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System of AQC functional equations in non-Archimedean normed spaces

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Abstract. In 1897, Hensel introduced a normed space which does not have the Archimedean property. During the last three decades theory of non–Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p–adic strings and superstrings. In this paper, we prove the generalized Hyers–Ulam–Rassias stability for a system of additive, quadratic and cubic functional equations in non–Archimedean normed spaces.

Keywords: Additive, quadratic, cubic, functional equation, non–Archimedean spaces, generalized Hyers–Ulam–Rassias stability.

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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$" If the problem accepts a solution?, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [23] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [11]. It gave rise the stability theory for functional equations. Hyers' theorem was generalized by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach (also, see [8, 16]).

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The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is related to a symmetric bi–additive function [1, 15]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1) is said to be a quadratic function. It is well known that a function $f: X \to Y$ where X and Y are real vector spaces, is quadratic if and only if there exists a unique symmetric bi–additive function B such that $f(x) = B(x,x)$ for all x. The bi–additive function B is given by $B(x,y) = \frac{1}{4}(f(x+y) - f(x-y))$.

The Hyers–Ulam stability problem for the quadratic functional equation was solved by Skof [22]. Czerwik[4] proved the Hyers-Ulam-Rassias stability of the equation (1). Later, Jung [14] generalized the results obtained by Skof and Czerwik. Eshaghi Gordji and Khodaei [7] obtained the general solution and the generalized Hyers–Ulam–Rassias stability of the following quadratic functional equation for $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$,

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y). \tag{2}$$

Jun and Kim [12] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{3}$$

and proved the general solution and the generalized Hyers–Ulam–Rassias stability of the functional equation (3). They proved that a function $f: X \to Y$, where X and Y are real vector spaces, is a solution of (3) if and only if there exists a unique function $C: X \times X \times X \longrightarrow Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$. Moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. The function C is given by $C(x,y,z) = \frac{1}{24}(f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z))$ for all $x, y, z \in X$. Obviously, the function $f(x) = cx^3$ satisfies the functional equation (3), which is called the cubic functional equation. Jun et al. [13] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(ax+by)+f(ax-by)=ab^2(f(x+y)+f(x-y))+2a(a^2-b^2)f(x),$$

where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$.

In 1897, Hensel [10] has introduced a normed space which does not have the Archimedean property. During the last three decades theory of non–Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p–adic strings and superstrings [17]. Although many results in the classical normed space theory have a non–Archimedean counterpart, but their proofs are essentially different and require an entirely new kind of intuition [2, 5, 6, 8, 18, 20, 24].

Let $\mathbb{K}$ be a field. A non–Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot|: \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

- (i) $|a| \geqslant 0$ and equality holds if and only if $a = 0$,
- (ii) $|ab| = |a||b|$,
- (iii) $|a+b| \leqslant \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leqslant 1$ for each integer n. We always assume in addition that $|\cdot|$ is non trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non–Archimedean non–trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:

$(NA1)$ $\|x\| = 0$ if and only if $x = 0$;
$(NA2)$ $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
$(NA3)$ the strong triangle inequality (ultrametric); namely, $\|x+y\| \leqslant \max\{\|x\|, \|y\|\}$ $(x, y \in X)$.

Then $(X, \|\cdot\|)$ is called a non–Archimedean normed space.

It follows from $(NA3)$ that $\|x_m - x_l\| \leqslant \max\{\|x_{\jmath+1} - x_\jmath\| : l \leqslant \jmath \leqslant m-1\}$ with $m > l$. Therefore, a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non–Archimedean normed space. If each Cauchy sequence is convergent, then the non–Archimedean normed space is said to be complete and is called non–Archimedean Banach space.

In mathematics, especially in the field of group theory, a divisible group is an abelian group in which every element can, in some sense, be divided by positive integers, or more accurately, every element is an nth multiple for each positive integer n. Let n be a positive integer and G an abelian group. An element $x \in G$ is said to be divisible by n if there is $y \in G$ such that $x = ny$. An abelian group G is divisible if and only if for every positive integer n and every g in G, there exists y in G such that $ny = g$.

Throughout this paper, unless otherwise explicitly stated, we assume that $i, j, k, m, n, p, s, t \in \mathbb{N} \cup \{0\}$ with $1 \leqslant i \leqslant s < j \leqslant t < k \leqslant n$, X be a non–Archimedean Banach space and G be a divisible group with the identity element 0. Also, let $f : G^n \to X$ be a function. We consider the following generalized system of additive, quadratic and cubic functional equations:

$$\begin{cases} f(a_1x_1 + b_1y_1, x_2, ..., x_n) = a_1f(x_1, ..., x_n) + b_1f(y_1, x_2, ..., x_n), \\ \quad \vdots \\ f(x_1, ..., x_{s-1}, a_sx_s + b_sy_s, x_{s+1}, ..., x_n) = \\ a_sf(x_1, ..., x_n) + b_sf(x_1, ..., x_{s-1}, y_s, x_{s+1}, ..., x_n), \\ f(x_1, ..., x_s, a_{s+1}x_{s+1} + b_{s+1}y_{s+1}, x_{s+2}, ..., x_n) + f(x_1, ..., x_s, a_{s+1}x_{s+1} - b_{s+1}y_{s+1}, x_{s+2}, ..., x_n) = \\ 2a_{s+1}^2f(x_1, ..., x_n) + 2b_{s+1}^2f(x_1, ..., x_s, y_{s+1}, x_{s+2}, ..., x_n), \\ \quad \vdots \\ f(x_1, ..., x_{t-1}, a_tx_t + b_ty_t, x_{t+1}, ..., x_n) + f(x_1, ..., x_{t-1}, a_tx_t - b_ty_t, x_{t+1}, ..., x_n) = \\ 2a_t^2f(x_1, ..., x_n) + 2b_t^2f(x_1, ..., x_{t-1}, y_t, x_{t+1}, ..., x_n), \\ f(x_1, ..., x_t, a_{t+1}x_{t+1} + b_{t+1}y_{t+1}, x_{t+2}, ..., x_n) + f(x_1, ..., x_t, a_{t+1}x_{t+1} - b_{t+1}y_{t+1}, x_{t+2}, ..., x_n) = \\ a_{t+1}b_{t+1}^2(f(x_1, ..., x_t, x_{t+1} + y_{t+1}, x_{t+2}, ..., x_n) + \\ f(x_1, ..., x_t, x_{t+1} - y_{t+1}, x_{t+2}, ..., x_n)) + 2a_{t+1}(a_{t+1}^2 - b_{t+1}^2)f(x_1, ..., x_n), \\ \quad \vdots \\ f(x_1, ..., x_{n-1}, a_nx_n + b_ny_n) + f(x_1, ..., x_{n-1}, a_nx_n - b_ny_n) = \\ a_nb_n^2(f(x_1, ..., x_{n-1}, x_n + y_n) + f(x_1, ..., x_{n-1}, x_n - y_n)) + 2a_n(a_n^2 - b_n^2)f(x_1, ..., x_n), \end{cases} \tag{4}$$

