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Computational aspect to the nearest southeast submatrix that makes multiple a prescribed eigenvalue

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Abstract. Given four complex matrices A, B, C and D where $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ and let the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a normal matrix and assume that λ is a given complex number that is not eigenvalue of matrix A. We present a method to calculate the distance norm (with respect to 2-norm) from D to the set of matrices $X \in C^{m\times m}$ such that, λ be a multiple eigenvalue of matrix $\begin{pmatrix} A & B \\ C & X \end{pmatrix}$. We also find the nearest matrix X to the matrix D.

Keywords: Normal matrix, multiple eigenvalues, singular value, distance matrices.

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1. Introduction

In paper[4], A.N. Malyshev obtained the following formula for the 2-norm distance $\mathrm{rsep}(A)$ from a complex $n \times n$ matrix to a closest matrix with a multiple eigenvalue:

$$\mathrm{rsep}(A) = \min_{\lambda \in \mathbf{C}} \max_{\gamma \geq 0} \sigma_{2n-1}(G(\gamma)),$$

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where,

$$G(\gamma) = \begin{pmatrix} \lambda I - A & \gamma I \\ 0 & \lambda I - A \end{pmatrix},$$

and $\sigma_{2n-1}(G(\gamma))$ is the penultimate singular value of the matrix G, assuming that the singular values are numbered in decreasing order. Ikramov and Nazari in [2] introduced a correction for Malyshev's formula for a normal matrix. In recent paper [5] Nazari and Rajabi used the same correction to [2] for the paper of Lippert [3] for normal matrices.

In the recent paper [1], Gracia and Velasco obtained the following formula for the 2-norm distance D from a complex $m \times m$ matrix to a closest matrix with a multiple eigenvalue:

$$\min_{X \in C^{m\times m}, m(\lambda, M(\alpha,X)) \geqslant 2} \parallel X - D \parallel = \sup_{t\in R} \sigma_{2m-1}(S_2(t)), \tag{1}$$

where

$$M(\alpha, X) = \begin{pmatrix} A & B \\ C & X \end{pmatrix}, \qquad \text{and} \qquad S_2(t) = \begin{pmatrix} M & tN \\ 0 & M \end{pmatrix},$$

that

$$M = (D - \lambda I_m) - C(A - \lambda I_n)^{-1}B, \tag{2}$$

$$N = I_m + C(A - \lambda I_n)^{-2}B, \tag{3}$$

where λ is not eigenvalue of matrix A and $\sigma_{2m-1}(S_2(t))$ is the penultimate singular value of the matrix $S_2(t)$, where $\alpha = (A, B, C) \in L_{n,m} = \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times n}$. In the two Theorems that follows, we briefly describe the article of Gracia and Velasco.

Theorem 1.1 Let $t^\star > 0$ be a local optimizer of function $s_2(t) = \sigma_{2m-1}(S_2(t))$. Suppose $\sigma^\star = s_2(t^\star) > 0$, then there exists a pair of normalized singular vectors associated with the singular value $t^\star$ of $s_2(t^\star)$, namely a left vector

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1, u_2 \in \mathbb{C}^m$$

and a corresponding right vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbb{C}^m$$

such that

$$\mathrm{Re}(u_1^\star N v_2) = 0. \tag{4}$$

where the matrix N is defined by (3). Moreover, the matrices

$$U = \begin{pmatrix} u_1 \ u_2 \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 \ v_2 \end{pmatrix}, \tag{5}$$

satisfy the relation

$$V^{\star}V = U^{\star}U \in \mathbb{C}^{2\times 2}. \tag{6}$$

Theorem 1.2 Let λ is not eigenvalue of matrix A and $t^{\star}$ in Theorem (1.1) is a positive number. The matrix $D+\Delta$, where

$$\Delta = -\sigma^{\star} U V^{\dagger}, \tag{7}$$

is the closest (with respect to the 2-norm) matrix to matrix X, such that the matrix $\begin{pmatrix} A & B \\ C & X \end{pmatrix}$ having multiple eigenvalue λ and

$$\| \Delta \|_2 = \sigma^{\star}, \tag{8}$$

where denote by $V^{\dagger}$ the Moore-Penrose inverse matrix of V.

