $\rho \in I^X$, $\delta(\bar{1}-\rho,\lambda) < 1-s$ and $\rho(x) < t$. Since $\delta(\bar{1}-\rho,\lambda) < 1-s < 1-r$. Then, there exists $\rho \in I^X$, $C_\delta(\lambda,r) \leqslant \rho$. It follows $C_\delta(\lambda,r)(x) < t$. This however contradicts (22). Hence, the result.

(C5) From (C2), it is suffices to show that $C_\delta(C_\delta(\lambda,r),r) \leqslant C_\delta(\lambda,r)$. Suppose the contrary, i.e. there exist $x \in X$ and $t \in I_0$ such that

$$C_\delta(C_\delta(\lambda,r),r)(x) > t > C_\delta(\lambda,r)(x). \tag{23}$$

Then, there exists $\rho \in I^X$ such that $C_\delta(C_\delta(\lambda,r),r)(x) > t > \rho(x)$, $\delta(\bar{1}-\rho,\lambda) < 1-r$. Since $\delta(\bar{1}-\rho,\lambda) < 1-r$, then by axiom (FQP3$^*(i)$), there exists $\eta \in I^X$ such that $\delta(\bar{1}-\rho,\eta) < 1-r$ and $\delta(\bar{1}-\eta,\lambda) < 1-r$. Hence, $C_\delta(\lambda,r) \leqslant \eta$. It follows from Lemma 5.2, $\delta(\bar{1}-\rho,C_\delta(\lambda,r)) < 1-r$. Therefore, $C_\delta(C_\delta(\lambda,r),r) \leqslant \rho$ implies $C_\delta(C_\delta(\lambda,r),r)(x) \leqslant \rho(x) < t$. This contradicts (23). Hence, C_δ is a fuzzy closure operator. From Definition 2.3, C_δ generates a smooth topology $\tau_\delta : I^X \longrightarrow I$ as in (4).

■

Corollary 5.4 Let (X, δ) be a P-smooth quasi-proximity space. Then the mapping $C_{\delta^{-1}} : I^X \times I_1 \longrightarrow I^X$ given by

$$C_{\delta^{-1}}(\lambda, r) = \bigwedge\{\rho \in I^X \mid \delta(\lambda, \bar{1} - \rho) < 1 - r\}, \tag{24}$$

is a fuzzy closure operator. Furthermore, $C_{\delta^{-1}}$ generates a smooth topology $\tau_{\delta^{-1}} : I^X \longrightarrow I$ as (4).

Proposition 5.5 Let (X, δ) be a P-smooth quasi-proximity space, $\lambda, \rho \in I^X$, $x_t \in Pt(X)$ and $r \in I_1$. Then

(1) $x_t \; q \; C_\delta(\lambda, r)$ if and only if $\delta(x_t, \lambda) \geqslant 1 - r$.
(2) $x_t \; q \; C_{\delta^{-1}}(\lambda, r)$ if and only if $\delta(\lambda, x_t) \geqslant 1 - r$.
(3) $\delta(\lambda, \rho) = \delta(C_{\delta^{-1}}(\lambda, r), C_\delta(\rho, r))$.

Proof.

(1) Let $x_t \; q \; C_\delta(\lambda, r)$ and suppose $\delta(x_t, \lambda) < 1-r$. Then, $\delta(x_t, \lambda) = \delta(\bar{1}-(\bar{1}-x_t), \lambda) < 1-r$, this implies $C_\delta(\lambda, r) \leqslant \bar{1}-x_t$ and thus $x_t \; \bar{q} \; C_\delta(\lambda, r)$ which is a contradiction. Hence, $\delta(x_t, \lambda) \geqslant 1 - r$.

Conversely, let $\delta(x_t, \lambda) \geqslant 1 - r$, suppose $x_t \; \bar{q} \; C_\delta(\lambda, r)$. Then from (21), there exists $\rho \in I^X$ with $\delta(\bar{1} - \rho, \lambda) < 1 - r$ such that $x_t \; \bar{q} \; \rho$ implies $x_t \leqslant \bar{1} - \rho$. From Remark *1*(2), we have $\delta(x_t, \lambda) < 1 - r$ which is a contradiction with hypothesis.