for all $x_i, y_i \in G$ and $a_i, b_i \in \mathbb{Z} \setminus \{0\}$, $i = 1, ..., n$, and $a_i \neq \pm 1, \pm b_i$, $i = t+1, ..., n$. In this paper, we prove the generalized Hyers–Ulam–Rassias stability of system (4) in non–Archimedean Banach spaces.

2. Main results

We prove the following main result, which can be regarded as an extension of the main results of Park [19] and Ciepliński [3].

Theorem 2.1 Let $\varphi_i : G^{2n} \to [0,\infty)$ be a function such that

$$\begin{cases} \lim_{m\to\infty} \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m} a_{s+1}^{2m}...a_t^{2m} a_{t+1}^{3m}...a_n^{3m}|} \\ \varphi_i(2^m a_1^m x_1, y_1, ..., 2^m a_{i-1}^m x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_i^m y_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1} \\ , ..., 2^m a_t^m x_t, y_t, a_{t+1}^m x_{t+1}, y_{t+1}, ..., a_n^m x_n, y_n) = 0; \\ \lim_{m\to\infty} \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m} a_{s+1}^{2m}...a_t^{2m} a_{t+1}^{3m}...a_n^{3m}|} \\ \varphi_k(2^m a_1^m x_1, y_1, ..., 2^m a_t^m x_t, y_t, a_{t+1}^m x_{t+1}, y_{t+1} \\ , ..., a_{k-1}^m x_{k-1}, y_{k-1}, a_k^m x_k, a_k^m y_k, a_{k+1}^m x_{k+1}, y_{k+1}, ..., a_n^m x_n, y_n) = 0; \end{cases} \tag{5}$$

for $i \in \{1, ..., t\}$ and $k \in \{t+1, ..., n\}$. Also assume that

$$\begin{aligned} &\lim_{m\to\infty} \max\Big\{ \max\{ \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m} a_{s+1}^{2m}...a_t^{2m} 2 a_{t+1}^{3(m+1)}...a_k^{3(m+1)} a_{k+1}^{3m}...a_n^{3m}|} \\ &\varphi_k(2^m a_1^m x_1, y_1, ..., 2^m a_t^m x_t, y_t, a_{t+1}^{m+1} x_{t+1}, y_{t+1}, ..., \\ &a_{k-1}^{m+1} x_{k-1}, y_{k-1}, a_k^m x_k, 0, a_{k+1}^m x_{k+1}, y_{k+1}, ..., a_n^m x_n, y_n) \ : \ k = t+1, ..., n\}, \\ &\frac{1}{|a_{t+1}^3...a_n^3|} \max\{ \frac{1}{|2^{sm}a_1^m...a_s^m 4^{j-s+(t-s)m} a_{s+1}^{2(m+1)}...a_j^{2(m+1)} a_{j+1}^{2m}...a_t^{2m} a_{t+1}^{3m}...a_n^{3m}|} \\ &\varphi_j(2^m a_1^m x_1, y_1, ..., 2^m a_s^m x_s, y_s, 2^{m+1} a_{s+1}^{m+1} x_{s+1}, y_{s+1}, ..., 2^{m+1} a_{j-1}^{m+1} x_{j-1}, y_{j-1}, 2^m a_j^m x_j, \\ &2^m a_j^m x_j, 2^m a_{j+1}^m x_{j+1}, y_{j+1}, ..., 2^m a_t^m x_t, y_t, a_{t+1}^{m+1} x_{t+1}, y_{t+1}, ..., a_n^{m+1} x_n, y_n) \ : \ j = s+1, ..., t\}, \\ &\frac{1}{|4^{t-s} a_{s+1}^2...a_t^2 a_{t+1}^3...a_n^3|} \max\{ \frac{1}{|2^{i+sm} a_1^{m+1}...a_i^{m+1} a_{i+1}^m...a_s^m 4^{(t-s)m} a_{s+1}^{2m}...a_t^{2m} a_{t+1}^{3m}...a_n^{3m}|} \\ &\varphi_i(2^{m+1} a_1^{m+1} x_1, y_1, ..., 2^{m+1} a_{i-1}^{m+1} x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_i^m x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1}, ..., 2^m a_s^m x_s, \\ &y_s, 2^{m+1} a_{s+1}^{m+1} x_{s+1}, y_{s+1}, ..., 2^{m+1} a_t^{m+1} x_t, y_t, a_{t+1}^{m+1} x_{t+1}, y_{t+1}, ..., a_n^{m+1} x_n, y_n) : \ i = 1, ..., s\} \Big\} = 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\Phi(x_1, y_1, ..., x_n, y_n) = \lim_{p\to\infty} \max\Big\{ \max\Big\{ \\ &\max\{ \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m} a_{s+1}^{2m}...a_t^{2m} 2 a_{t+1}^{3(m+1)}...a_k^{3(m+1)} a_{k+1}^{3m}...a_n^{3m}|} \varphi_k(2^m a_1^m x_1, y_1, ..., 2^m a_t^m x_t, \\ &y_t, a_{t+1}^{m+1} x_{t+1}, y_{t+1}, ..., a_{k-1}^{m+1} x_{k-1}, y_{k-1}, a_k^m x_k, 0, a_{k+1}^m x_{k+1}, y_{k+1}, ..., a_n^m x_n, y_n) : k = t+1, ..., n\}, \\ &\frac{1}{|a_{t+1}^3...a_n^3|} \max\{ \frac{1}{|2^{sm}a_1^m...a_s^m 4^{j-s+(t-s)m} a_{s+1}^{2(m+1)}...a_j^{2(m+1)} a_{j+1}^{2m}...a_t^{2m} a_{t+1}^{3m}...a_n^{3m}|} \\ &\varphi_j(2^m a_1^m x_1, y_1, ..., 2^m a_s^m x_s, y_s, 2^{m+1} a_{s+1}^{m+1} x_{s+1}, y_{s+1}, ..., 2^{m+1} a_{j-1}^{m+1} x_{j-1}, y_{j-1}, 2^m a_j^m x_j, \\ &2^m a_j^m x_j, 2^m a_{j+1}^m x_{j+1}, y_{j+1}, ..., 2^m a_t^m x_t, y_t, a_{t+1}^{m+1} x_{t+1}, y_{t+1}, ..., a_n^{m+1} x_n, y_n) : \ j = s+1, ..., t\}, \\ &\frac{1}{|4^{t-s} a_{s+1}^2...a_t^2 a_{t+1}^3...a_n^3|} \max\{ \frac{1}{|2^{i+sm} a_1^{m+1}...a_i^{m+1} a_{i+1}^m...a_s^m 4^{(t-s)m} a_{s+1}^{2m}...a_t^{2m} a_{t+1}^{3m}...a_n^{3m}|} \\ &\varphi_i(2^{m+1} a_1^{m+1} x_1, y_1, ..., 2^{m+1} a_{i-1}^{m+1} x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1}, ..., 2^m a_s^m x_s, y_s, 2^{m+1} a_{s+1}^{m+1} x_{s+1}, \\ &y_{s+1}, ..., 2^{m+1} a_t^{m+1} x_t, y_t, a_{t+1}^{m+1} x_{t+1}, y_{t+1}, ..., a_n^{m+1} x_n, y_n) : i = 1, ..., s\} \Big\}, m = 0, 1, ..., p \Big\} < \infty, \end{aligned} \tag{7}$$

