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Unique common coupled fixed point theorem for four maps in S_b-metric spaces

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Abstract. In this paper we prove a unique common coupled fixed point theorem for two pairs of w-compatible mappings in S_b-metric spaces satisfying a contractive type condition. We furnish an example to support our main theorem. We also give a corollary for Jungck type maps.

Keywords: S_b-Metric space, w-compatible pairs, S_b-completeness, coupled fixed point.

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1. Introduction

In 2012, Sedghi et al. [10] introduced the notion of S-metric space and proved several results, for example, refer [7, 11]. On the other hand, the concept of b-metric space was introduced by Czerwik [2].

Recently, Sedghi et al. [8] defined S_b-metric spaces by using the concepts of S and b-metric spaces and proved common fixed point theorem for four maps in S_b-metric spaces.

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Bhaskar and Lakshmikantham [3] introduced the notion of coupled fixed point and proved some coupled fixed point results . Several authors proved coupled fixed point theorems in various spaces,for example, see the references[5, 6] and the references therein.

The aim of this paper is to prove a unique common coupled fixed point theorem for four mappings in S_b-metric spaces. Throughout this paper $\mathcal{R}^+$ and $\mathcal{N}$ denote the set of all non-negative real numbers and positive integers respectively. First we recall some definitions, lemmas and examples.

Definition 1.1 ([10]) Let X be a non-empty set. A S-metric on X is a function $S : X^3 \to \mathcal{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$,

$(S1):$ $0 < S(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,
$(S2):$ $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
$(S3):$ $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

Then the pair (X, S) is called a S-metric space.

Definition 1.2 ([2]) Let X be a non-empty set and $s \geqslant 1$ a given real number. A function $d : X \times X \to \mathcal{R}^+$ is called a b-metric if the following axioms are satisfied for all $x, y, z \in X$,

(b_1) $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space.

Definition 1.3 ([8]) Let X be a non-empty set and $b \geqslant 1$ be given real number. Suppose that a mapping $S_b : X^3 \to \mathcal{R}^+$ be a function satisfying the following properties:

(S_b1) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z$,
(S_b2) $S_b(x, y, z) = 0 \Leftrightarrow x = y = z$,
(S_b3) $S_b(x, y, z) \leqslant b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$ for all $x, y, z, a \in X$.

Then the function S_b is called a S_b-metric on X and the pair (X, S_b) is called a S_b-metric space.

Remark 1 *([8])It should be noted that, the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed each S-metric space is a S_b-metric space with $b = 1$.*

Following example shows that a S_b-metric on X need not be a S-metric on X.

***Example* 1.4** ([8]) Let (X, S_b) be a S_b-metric space and $S_b(x, y, z) = S(x, y, z)^p$, where $p > 1$ is a real number. Note that S_b is a S_b-metric with $b = 2^{2(p-1)}$. Also, (X, S_b) is not necessarily a S-metric space.

Definition 1.5 ([8]) Let (X, S_b) be a S_b-metric space. Then, for $x \in X$ and $r > 0$, we defined the open ball $B_{S_b}(x, r)$ and closed ball $B_{S_b}[x, r]$ with center x and radius r as follows respectively:

$$B_{S_b}(x, r) = \{y \in X : S_b(y, y, x) < r\},$$
$$B_{S_b}[x, r] = \{y \in X : S_b(y, y, x) \leq r\}.$$

Lemma 1.6 ([8])In a S_b-metric space, we have

$$S_b(x, x, y) \leq bS_b(y, y, x)$$

and

$$S_b(y, y, x) \leq bS_b(x, x, y).$$

Lemma 1.7 ([8])In a S_b-metric space, we have

$$S_b(x, x, z) \leq 2bS_b(x, x, y) + b^2 S_b(y, y, z).$$

Definition 1.8 ([8])If (X, S_b) be a S_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(x_n, x_n, x_m) < \epsilon$ for each $m, n \geqslant n_0$.
(2) S_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(x_n, x_n, x) < \epsilon$ or $S_b(x, , x, x_n) < \epsilon$ for all $n \geqslant n_0$ and we denote by $\lim_{n\to\infty} x_n = x$.