Let $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. By a similar method that introduced in [2], we discuss some issues related to the computer implementation of this method. It turns out that the case of a general matrix G is substantially different from that of a normal matrix G.

2. Normal matrix

Let G be a normal matrix. Let

$$A = \begin{pmatrix} 12 & 7 & 7 \\ 7 & 16 & 10 \\ 7 & 10 & 12 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 5 \\ 3 & 3 \\ 11 & 11 \end{pmatrix},$$

$$C = \begin{pmatrix} 5 & 3 & 11 \\ 5 & 3 & 11 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix},$$

then it is easy to see that the matrix $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is normal matrix. Assume that $\lambda = 0$. By MATLAB software with long format for computation, we found the values $t^{\star} = 7.60093750000001$, $\sigma^{\star} = 7.60093621516323$. The singular value σ_{2m-2} equals 7.60093750000000. These two values are approximately the same, namely

$$\sigma_{2m-1}(S_2(t^{\star})) \simeq \sigma_{2m-2}(S_2(t^{\star})).$$

Thus, in the optimal matrix $S_2(t^{\star})$, the value $\sigma^{\star}$ is iterated. Let $u^{(2m-1)}$, $v^{(2m-1)}$ and $u^{(2m-2)}$, $v^{(2m-2)}$ be the pairs of singular vectors of $S_2(t^{\star})$ associated with σ_{2m-1} and σ_{2m-2}, respectively, that MATLAB gives us. An attempt to use any of these pairs for implementing the construction described in Theorem (1.2) leads to catastrophic results. Namely, for the matrix

$$\Delta^{2m-1} = -\sigma^{\star} U^{(2m-1)} V^{(2m-1)\dagger},$$

we obtain

$$\| \Delta^{2m-1} \|= 2.345558000417827 \times 10^{16},$$

while $\Delta^{2m-2} = \sigma^{\star} U^{(2m-2)} V^{(2m-2)\dagger}$ has the norm

$$\| \Delta^{2m-2} \|= 2.345558396903085 \times 10^{16}.$$

It is easy to find the reason why equality (8) is violated in both cases. The value of $u_1^{\star 2m-1} N v_2^{2m-1}$, for two vectors $u^{(2m-1)} = \begin{pmatrix} u_1^{2m-1} \\ u_2^{2m-1} \end{pmatrix}$ and $v^{(2m-1)} = \begin{pmatrix} v_1^{2m-1} \\ u_2^{2m-1} \end{pmatrix}$, is

$$-0.78923059741806$$

and for the pair vectors $u^{(2m-2)} = \begin{pmatrix} u_1^{2m-2} \\ u_2^{2m-2} \end{pmatrix}$ and $v^{(2m-1)} = \begin{pmatrix} v_1^{2m-2} \\ u_2^{2m-2} \end{pmatrix}$, is

$$1.00000000000000.$$

In any case above, equality (4), even approximately does not hold. It follows that equality (6) is violated.i.e.

$$U^{*}U = \begin{pmatrix} 0.10538463458652 \; 0.30704838931280 \\ 0.30704838931280 \; 0.89461536541348 \end{pmatrix},$$

$$V^{*}V = \begin{pmatrix} 0.89461536541348 \; 0.30704838931280 \\ 0.30704838931280 \; 0.10538463458652 \end{pmatrix}.$$

If we calculate the eigenvalues of matrix G, we see that

$$\text{The eigenvalues of matrix } G = \\ 10^{7} \times \left\{ \begin{array}{c} 0.00000371178081, \\ 0.00000065233548, \\ -0.00000139525368, \\ 0.00000069856511 + 4.06720386782124i, \\ 0.00000069856511 - 4.06720386782124i \end{array} \right\}.$$

Since $\lambda = 0$, by Theorem (1.2) we must have a multiple eigenvalue zero in matrix G, and all of eigenvalue that calculated above far from zero.