for all $x_i, y_i \in G$ and $a_i, b_i \in \mathbb{Z}\setminus\{0\},\ i = 1, ..., n$. Let $f : G^n \to X$ be a mapping satisfying

$$\begin{cases} \|f(a_1x_1 + b_1y_1, x_2, ..., x_n) - a_1f(x_1, x_2, ..., x_n) - \\ b_1f(y_1, x_2, ..., x_n)\| \leqslant \varphi_1(x_1, y_1, x_2, y_2, ..., x_n, y_n), \\ \quad \vdots \\ \|f(x_1, x_2, ..., x_{s-1}, a_sx_s + b_sy_s, x_{s+1}, ..., x_n) - a_sf(x_1, x_2, ..., x_n) - \\ b_sf(x_1, x_2, ..., x_{s-1}, y_s, x_{s+1}, ..., x_n)\| \leqslant \varphi_s(x_1, y_1, x_2, y_2, ..., x_n, y_n), \\ \|f(x_1, x_2, ..., x_s, a_{s+1}x_{s+1} + b_{s+1}y_{s+1}, x_{s+2}, ..., x_n) + \\ f(x_1, x_2, ..., x_s, a_{s+1}x_{s+1} - b_{s+1}y_{s+1}, x_{s+2}, ..., x_n) - 2a_{s+1}^2f(x_1, x_2, ..., x_n) - \\ 2b_{s+1}^2f(x_1, x_2, ..., x_s, y_{s+1}, x_{s+2}, ..., x_n)\| \leqslant \varphi_{s+1}(x_1, y_1, x_2, y_2, ..., x_n, y_n), \\ \quad \vdots \\ \|f(x_1, x_2, ..., x_{t-1}, a_tx_t + b_ty_t, x_{t+1}, ..., x_n) + \\ f(x_1, x_2, ..., x_{t-1}, a_tx_t - b_ty_t, x_{t+1}, ..., x_n) - 2a_t^2f(x_1, x_2, ..., x_n) - \\ 2b_t^2f(x_1, x_2, ..., x_{t-1}, y_t, x_{t+1}, ..., x_n)\| \leqslant \varphi_t(x_1, y_1, x_2, y_2, ..., x_n, y_n), \\ \|f(x_1, x_2, ..., x_t, a_{t+1}x_{t+1} + b_{t+1}y_{t+1}, x_{t+2}, ..., x_n) + \\ f(x_1, x_2, ..., x_t, a_{t+1}x_{t+1} - b_{t+1}y_{t+1}, x_{t+2}, ..., x_n) \\ -a_{t+1}b_{t+1}^2(f(x_1, x_2, ..., x_t, x_{t+1} + y_{t+1}, x_{t+2}, ..., x_n) + \\ f(x_1, x_2, ..., x_t, x_{t+1} - y_{t+1}, x_{t+2}, ..., x_n)) \\ -2a_{t+1}(a_{t+1}^2 - b_{t+1}^2)f(x_1, x_2, ..., x_n)\| \leqslant \varphi_{t+1}(x_1, y_1, x_2, y_2, ..., x_n, y_n), \\ \quad \vdots \\ \|f(x_1, x_2, ..., x_{n-1}, a_nx_n + b_ny_n) + f(x_1, x_2, ..., x_{n-1}, a_nx_n - b_ny_n) - \\ a_nb_n^2(f(x_1, x_2, ..., x_{n-1}, x_n + y_n) + f(x_1, x_2, ..., x_{n-1}, x_n - y_n)) - \\ 2a_n(a_n^2 - b_n^2)f(x_1, x_2, ..., x_n)\| \leqslant \varphi_n(x_1, y_1, x_2, y_2, ..., x_n, y_n), \end{cases} \tag{8}$$

for all $x_i, y_i \in G$ and $a_i, b_i \in \mathbb{Z}\setminus\{0\},\ i = 1, ..., n,$ and $a_i \neq \pm1, \pm b_i\ ,\ i = t+1, ..., n$. Assume that $f(x_1, x_2, ..., x_n) = 0$ if $x_j = 0$ for some $j = s+1, ..., t$. Then there exists a unique mapping $T : G^n \to X$ satisfying (4) and

$$\|f(x_1, ..., x_n) - T(x_1, ..., x_n)\| \leqslant \Phi(x_1, y_1, ..., x_n, y_n) \tag{9}$$

for all $x_i, y_i \in G,\ i = 1, ..., n$.