Definition 1.9 ([8]) A S_b-metric space (X, S_b) is called complete if every S_b-Cauchy sequence is S_b-convergent in X.

Lemma 1.10 ([9]) If (X, S_b) be a S_b-metric space with $b \geqslant 1$ and suppose that $\{x_n\}$ is a S_b-convergent to x, then we have
(i) $\frac{1}{2b} S_b(y, x, x) \leq \lim_{n\to\infty} \inf\ S_b(y, y, x_n) \leq \lim_{n\to\infty} \sup\ S_b(y, y, x_n) \leq 2bS_b(y, y, x)$
and
(ii) $\frac{1}{b^2} S_b(x, x, y) \leq \lim_{n\to\infty} \inf\ S_b(x_n, x_n, y) \leq \lim_{n\to\infty} \sup\ S_b(x_n, x_n, y) \leq b^2 S_b(x, x, y)$ for all $y \in X$.
In particular, if $x = y$, then we have $\lim_{n\to\infty} S_b(x_n, x_n, y) = 0$.

Definition 1.11 ([3]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.12 ([4]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called

(i) a coupled coincident point of mappings $F : X \times X \to X$ and $f : X \to X$ if $fx = F(x, y)$ and $fy = F(y, x)$,
(ii) a common coupled fixed point of mappings $F : X \times X \to X$ and $f : X \to X$ if $x = fx = F(x, y)$ and $y = fy = F(y, x)$.

Definition 1.13 ([1]) Let X be a nonempty set and $F : X \times X \to X$ and $f : X \to X$.Then $\{F, f\}$ is said to be w-compatible pair if $f(F(x, y)) = F(fx, fy)$ and $f(F(y, x)) = F(fy, fx)$ whenever there exist $x, y \in X$ with $fx = F(x, y)$ and $fy = F(y, x)$.

2. Main Result

Now we give our main result.

Theorem 2.1 Let (X, S_b) be a S_b-metric space. Suppose that $f, g : X \times X \to X$ and $F, G : X \to X$ be satisfying

(2.1.1) $f(X \times X) \subseteq G(X), g(X \times X) \subseteq F(X)$,
(2.1.2) $\{f, F\}$ and $\{g, G\}$ are w-compatible pairs,
(2.1.3) One of $F(X)$ or $G(X)$ is S_b-complete subspace of X,

(2.1.4) $S_b\big(f(x,y), f(x,y), g(u,v)\big)$

$$\leqslant k \max \left\{ \begin{array}{c} S_b(Fx,Fx,Gu), S_b(Fy,Fy,Gv), S_b(f(x,y),f(x,y),Fx), \\ S_b(f(y,x),f(y,x),Fy), S_b(g(u,v),g(u,v),Gu), S_b(g(v,u),g(v,u),Gv), \\ \frac{1}{4b^2}\left[S_b(f(x,y),f(x,y),Gu) + S_b(g(u,v),g(u,v),Fx) \right], \\ \frac{1}{4b^2}\left[S_b(f(y,x),f(y,x),Gv) + S_b(g(v,u),g(v,u),Fy) \right] \end{array} \right\},$$

for all $x, y, u, v \in X$, where $0 \leq k < \frac{1}{4b^5}$.

Then f, g, F and G have a unique common coupled fixed point in $X \times X$.