(2) Let $x_t \; q \; C_{\delta^{-1}}(\lambda, r)$ and suppose $\delta(\lambda, x_t) < 1 - r$. Then, $\delta(\lambda, \bar{1} - (\bar{1} - x_t)) < 1 - r$, from (24) we have, $C_{\delta^{-1}}(\lambda, r) \leqslant \bar{1} - x_t$. This implies $x_t \; \bar{q} \; C_{\delta^{-1}}(\lambda, r)$ which is a contradiction. Hence, $\delta(\lambda, x_t) \geqslant 1 - r$.

Conversely, similar of part (1).

(3) The necessity comes directly from Remark *1*(2). For sufficiency, suppose that $\delta(\lambda, \rho) < 1 - r$. Then from $(FQP3^*(i))$, there exists $\eta \in I^X$ such that $\delta(\lambda, \eta) < 1-r$ and $\delta(\bar{1}-\eta, \rho) < 1-r$. If $x_t \in \eta$, then $\delta(\lambda, x_t) < 1-r$, so $x_t \; \bar{q} \; C_{\delta^{-1}}(\lambda, r)$ and hence $x_t \leqslant \bar{1}-C_{\delta^{-1}}(\lambda, r)$. Then $\eta \leqslant \bar{1}-C_{\delta^{-1}}(\lambda, r)$ implies that $C_{\delta^{-1}}(\lambda, r) \leqslant \bar{1}-\eta$. Thus, from Remark *1*(2), we get, $\delta(C_{\delta^{-1}}(\lambda, r), \rho) < 1 - r$. Again by applying $((FQP3^*(i)))$, we have, $\nu \in I^X$ such that $\delta(C_{\delta^{-1}}(\lambda, r), \bar{1}-\nu) < 1-r$ and $\delta(\nu, \rho) < 1-r$. If $x_t \in \nu$, then $\delta(x_t, \rho) < 1-r$. So, $x_t \; \bar{q} \; C_\delta(\rho, r)$ and hence $x_t \leqslant \bar{1}-C_\delta(\rho, r)$. Then $\nu \leqslant \bar{1} - C_\delta(\rho, r)$ which implies $C_\delta(\rho, r) \leqslant \bar{1} - \nu$, and from Remark *1*(2), we get, $\delta(C_{\delta^{-1}}(\lambda, r), C_\delta(\rho, r)) < 1 - r$. Therefore, $\delta(\lambda, \rho) = \delta(C_{\delta^{-1}}(\lambda, r), C_\delta(\rho, r))$.

■

Theorem 5.6 If (X, δ) is a P-smooth quasi-proximity space. Then, $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPR_i-space, $i = 0, 1, 2, 3$.

Proof.

(0) Let $x_t \; \bar{q} \; C_\delta(y_m, r)$. Then, $\delta(x_t, y_m) < 1-r$. Since $\delta^{-1}(y_m, x_t) = \delta(x_t, y_m) < 1-r$, from Proposition 5.4(2), $y_m \; \bar{q} \; C_{\delta^{-1}}(x_t, r)$. Similarly, if $x_t \; \bar{q} \; C_{\delta^{-1}}(y_m, r)$, we have $y_m \; \bar{q} \; C_\delta(x_t, r)$. Hence, $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPR_0-space.

(1) Let $x_t \; \bar{q} \; C_\delta(y_m, r)$. Then $\delta(x_t, y_m) < 1 - r$, by $(FQP3^*(i))$, there exists $\eta \in I^X$ such that $\delta(x_t, \eta) < 1 - r$ and $\delta(\bar{1} - \eta, y_m) < 1 - r$. Hence, $x_t \; \bar{q} \; C_\delta(\eta, r)$ and $y_m \; \bar{q} \; C_{\delta^{-1}}(\bar{1}-\eta, r)$. Then, $x_t \leqslant \bar{1}-C_\delta(\eta, r) = I_\delta(\bar{1}-\eta, r)$, $y_m \leqslant \bar{1}-C_{\delta^{-1}}(\bar{1}-\eta, r) = I_{\delta^{-1}}(\eta, r)$ and $I_\delta(\bar{1} - \eta, r) \; \bar{q} \; I_{\delta^{-1}}(\eta, r)$. Similarly, if $x_t \; \bar{q} \; C_{\delta^{-1}}(y_m, r)$. Then there exists an r-τ_δ-open fuzzy set $I_\delta(\bar{1} - \nu, r)$ and an r-$\tau_{\delta^{-1}}$-open fuzzy set $I_{\delta^{-1}}(\nu, r)$ such that $x_t \in I_{\delta^{-1}}(\nu, r)$, $y_m \in I_\delta(\bar{1} - \nu, r)$ and $I_{\delta^{-1}}(\nu, r) \; \bar{q} \; I_\delta(\bar{1} - \nu, r)$. Hence, $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPR_1-space.