Proof. Fix $i \in \{1, 2, ..., s\}$ and consider the following inequality.

$$\begin{aligned} &\|f(x_1, x_2, ..., a_ix_i + b_iy_i, ..., x_n) - a_if(x_1, x_2, ..., x_i, ..., x_n) - b_if(x_1, x_2, ..., y_i, ..., x_n)\| \\ &\leqslant \varphi_i(x_1, y_1, x_2, y_2, ..., x_n, y_n). \end{aligned} \tag{10}$$

Putting $x_i = y_i$ and $a_i = b_i$ in (10) we get

$$\|f(x_1, x_2, ..., x_i, ..., x_n) - \frac{1}{2a_i}f(x_1, x_2, ..., 2a_ix_i, ..., x_n)\| \leqslant \frac{1}{|2a_i|}\varphi_i(x_1, y_1, x_2, y_2, ..., x_i, x_i, ..., x_n, y_n).$$

Therefore, one can obtain

$$\begin{aligned} &\|\frac{1}{2^{i-1}a_1...a_{i-1}}f(2a_1x_1, 2a_2x_2, ..., 2a_{i-1}x_{i-1}, x_i, ..., x_n) \\ &- \frac{1}{2^ia_1...a_{i-1}a_i}f(2a_1x_1, 2a_2x_2, ..., 2a_ix_i, x_{i+1}, ..., x_n)\| \\ \leqslant &\frac{1}{|2^ia_1...a_{i-1}a_i|}\varphi_i(2a_1x_1, y_1, ..., 2a_{i-1}x_{i-1}, y_{i-1}, x_i, x_i, x_{i+1}, y_{i+1}, ..., x_n, y_n). \end{aligned}$$

So, we have

$$\|f(x_1,x_2,...,x_n)-\frac{1}{2^s a_1...a_s}f(2a_1x_1,2a_2x_2,...,2a_sx_s,x_{s+1},...,x_n)\|\leqslant \max\{\frac{1}{|2^i a_1...a_{i-1}a_i|} \varphi_i(2a_1x_1,y_1,...,2a_{i-1}x_{i-1},y_{i-1},x_i,x_i,x_{i+1},y_{i+1},...,x_n,y_n)\ :\ i=1,...,s\}. \tag{11}$$

Now, fix $j\in\{s+1,s+2,...,t\}$ and consider the following inequality.

$$\begin{aligned}&\|f(x_1,x_2,...,a_jx_j+b_jy_j,...,x_n)+f(x_1,x_2,...,a_jx_j-b_jy_j,...,x_n)-2a_j^2f(x_1,x_2,...,x_n)\\&-2b_j^2f(x_1,x_2,...,y_j,...,x_n)\|\leqslant\varphi_j(x_1,y_1,...,x_j,y_j,...,x_n,y_n).\end{aligned}\tag{12}$$

Putting $y_j=x_j$ and $b_j=a_j$ in (12), we get

$$\|f(x_1,x_2,...,x_j,...,x_n)-\frac{1}{4a_j^2}f(x_1,x_2,...,2a_jx_j,...,x_n)\|\leqslant\frac{1}{|4a_j^2|}\varphi_j(x_1,y_1,...,x_j,x_j,...,x_n,y_n).$$

Hence,

$$\begin{aligned}&\|\frac{1}{4^{j-s-1}a_{s+1}^2...a_{j-1}^2}f(x_1,...,x_s,2a_{s+1}x_{s+1},...,2a_{j-1}x_{j-1},x_j,x_{j+1},...,x_n)\\&-\frac{1}{4^{j-s}a_{s+1}^2...a_{j-1}^2a_j^2}f(x_1,...,x_s,2a_{s+1}x_{s+1},...,2a_jx_j,x_{j+1},...,x_n)\|\leqslant\\&\frac{1}{|4^{j-s}a_{s+1}^2...a_{j-1}^2a_j^2|}\varphi_j(x_1,y_1,...,x_s,y_s,2a_{s+1}x_{s+1},y_{s+1},...,2a_{j-1}x_{j-1},y_{j-1},\\&x_j,x_j,x_{j+1},y_{j+1},...,x_n,y_n).\end{aligned}$$

So, we have

$$\begin{aligned}&\|f(x_1,x_2,...,x_n)-\frac{1}{4^{t-s}a_{s+1}^2...a_t^2}f(x_1,...,x_s,2a_{s+1}x_{s+1},...,2a_tx_t,x_{t+1},...,x_n)\|\\&\leqslant\max\{\frac{1}{|4^{j-s}a_{s+1}^2...a_{j-1}^2a_j^2|}\varphi_j(x_1,y_1,...,x_s,y_s,2a_{s+1}x_{s+1},y_{s+1},\\&...,2a_{j-1}x_{j-1},y_{j-1},x_j,x_j,x_{j+1},y_{j+1},...,x_n,y_n)\ :\ j=s+1,...,t\}.\end{aligned}\tag{13}$$

Now, fix $k\in\{t+1,t+2,...,n\}$ and consider the following inequality

$$\begin{aligned}&\|f(x_1,...,a_kx_k+b_ky_k,...,x_n)+f(x_1,...,a_kx_k-b_ky_k,...,x_n)-a_kb_k^2(f(x_1,...,x_k+y_k,...,x_n)\\&+f(x_1,...,x_k-y_k,...,x_n))-2a_k(a_k^2-b_k^2)f(x_1,...,x_n)\|\leqslant\varphi_k(x_1,y_1,...,x_k,y_k,...,x_n,y_n).\end{aligned}\tag{14}$$