The situation can be rectified as follows. Consider the number

$$\sigma^{\star} = \sigma_{2m-1}(S_2(t))$$

as a double singular value of $S_2(t)$ and the vectors $u^{(2m-1)}$ and $u^{(2m-2)}$ as an orthonormal basis in the left singular subspace associated with $\sigma^{\star}$. In this subspace, we look for a normalized vector

$$u = \alpha u^{(2m-1)} + \beta u^{(2m-2)}, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{9}$$

and combined with the associated right singular vector

$$v = \alpha v^{(2m-1)} + \beta v^{(2m-2)} \tag{10}$$

in order to satisfy relation (4). From (4) we have

$$\mathrm{Re}(u_1^\star N v_2) = 0. \tag{11}$$

Substituting (9) and (10) into (11), we achieve the relation

$$\left(\bar{\alpha}\ \bar{\beta}\right) \mathrm{Re}W \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \tag{12}$$

in which

$$W = \begin{pmatrix} u_1^{(2m-1)H} * N * v_2^{(2m-1)} & u_1^{(2m-1)H} * N * v_2^{(2m-2)} \\ u_1^{(2m-2)H} * N * v_2^{(2m-1)} & u_1^{(2m-2)H} * N * v_2^{(2m-2)} \end{pmatrix} \tag{13}$$

and

$$W_r = \mathrm{Re}(W) = \begin{pmatrix} \mathrm{Re}(W_{11}) & \frac{(\bar{W}_{21}+W_{12})}{2} \\ \frac{(\bar{W}_{12}+W_{21})}{2} & \mathrm{Re}(W_{22}) \end{pmatrix} \tag{14}$$

The existence of a nontrivial solution for Eq. (12) is ensured by the fact that the Hermitian matrix (12) is indefinite. In fact, let us call $g(t) = \sigma_{2m-2}(S_2(t))$. Let $\mu_1 \geqslant \mu_2$ be the eigenvalues of the matrix ReW. Then the right derivatives of the functions S_2 and g at $t^\star$ are equal to μ_2 and μ_1

$$S_2^{'}(t^{\star +}) = \mu_2, \qquad g^{'}(t^{\star +}) = \mu_1$$

respectively. Since S_2 is decreasing and g is increasing at right of $t^\star$, we deduce that

$$\mu_2 < 0 \text{ and } \mu_1 > 0.$$

The numbers α and β can be found, for example, in following manner. Let

$$W_r = PMP^\star, \qquad M = \mathrm{diag}(\mu_1, \mu_2),$$

be the spectral decomposition of W. Set

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} T \\ K \end{pmatrix}, \tag{15}$$

and recast (12) as

$$\mu_1 |t|^2 + \mu_2 |\sigma|^2 = 0, \quad |t|^2 + |\sigma|^2 = 1. \tag{16}$$

The pair

$$\left(\frac{|\mu_2|}{|\mu_1| + |\mu_2|}\right)^{\frac{1}{2}}, \qquad \left(\frac{|\mu_1|}{|\mu_1| + |\mu_2|}\right)^{\frac{1}{2}}$$

is a solution to system (16). (Recall again that μ_1 and μ_2 are numbers of different signs.) Using (15), we obtain the corresponding pair α, β. In the example above with matrix G, this technique yields

$$\alpha = -0.74759578100550, \qquad \beta = 0.66415400941556$$

For the corresponding singular vectors (9) and (10), we have

$$u_1^{\star} N v_2 = -2.238877486182567 \times 10^{-17}$$

The matrix Δ constructed from these vectors has the norm

$$7.60093681855830$$

which is in very good agreement with $\sigma^{\star}$. Finally, we found

$$U^{\star}U = \begin{pmatrix} 0.50000003728108 & 0.17160917645601 \\ 0.17160917645601 & 0.49999996271892 \end{pmatrix},$$

$$V^{\star}V = \begin{pmatrix} 0.49999996271892 & 0.17160917645601 \\ 0.17160917645601 & 0.50000003728108 \end{pmatrix}$$

it follows that $U^{\star}U \simeq V^{\star}V$ and the eigenvalues of matrix G as follows:

$$\text{The eigenvalues of matrix } G = \begin{Bmatrix} 38.23444569157454, \\ 8.77515531971971, \\ 6.20796134531682, \\ 0.00000000420761, \\ -0.00000113313609 \end{Bmatrix}.$$

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