Proof. Let $x_0, y_0 \in X$. From (2.1.1), we can construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ such that

$$\begin{aligned} f(x_{2n}, y_{2n}) &= Gx_{2n+1} = z_{2n}, \\ f(y_{2n}, x_{2n}) &= Gy_{2n+1} = w_{2n}, \\ g(x_{2n+1}, y_{2n+1}) &= Fx_{2n+2} = z_{2n+1}, \\ g(y_{2n+1}, x_{2n+1}) &= Fy_{2n+2} = w_{2n+1}, \quad n = 0, 1, 2, \cdots. \end{aligned}$$

Case(i): Suppose $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$ for some m. Put

$$S_{2m} = \max\{S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\}.$$

From (2.1.4), we have

$$\begin{aligned} &S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}) \\ &\quad = S_b(f(x_{2m+2}, y_{2m+2}), f(x_{2m+2}, y_{2m+2}), g(x_{2m+1}, y_{2m+1})) \\ &\quad \leqslant k \max \left\{ \begin{array}{c} S_b(Fx_{2m+2}, Fx_{2m+2}, Gx_{2m+1}), S_b(Fy_{2m+2}, Fy_{2m+2}, Gy_{2m+1}), \\ S_b(f(x_{2m+2}, y_{2m+2}), f(x_{2m+2}, y_{2m+2}), Fx_{2m+2}), \\ S_b(f(y_{2m+2}, x_{2m+2}), f(y_{2m+2}, x_{2m+2}), Fy_{2m+2}), \\ S_b(g(x_{2m+1}, y_{2m+1}), g(x_{2m+1}, y_{2m+1}), Gx_{2m+1}), \\ S_b(g(y_{2m+1}, x_{2m+1}), g(y_{2m+1}, x_{2m+1}), Gy_{2m+1}), \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(x_{2m+2}, y_{2m+2}), f(x_{2m+2}, y_{2m+2}), Gx_{2m+1}) \\ +S_b(g(x_{2m+1}, y_{2m+1}), g(x_{2m+1}, y_{2m+1}), Fx_{2m+2}) \end{array} \right], \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(y_{2m+2}, x_{2m+2}), f(y_{2m+2}, x_{2m+2}), Gy_{2m+1}) \\ +S_b(g(y_{2m+1}, x_{2m+1}), g(y_{2m+1}, x_{2m+1}), Fy_{2m+2}) \end{array} \right] \end{array} \right\} \\ &\quad = k \max \left\{ \begin{array}{l} S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), S_b(z_{2m+2}, z_{2m+2}, z_{2m+1}), \\ S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}), S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m}), \\ \frac{1}{4b^2}[S_b(z_{2m+2}, z_{2m+2}, z_{2m}) + S_b(z_{2m+1}, z_{2m+1}, z_{2m+1})], \\ \frac{1}{4b^2}[S_b(w_{2m+2}, w_{2m+2}, w_{2m}) + S_b(w_{2m+1}, w_{2m+1}, w_{2m+1})] \end{array} \right\} \\ &\quad \leqslant kS_{2m+1}. \end{aligned}$$

Similarly, we can prove

$$S_b(w_{2m+2}, w_{2m+2}, w_{2m+1}) \leq kS_{2m+1}.$$

It follows that $z_{2m+2} = z_{2m+1}$ and $w_{2m+2} = w_{2m+1}$. Continuing this process we can conclude that $z_{2m+k} = z_{2m}$ and $w_{2m+k} = w_{2m}$ for all $k \geqslant 0$. It follows that $\{z_m\}$ and

$\{w_m\}$ are Cauchy sequences.
Case (ii): Assume that $z_n \neq z_{n+1}$ or $w_n \neq w_{n+1}$ for all n. From (2.1.4), we have

$$\begin{aligned}
&S_b(z_{2n+2}, z_{2n+2}, z_{2n+1})\\
&\quad = S_b\big(f(x_{2n+2}, y_{2n+2}), f(x_{2n+2}, y_{2n+2}), g(x_{2n+1}, y_{2n+1})\big)\\
&\quad \leqslant k \max \left\{ \begin{array}{c} S_b(Fx_{2n+2}, Fx_{2n+2}, Gx_{2n+1}), S_b(Fy_{2n+2}, Fy_{2n+2}, Gy_{2n+1}),\\ S_b(f(x_{2n+2}, y_{2n+2}), f(x_{2n+2}, y_{2n+2}), Fx_{2n+2}),\\ S_b(f(y_{2n+2}, x_{2n+2}), f(y_{2n+2}, x_{2n+2}), Fy_{2n+2}),\\ S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Gx_{2n+1}),\\ S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Gy_{2n+1}),\\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(x_{2n+2}, y_{2n+2}), f(x_{2n+2}, y_{2n+2}), Gx_{2n+1})\\ +S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Fx_{2n+2})\end{array}\right],\\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(y_{2n+2}, x_{2n+2}), f(y_{2n+2}, x_{2n+2}), Gy_{2n+1})\\ +S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Fy_{2n+2})\end{array}\right] \end{array} \right\}
\end{aligned}$$