Putting $y_i=0$ in (14) to get

$$\|f(x_1,...,x_k,...,x_n)-\frac{1}{a_k^3}f(x_1,x_2,...,a_kx_k,...,x_n)\|\leqslant\frac{1}{|2a_k^3|}\varphi_k(x_1,y_1,...,x_k,0,...,x_n,y_n).$$

Therefore, one can obtain

$$\begin{aligned}&\|\frac{1}{a_{t+1}^3...a_{k-1}^3}f(x_1,...,x_t,a_{t+1}x_{t+1},...,a_{k-1}x_{k-1},x_k,x_{k+1},...,x_n)-\frac{1}{a_{t+1}^3...a_{k-1}^3a_k^3}\\&f(x_1,...,x_t,a_{t+1}x_{t+1},...,a_{k-1}x_{k-1},a_kx_k,x_{k+1},...,x_n)\|\leqslant\frac{1}{|2a_{t+1}^3...a_{k-1}^3a_k^3|}\\&\varphi_k(x_1,y_1,...,x_t,y_t,a_{t+1}x_{t+1},y_{t+1},...,a_{k-1}x_{k-1},y_{k-1},x_k,0,x_{k+1},y_{k+1},...,x_n,y_n).\end{aligned}$$

So, we have

$$\|f(x_1,...,x_n) - \frac{1}{a_{t+1}^3...a_n^3} f(x_1,...,x_t,a_{t+1}x_{t+1},...,a_nx_n)\| \leqslant \max\{\frac{1}{|2a_{t+1}^3...a_{k-1}^3a_k^3|} \tag{15}$$
$$\varphi_k(x_1,y_1,...,x_t,y_t,a_{t+1}x_{t+1},y_{t+1},...,a_{k-1}x_{k-1},y_{k-1},x_k,0,x_{k+1},y_{k+1},...,x_n,y_n) : k=t+1,...,n\}.$$

By (11), (13) and (15) we get

$$\|f(x_1,x_2,...,x_n) - \frac{1}{2^s a_1...a_s 4^{t-s} a_{s+1}^2...a_t^2 a_{t+1}^3...a_n^3} f(2a_1x_1,...,2a_tx_t,a_{t+1}x_{t+1},...,a_nx_n)\|$$
$$\leqslant \max\{\|f(x_1,...,x_n) - \frac{1}{a_{t+1}^3...a_n^3} f(x_1,...,x_t,a_{t+1}x_{t+1},...,a_nx_n)\|, \|\frac{1}{a_{t+1}^3...a_n^3} f(x_1,...,x_t,a_{t+1}x_{t+1},...,a_nx_n)$$
$$- \frac{1}{4^{t-s}a_{s+1}^2...a_t^2a_{t+1}^3...a_n^3} f(x_1,...,x_s,2a_{s+1}x_{s+1},...,2a_tx_t,a_{t+1}x_{t+1},...,a_nx_n)\|,$$
$$\|\frac{1}{4^{t-s}a_{s+1}^2...a_t^2a_{t+1}^3...a_n^3} f(x_1,...,x_s,2a_{s+1}x_{s+1},...,2a_tx_t,a_{t+1}x_{t+1},...,a_nx_n)$$
$$- \frac{1}{2^s a_1...a_s 4^{t-s} a_{s+1}^2...a_t^2 a_{t+1}^3...a_n^3} f(2a_1x_1,...,2a_tx_t,a_{t+1}x_{t+1},...,a_nx_n)\|\}$$
$$\leqslant \max\Big\{ \max\{\frac{1}{|2a_{t+1}^3...a_{k-1}^3a_k^3|}\varphi_k(x_1,y_1,...,x_t,y_t,$$
$$a_{t+1}x_{t+1},y_{t+1},...,a_{k-1}x_{k-1},y_{k-1},x_k,0,x_{k+1},y_{k+1},...,x_n,y_n) : k=t+1,...,n\},$$
$$\frac{1}{|a_{t+1}^3...a_n^3|}\max\{\frac{1}{|4^{j-s}a_{s+1}^2...a_{j-1}^2a_j^2|}\varphi_j(x_1,y_1,...,x_s,y_s,2a_{s+1}x_{s+1},y_{s+1},...,2a_{j-1}x_{j-1},$$
$$y_{j-1},x_j,x_j,x_{j+1},y_{j+1},...,x_t,y_t,a_{t+1}x_{t+1},y_{t+1},...,a_nx_n,y_n) : j=s+1,...,t\},$$
$$\frac{1}{|4^{t-s}a_{s+1}^2...a_t^2a_{t+1}^3...a_n^3|}\max\{\frac{1}{|2^ia_1...a_{i-1}a_i|}\varphi_i(2a_1x_1,y_1,...,2a_{i-1}x_{i-1},y_{i-1},x_i,x_i,$$
$$x_{i+1},y_{i+1},...,x_s,y_s,2a_{s+1}x_{s+1},...,2a_tx_t,a_{t+1}x_{t+1},...,a_nx_n) \ : \ i=1,...,s\}\Big\}.$$