(2) Let $x_t \bar{q} \rho = C_{\delta^{-1}}(\rho, r)$. Then, from Proposition 5.4(2), $\delta(\rho, x_t) < 1 - r$, and by $(FQP3^*(ii))$, there exists $\eta \in I^X$ such that $\delta(\rho, \eta) < 1-r$ and $\delta(\bar{1}-\eta, x_t) < 1-r$. Hence, $\rho \bar{q} C_\delta(\eta, r)$ and $x_t \bar{q} C_{\delta^{-1}}(\bar{1} - \eta, r)$. This implies $\rho \leqslant I_\delta(\bar{1} - \eta, r)$, $x_t \in I_{\delta^{-1}}(\eta, r)$ and $I_\delta(\bar{1} - \eta, r) \bar{q} I_{\delta_{-1}}(\eta, r)$. Similarly, if $x_t \bar{q} C_\delta(\rho, r)$, then there exists $\nu \in I^X$ such that $x_t \in I_\delta(\bar{1} - \nu, r)$ and $\rho \in I_{\delta^{-1}}(\nu, r)$ and $I_\delta(\bar{1} - \nu, r) \bar{q} I_{\delta^{-1}}(\nu, r)$. Hence, $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPR_2-space.

(3) Similar to part (2).

■

Definition 5.7 A P-smooth quasi-proximity δ is said to be separated if and only if

$$x_t \bar{q} y_m \text{ implies } \delta(x_t, y_m) \neq 1 \tag{25}$$

for all $x_t, y_m \in Pt(X)$.

Theorem 5.8 If (X, δ) is a separated P-smooth quasi-proximity space. Then, $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPT_i-space, $i = 0, 1, 2, 3, 4$.

Proof.

(0) Let $x_t \bar{q} y_m$. Since δ is a separated, then $\delta(x_t, y_m) \neq 1$. Then, there exists $r \in I_1$ such that $\delta(x_t, y_m) < 1 - r$. It follows that $y_m \bar{q} C_{\delta^{-1}}(x_t, r)$. Since $x_t \in I_{\delta^{-1}}(x_t, r) \leqslant C_{\delta^{-1}}(x_t, r)$. Then, there exists $\lambda = I_{\delta^{-1}}(x_t, r)$ such that $x_t \in \lambda$, $y_m \bar{q} \lambda$. Hence $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPT_0-space.

(1) Similar to part (0).

(2) Let $x_t \bar{q} y_m$. Since δ is separated, then $\delta(x_t, y_m) \neq 1$. Then, there exists $r \in I_1$ such that $\delta(x_t, y_m) < 1 - r$. From axiom (FQP3* (i)), there exits $\eta \in I^X$ such that $\delta(x_t, \eta) < 1 - r$ and $\delta(\bar{1} - \eta, y_m) < 1 - r$. This implies $x_t \bar{q} C_\delta(\eta, r)$ and $y_m \bar{q} C_{\delta^{-1}}(\bar{1} - \eta, r)$ which yields $x_t \in \bar{1} - C_\delta(\eta, r) = I_\delta(\bar{1} - \eta, r)$, $y_m \in \bar{1} - C_{\delta^{-1}}(\bar{1} - \eta, r) = I_{\delta^{-1}}(\eta, r)$ and $I_\delta(\bar{1} - \eta, r) \bar{q} I_{\delta^{-1}}(\eta, r)$. Hence $(X, \tau_\delta, \tau_{\delta^{-1}})$ is FPT_2-space.

(3) It is follows directly from Theorem 4.7(2) and part (1).

(4) It is follows directly from Theorem 4.7(3) and part (1).

■

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