$$= k \max \left\{ \begin{array}{l} S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}),\\ S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}),\\ \frac{1}{4b^2}[S_b(z_{2n+2}, z_{2n+2}, z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})],\\ \frac{1}{4b^2}[S_b(w_{2n+2}, w_{2n+2}, w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})] \end{array} \right\}.$$

But

$$\begin{aligned}
&\tfrac{1}{4b^2}\,[S_b(z_{2n+2}, z_{2n+2}, z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, z_{2n+1})]\\
&\quad \leq \frac{1}{4b^2}[2bS_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) + bS_b(z_{2n}, z_{2n}, z_{2n+1})]\\
&\quad \leq \frac{1}{4b^2}[2bS_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) + b^2S_b(z_{2n+1}, z_{2n+1}, z_{2n})]\\
&\quad \leq \max\{S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}), S_b(z_{2n+1}, z_{2n+1}, z_{2n})\}\\
&\quad \leq \max\{S_{2n+1}, S_{2n}\}
\end{aligned}$$

Similarly,

$$\tfrac{1}{4b^2}[S_b(w_{2n+2}, w_{2n+2}, w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, w_{2n+1})] \leq \max\{S_{2n+1}, S_{2n}\}.$$

Hence,

$$S_b(z_{2n+2}, z_{2n+2}, z_{2n+1}) \leq k \max\{S_{2n+1}, S_{2n}\}.$$

Similarly,

$$S_b(w_{2n+2}, w_{2n+2}, w_{2n+1}) \leq k \max\{S_{2n+1}, S_{2n}\}.$$

Hence, it is clear that

$$S_{2n+1} \leq k \max\{S_{2n+1}, S_{2n}\}.$$

If S_{2n+1} is maximum, then we get contradiction. Hence S_{2n} is maximum. Therefore

$$S_{2n+1} \leqslant kS_{2n} < S_{2n}. \tag{1}$$

Similarly, we can conclude that $S_{2n} < S_{2n-1}$. Thus, $\{S_n\}$ is non-increasing sequence of non-negative real numbers and hence converges to $r \geqslant 0$. Suppose $r > 0$. Letting $n \to \infty$ in (1), we have

$$r \leq kr < r.$$

It is a contradiction. Hence $r = 0$. Thus,

$$\lim_{n\to\infty} S_b(z_{n+1}, z_{n+1}, z_n) = 0 \tag{2}$$

and

$$\lim_{n\to\infty} S_b(w_{n+1}, w_{n+1}, w_n) = 0. \tag{3}$$

Now, we prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences in (X, S_b). On contrary suppose that $\{z_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy then there exist $\epsilon > 0$ and monotonically increasing sequence of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$.

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \geqslant \epsilon \tag{4}$$

and

$$\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k-2}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k-2})\} < \epsilon. \tag{5}$$

From (4) and (5), we have that

$$\begin{aligned}
\epsilon &\leq \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\leq 2b \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+2}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+2})\} \\
&\quad + b \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k+2}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k+2})\} \\
&\leq 4b^2 \max\{S_b(z_{2m_k}, z_{2m_k}, z_{2m_k+1}), S_b(w_{2m_k}, w_{2m_k}, w_{2m_k+1})\} \\
&\quad\quad + 2b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1})\} \\
&\quad\quad + 2b^2 \max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k+1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k+1})\} \\
&\quad\quad + b^2 \max\{S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1})\}.
\end{aligned} \tag{6}$$