It follows that

$$\|\frac{1}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} f(2^ma_1^mx_1,...,2^ma_t^mx_t,a_{t+1}^mx_{t+1},...,a_n^mx_n)$$
$$- \frac{1}{2^{s(m+1)}a_1^{m+1}...a_s^{m+1}4^{(t-s)(m+1)}a_{s+1}^{2(m+1)}...a_t^{2(m+1)}a_{t+1}^{3(m+1)}...a_n^{3(m+1)}}$$
$$f(2^{m+1}a_1^{m+1}x_1,...,2^{m+1}a_t^{m+1}x_t,a_{t+1}^{m+1}x_{t+1},...,a_n^{m+1}x_n)\| \leqslant$$
$$\max\Big\{ \max\{\frac{1}{|2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}2a_{t+1}^{3(m+1)}...a_k^{3(m+1)}a_{k+1}^{3m}...a_n^{3m}|}\varphi_k(2^ma_1^mx_1,y_1,...,$$
$$2^ma_t^mx_t,y_t,a_{t+1}^{m+1}x_{t+1},y_{t+1},...,a_{k-1}^{m+1}x_{k-1},y_{k-1},a_k^mx_k,0,a_{k+1}^mx_{k+1},y_{k+1},...,a_n^mx_n,y_n) : k=t+1,...,n\},$$
$$\frac{1}{|a_{t+1}^3...a_n^3|}\max\{\frac{1}{|2^{sm}a_1^m...a_s^m4^{j-s+(t-s)m}a_{s+1}^{2(m+1)}...a_j^{2(m+1)}a_{j+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|}\varphi_j(2^ma_1^mx_1, \tag{16}$$
$$y_1,...,2^ma_s^mx_s,y_s,2^{m+1}a_{s+1}^{m+1}x_{s+1},y_{s+1},...,2^{m+1}a_{j-1}^{m+1}x_{j-1},y_{j-1},2^ma_j^mx_j,$$
$$2^ma_j^mx_j,2^ma_{j+1}^mx_{j+1},y_{j+1},...,2^ma_t^mx_t,y_t,a_{t+1}^{m+1}x_{t+1},y_{t+1},...,a_n^{m+1}x_n,y_n) : j=s+1,...,t\},$$
$$\frac{1}{|4^{t-s}a_{s+1}^2...a_t^2a_{t+1}^3...a_n^3|}\max\{\frac{1}{|2^{i+sm}a_1^{m+1}...a_i^{m+1}a_{i+1}^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|}$$
$$\varphi_i(2^{m+1}a_1^{m+1}x_1,y_1,...,2^{m+1}a_{i-1}^{m+1}x_{i-1},y_{i-1},2^ma_i^mx_i,2^ma_i^mx_i,2^ma_{i+1}^mx_{i+1},y_{i+1},...,2^ma_s^mx_s,$$
$$y_s,2^{m+1}a_{s+1}^{m+1}x_{s+1},y_{s+1},...,2^{m+1}a_t^{m+1}x_t,y_t,a_{t+1}^{m+1}x_{t+1},y_{t+1},...,a_n^{m+1}x_n,y_n) : i=1,...,s\}\Big\},$$

for all $m \in \mathbb{N} \cup \{0\}$. It follows from (16) and (6) that the sequence

$$\{\frac{1}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} f(2^ma_1^mx_1,...,2^ma_t^mx_t,a_{t+1}^mx_{t+1},...,a_n^mx_n)\}$$

is Cauchy. By completeness of X, it is a convergent sequence. Therefore we can define $T : G^n \to X$ by

$$T(x_1, ..., x_n) := \lim_{m\to\infty} \frac{1}{2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} f(2^m a_1^m x_1, ..., 2^m a_t^m x_t, a_{t+1}^m x_{t+1}, ..., a_n^m x_n), \tag{17}$$

for all $x_i, y_i \in G$ and $a_i \in \mathbb{Z}\setminus\{0\}$, $i = 1, ..., n$, and $a_i \neq \pm 1, \pm b_i$, $i = t+1, ..., n$. Using induction with (16) one can show that

$$\begin{aligned}
&\|f(x_1, ..., x_n) - \frac{1}{2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} f(2^m a_1^m x_1, ..., 2^m a_t^m x_t, a_{t+1}^m x_{t+1}, ..., a_n^m x_n)\| \\
&\leqslant \max\Big\{ \max\Big\{ \max\{ \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m} 2a_{t+1}^{3(m+1)}...a_k^{3(m+1)}a_{k+1}^{3m}...a_n^{3m}|} \\
&\quad \varphi_k(2^m a_1^m x_1, y_1, ..., 2^m a_t^m x_t, y_t, a_{t+1}^{m+1}x_{t+1}, y_{t+1}, ..., a_{k-1}^{m+1}x_{k-1}, y_{k-1}, a_k^m x_k, 0, a_{k+1}^m x_{k+1} \\
&\quad , y_{k+1}, ..., a_n^m x_n, y_n) : k = t+1, ..., n\}, \\
&\frac{1}{|a_{t+1}^3...a_n^3|} \max\{ \frac{1}{|2^{sm}a_1^m...a_s^m 4^{j-s+(t-s)m}a_{s+1}^{2(m+1)}...a_j^{2(m+1)}a_{j+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|} \\
&\quad \varphi_j(2^m a_1^m x_1, y_1, ..., 2^m a_s^m x_s, y_s, 2^{m+1}a_{s+1}^{m+1}x_{s+1}, y_{s+1}, ..., 2^{m+1}a_{j-1}^{m+1}x_{j-1}, y_{j-1}, 2^m a_j^m x_j, \\
&\quad 2^m a_j^m x_j, 2^m a_{j+1}^m x_{j+1}, y_{j+1}, ..., 2^m a_t^m x_t, y_t, a_{t+1}^{m+1}x_{t+1}, y_{t+1}, ..., a_n^{m+1}x_n, y_n) : j = s+1, ..., t\}, \\
&\frac{1}{|4^{t-s}a_{s+1}^2...a_t^2 a_{t+1}^3...a_n^3|} \max\{ \frac{1}{|2^{i+sm}a_1^{m+1}...a_i^{m+1}a_{i+1}^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|} \\
&\varphi_i(2^{m+1}a_1^{m+1}x_1, y_1, ..., 2^{m+1}a_{i-1}^{m+1}x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_i^m x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1} \\
&, ..., 2^m a_s^m x_s, y_s, 2^{m+1}a_{s+1}^{m+1}x_{s+1}, y_{s+1}, ..., 2^{m+1}a_t^{m+1}x_t, y_t, a_{t+1}^{m+1}x_{t+1}, y_{t+1}, ..., a_n^{m+1}x_n, y_n) \\
&\quad : i = 1, ..., s\}\Big\} : m = 0, 1, ..., p \Big\}
\end{aligned} \tag{18}$$