From (2.1.4), we have

$$\begin{aligned}
&S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}) \\
&\quad = S_b(f(x_{2m_k+2}, y_{2m_k+2}), f(x_{2m_k+2}, y_{2m_k+2}), g(x_{2n_k+1}, y_{2n_k+1})) \\
&\quad \leqslant k \max \left\{ \begin{array}{c}
S_b(Fx_{2m_k+2}, Fx_{2m_k+2}, Gx_{2n_k+1}), S_b(Fy_{2m_k+2}, Fy_{2m_k+2}, Gy_{2n_k+1}), \\
S_b(f(x_{2m_k+2}, y_{2m_k+2}), f(x_{2m_k+2}, y_{2m_k+2}), Fx_{2m_k+2}), \\
S_b(f(y_{2m_k+2}, x_{2m_k+2}), f(y_{2m_k+2}, x_{2m_k+2}), Fy_{2m_k+2}), \\
S_b(g(x_{2n_k+1}, y_{2n_k+1}), g(x_{2n_k+1}, y_{2n_k+1}), Gx_{2n_k+1}), \\
S_b(g(y_{2n_k+1}, x_{2n_k+1}), g(y_{2n_k+1}, x_{2n_k+1}), Gy_{2n_k+1}), \\
\frac{1}{4b^2}\left[\begin{array}{l} S_b(f(x_{2m_k+2}, y_{2m_k+2}), f(x_{2m_k+2}, y_{2m_k+2}), Gx_{2n_k+1}) \\ +S_b(g(x_{2n_k+1}, y_{2n_k+1}), g(x_{2n_k+1}, y_{2n_k+1}), Fx_{2m_k+2}) \end{array} \right], \\
\frac{1}{4b^2}\left[\begin{array}{l} S_b(f(y_{2m_k+2}, x_{2m_k+2}), f(y_{2m_k+2}, x_{2m_k+2}), Gy_{2n_k+1}) \\ +S_b(g(y_{2n_k+1}, x_{2n_k+1}), g(y_{2n_k+1}, x_{2n_k+1}), Fy_{2m_k+2}) \end{array} \right]
\end{array} \right\}
\end{aligned}$$

$$= k \max \left\{ \begin{array}{l}
S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\
S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\
S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\
\frac{1}{4b^2}\left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \right], \\
\frac{1}{4b^2}\left[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \right]
\end{array} \right\}.$$

Similarly,

$$\begin{aligned}
&S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \\
&\quad \leqslant k \max \left\{ \begin{array}{l}
S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\
S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\
S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\
\frac{1}{4b^2}\left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \right], \\
\frac{1}{4b^2}\left[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \right]
\end{array} \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\max\left\{ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \right\} \\
&\quad \leqslant k \max \left\{ \begin{array}{l}
S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k}), \\
S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2m_k+1}), \\
S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2n_k}), \\
\frac{1}{4b^2}\left[S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1}) \right], \\
\frac{1}{4b^2}\left[S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1}) \right]
\end{array} \right\}. \quad (7)
\end{aligned}$$

But

$$\begin{aligned}
&\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \\
&\quad \le 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\qquad + b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2m_k}), S_b(w_{2n_k}, w_{2n_k}, w_{2m_k})\} \\
&\quad \le 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\qquad + b^2\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_k}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_k})\} \\
&\quad \le 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\qquad + b^2\left(2b\max\{S_b(z_{2m_k}, z_{2m_k}, z_{2n_{k-2}}), S_b(w_{2m_k}, w_{2m_k}, w_{2n_{k-2}})\}\right) \\
&\qquad + b^2\left(b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_{k-2}}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_{k-2}})\}\right) \\
&\quad < 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\qquad + 2b^3\epsilon + b^3\left(2b\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\}\right) \\
&\qquad + b^3\left(b\max\{S_b(z_{2n_{k-2}}, z_{2n_k-2}, z_{2n_k-1}), S_b(w_{2n_k-2}, w_{2n_k-2}, w_{2n_k-1})\}\right) \\
&\quad \le 2b\max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2m_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2m_k})\} \\
&\qquad + 2b^3\epsilon + 2b^4\max\{S_b(z_{2n_k}, z_{2n_k}, z_{2n_k-1}), S_b(w_{2n_k}, w_{2n_k}, w_{2n_k-1})\} \\
&\qquad + b^5\max\{S_b(z_{2n_k-1}, z_{2n_k-1}, z_{2n_k-2}), S_b(w_{2n_k-1}, w_{2n_k-1}, w_{2n_k-2})\}.
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \max\{S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), S_b(w_{2m_k+1}, w_{2m_k+1}, w_{2n_k})\} \le 2b^3\epsilon.$$