for all $x_i, y_i \in G$, $i = 1, ..., n$ and $p \in \mathbb{N}\cup\{0\}$. By taking p to approach infinity in (18) and using (7) one obtains (9). For $i \in \{1, 2, ..., s\}$ and by (10) and (17), we get

$$\begin{aligned}
&\|T(x_1, x_2, ..., a_i x_i + b_i y_i, ..., x_n) - a_i T(x_1, x_2, ..., x_i, ..., x_n) - b_i T(x_1, x_2, ..., y_i, ..., x_n)\| \\
&= \lim_{m\to\infty} \| \frac{1}{2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} \\
&\quad f(2^m a_1^m x_1, ..., 2^m a_i^m (a_i x_i + b_i y_i), ..., 2^m a_t^m x_t, a_{t+1}^m x_{t+1}, ..., a_n^m x_n) \\
&\quad - \frac{a_i}{2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} f(2^m a_1^m x_1, ..., 2^m a_i^m x_i, ..., 2^m a_t^m x_t, a_{t+1}^m x_{t+1}, ..., a_n^m x_n) \\
&\quad - \frac{b_i}{2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}} f(2^m a_1^m x_1, ..., 2^m a_i^m y_i, ..., 2^m a_t^m x_t, a_{t+1}^m x_{t+1}, ..., a_n^m x_n)\| \\
&\leqslant \lim_{m\to\infty} \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|} \varphi_i(2^m a_1^m x_1, y_1, ..., 2^m a_i^m x_i, 2^m a_i^m y_i, ..., 2^m a_t^m x_t, y_t, \\
&\quad a_{t+1}^m x_{t+1}, y_{t+1}, ..., a_n^m x_n, y_n).
\end{aligned} \tag{19}$$

It follows from (12) and (17) that

$$\begin{aligned}
&\|T(x_1, x_2, ..., a_jx_j + b_jy_j, ..., x_n) + T(x_1, x_2, ..., a_jx_j - b_jy_j, ..., x_n) - 2a_j^2T(x_1, x_2, ..., x_j, ..., x_n)\\
&\quad - 2b_j^2T(x_1, x_2, ..., y_j, ..., x_n)\| = \lim_{m\to\infty} \|\frac{1}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}\\
&\quad f(2^ma_1^mx_1, ..., 2^ma_j^m(a_jx_j + b_jy_j), ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_n^mx_n)\\
&\quad + \frac{1}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}\\
&\quad f(2^ma_1^mx_1, ..., 2^ma_j^m(a_jx_j - b_jy_j), ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_n^mx_n)\\
&\quad - \frac{2a_j^2}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}f(2^ma_1^mx_1, ..., 2^ma_j^mx_j, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_n^mx_n)\\
&\quad - \frac{2b_j^2}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}f(2^ma_1^mx_1, ..., 2^ma_j^my_j, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_n^mx_n)\|\\
&\leqslant \lim_{m\to\infty} \frac{1}{|2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|}\\
&\quad \varphi_j(2^ma_1^mx_1, y_1, ..., 2^ma_j^mx_j, 2^ma_j^my_j, ..., 2^ma_t^mx_t, y_t, a_{t+1}^mx_{t+1}, y_{t+1}, ..., a_n^mx_n, y_n)
\end{aligned} \tag{20}$$

for all $j \in \{s+1, s+2, ..., t\}$. By (14) and (17), we get

$$\begin{aligned}
&\|T(x_1, ..., a_kx_k + b_ky_k, ..., x_n) + T(x_1, ..., a_kx_k - b_ky_k, ..., x_n) - a_kb_k^2(T(x_1, ..., x_k + y_k, ..., x_n)\\
&\quad + T(x_1, x_2, ..., x_k - y_k, ..., x_n)) - 2a_k(a_k^2 - b_k^2)f(x_1, x_2, ..., x_n)\|\\
&= \lim_{m\to\infty} \|\frac{1}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}\\
&\quad f(2^ma_1^mx_1, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_k^m(a_kx_k + b_ky_k), ..., a_n^mx_n)\\
&\quad + \frac{1}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}\\
&\quad f(2^ma_1^mx_1, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_k^m(a_kx_k - b_ky_k), ..., a_n^mx_n)\\
&\quad - \frac{a_kb_k^2}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}\\
&\quad (f(2^ma_1^mx_1, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_k^mx_k + a_k^my_k, ..., a_n^mx_n)\\
&\quad + f(2^ma_1^mx_1, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_k^mx_k - a_k^my_k, ..., a_n^mx_n))\\
&\quad - \frac{2a_k(a_k^2 - b_k^2)}{2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}}f(2^ma_1^mx_1, ..., 2^ma_t^mx_t, a_{t+1}^mx_{t+1}, ..., a_n^mx_n)\|\\
&\leqslant \lim_{m\to\infty} \frac{1}{|2^{sm}a_1^m...a_s^m4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|}\\
&\quad \varphi_k(2^ma_1^mx_1, y_1, ..., 2^ma_t^mx_t, y_t, a_{t+1}^mx_{t+1}, y_{t+1}, ..., a_k^mx_k, a_k^my_k, ..., a_n^mx_n, y_n)
\end{aligned} \tag{21}$$

for all $k \in \{t+1, t+2, ..., n\}$. By (5), (19), (20) and (21), we conclude that T satisfies (4).

To prove the uniqueness property of T, Let $T' : G^n \to X$ be a mapping $T' : G^n \to X$

which satisfies (4) and (9). Then we have

$$\begin{aligned}
&\|T(x_1,x_2,...,x_n)-T'(x_1,x_2,...,x_n)\|\\
&\leqslant \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|}\max\Big\{\|T(2^m a_1^m x_1,...,2^m a_t^m x_t,a_{t+1}^m x_{t+1},...,a_n^m x_n)\\
&\quad - f(2^m a_1^m x_1,...,2^m a_t^m x_t,a_{t+1}^m x_{t+1},...,a_n^m x_n)\|,\|f(2^m a_1^m x_1,...,2^m a_t^m x_t,a_{t+1}^m x_{t+1},...,a_n^m x_n)\\
&\quad - T'(2^m a_1^m x_1,...,2^m a_t^m x_t,a_{t+1}^m x_{t+1},...,a_n^m x_n)\|\Big\}\\
&\leqslant \frac{1}{|2^{sm}a_1^m...a_s^m 4^{(t-s)m}a_{s+1}^{2m}...a_t^{2m}a_{t+1}^{3m}...a_n^{3m}|}\max\Big\{\Phi(2^m a_1^m x_1,y_1,...,2^m a_t^m x_t,y_t,a_{t+1}^m x_{t+1},\\
&\quad y_{t+1},...,a_n^m x_n,y_n)\ ,\Phi(2^m a_1^m x_1,y_1,...,2^m a_t^m x_t,y_t,a_{t+1}^m x_{t+1},y_{t+1},...,a_n^m x_n,y_n)\Big\},
\end{aligned}$$

which tends to zero as $m\to\infty$ by (7). Therefore $T=T'$. This completes the proof. ■

The following theorem, general system of additive functional equations, is especial case of Theorem 2.1, which generalizes results of Ciepliński [3].

Theorem 2.2 Let $\varphi_i : G^{2n}\to[0,\infty)$ be a function such that

$$\lim_{m\to\infty}\frac{1}{|2^{nm}a_1^m...a_n^m|}\varphi_i(2^m a_1^m x_1,y_1,...,2^m a_i^m x_i,2^m a_i^m y_i,...,2^m a_n^m x_n,y_n)=0$$

for $i\in\{1,...,n\}$. Also, assume that

$$\begin{aligned}
\lim_{p\to\infty}\max\{&\frac{1}{|2^{i+np}a_1^{p+1}...a_i^{p+1}a_{i+1}^p...a_n^p|}\varphi_i(2^{p+1}a_1^{p+1}x_1,y_1,...,2^{p+1}a_{i-1}^{p+1}x_{i-1},\\
&y_{i-1},2^p a_i^p x_i,2^p a_i^p x_i,2^p a_{i+1}^p x_{i+1},y_{i+1},...,2^p a_n^p x_n,y_n):i=1,...,n\}=0
\end{aligned}$$

and

$$\begin{aligned}
\Phi(x_1,y_1,...,x_n,y_n)=\lim_{t\to\infty}\max\Big\{&\max\{\frac{1}{|2^{i+np}a_1^{p+1}...a_i^{p+1}a_{i+1}^p...a_n^p|}\\
&\varphi_i(2^{p+1}a_1^{p+1}x_1,y_1,...,2^{p+1}a_{i-1}^{p+1}x_{i-1},y_{i-1},2^p a_i^p x_i,2^p a_i^p x_i,2^p a_{i+1}^p x_{i+1},\\
&y_{i+1},...,2^p a_n^p x_n,y_n):i=1,...,n\},\ p=0,1,...,t\Big\}<\infty,
\end{aligned}$$

for all $x_i,y_i\in G$ and $a_i,b_i\in\mathbb{Z}\setminus\{0\}$, where $i=1,...,n$. Let $f:G^n\to X$ be a mapping satisfying

$$\begin{cases}
\|f(a_1x_1+b_1y_1,x_2,...,x_n)-a_1f(x_1,x_2,...,x_n)-b_1f(y_1,x_2,...,x_n)\|\\
\leqslant\varphi_1(x_1,y_1,x_2,y_2,...,x_n,y_n),\\
\quad\vdots\\
\|f(x_1,x_2,...,a_nx_n+b_ny_n)-a_nf(x_1,x_2,...,x_n)-b_nf(x_1,x_2,...,y_n)\|\\
\leqslant\varphi_n(x_1,y_1,x_2,y_2,...,x_n,y_n),
\end{cases}$$

for all $x_i,y_i\in G$ and $a_i,b_i\in\mathbb{Z}\setminus\{0\},\ i=1,...,n,$. Then there exists a unique mapping $T:G^n\to X$ satisfying additive system and

$$\|f(x_1,...,x_n)-T(x_1,...,x_n)\|\leqslant\Phi(x_1,y_1,...,x_n,y_n)$$

for all $x_i,y_i\in G,\ i=1,...,n.$

Park [19] proved the below theorem which is especial case of Theorem 2.1.

Theorem 2.3 Let B_i be normed spaces for $i = 1, ..., d$ and D be a Banach space. Let $f : \prod_{i=1}^{d} B_i \to D$ be a mapping for which there exists a function $\varphi : \prod_{i=1}^{d} B_i^2 \to [0, \infty)$ such that

$$\Phi(x_1, y_1, ..., x_d, y_d) = \sum_{j=0}^{\infty} \sum_{i=1}^{d} \frac{1}{4^{i+jd}} \varphi(2^{j+1}x_1, 0, ..., 2^{j+1}x_{i-1}, 0, 2^j x_i, 2^j y_i, 2^j x_{i+1}, 0, ..., 2^j x_d, 0) < \infty,$$

and

$$\| \sum_{i=1}^{d} f(x_1, ..., x_{i-1}, x_i + y_i, x_{i+1}, ..., x_n) + \sum_{i=1}^{d} f(x_1, ..., x_{i-1}, x_i - y_i, x_{i+1}, ..., x_n)$$
$$- 2df(x_1, x_2, ..., x_i, ..., x_n) - \sum_{i=1}^{d} 2f(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \| < \varphi(x_1, y_1, ..., x_d, y_d),$$

for all $x_i, y_i \in B_i$, $i = 1, ..., d$. Assume that $f(x_1, ..., x_n) = 0$ if $x_i = 0$ for any $i = 1, ..., d$. Then there exists a unique multi-quadratic mapping $M : \prod_{i=1}^{d} B_i \to D$ such that

$$\|f(x_1, ..., x_d) - M(x_1, ..., x_d)\| \leqslant \Phi(x_1, y_1, ..., x_d, y_d),$$

for all $x_i \in B_i$, $i = 1, ..., d$.

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