Also

$$\begin{aligned}
&\tfrac{1}{4b^2}\left(S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})\right) \\
&\quad \le \frac{1}{4b^2}\begin{pmatrix} 2b\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}) + b^2\ S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \\ +2b\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}) + b\ S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \end{pmatrix} \\
&\quad \le \frac{1}{b^2}\max\left\{\begin{array}{l} 2bS_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), b^2S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ 2b\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), b\ S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}) \end{array}\right\} \\
&\quad \le \max\left\{\begin{array}{l} 2\ S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2m_k+1}), S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ 2\ S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2n_k}), S_b(z_{2m_k+1}, z_{2m_k+1}, z_{2n_k}), \\ S_b(w_{2m_k+1}, w_{2m_k+1}, z_{2n_k}) \end{array}\right\}.
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \frac{1}{4b^2}(S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k}) + S_b(z_{2n_k+1}, z_{2n_k+1}, z_{2m_k+1})) \le 2b\epsilon.$$

Similarly,

$$\lim_{k\to\infty} \frac{1}{4b^2}(S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k}) + S_b(w_{2n_k+1}, w_{2n_k+1}, w_{2m_k+1})) \le 2b\epsilon.$$

Now, letting $k \to \infty$ in (7), we have

$$\lim_{k\to\infty} \max\left\{ \begin{array}{l} S_b(z_{2m_k+2}, z_{2m_k+2}, z_{2n_k+1}), \\ S_b(w_{2m_k+2}, w_{2m_k+2}, w_{2n_k+1}) \end{array} \right\} \leq k \max\left\{ 2b^3\epsilon, 0, 0, 2b\epsilon, 2b\epsilon \right\}$$
$$= k\ 2b^3\epsilon.$$

Hence, letting $k \to \infty$ in (6), we have

$$\epsilon \leq 0 + 0 + 0 + b^2\ k\ 2b^3\epsilon < \epsilon,$$

it is a contradiction. Hence, $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b-Cauchy sequences. In addition

$$\begin{aligned}
&\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2m+1}), S_b(w_{2n+1}, w_{2n+1}, w_{2m+1})\} \\
&\quad\leqslant 2b\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\
&\qquad + b\max\{S_b(z_{2m+1}, z_{2m+1}, z_{2n}), S_b(w_{2m+1}, w_{2m+1}, w_{2n})\} \\
&\quad\leqslant 2b\max\{S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n})\} \\
&\quad + 2b^2\max\{S_b(z_{2m+1}, z_{2m+1}, z_{2m}), S_b(w_{2m+1}, w_{2m+1}, w_{2m})\} \\
&\quad + b^2\max\{S_b(z_{2n}, z_{2n}, z_{2m}), S_b(w_{2n}, w_{2n}, w_{2m})\}.
\end{aligned}$$

From(2), (3) and since $\{z_{2n}\}$ and $\{w_{2n}\}$ are S_b -Cauchy sequences, it follows that $\{z_{2n+1}\}$ and $\{w_{2n+1}\}$ are also S_b-Cauchy sequences in (X, S_b). Thus, $\{z_n\}$ and $\{w_n\}$ are S_b-Cauchy sequences in (X, S_b). Suppose $F(X)$ is complete subspace of X. Then it follows that $\{z_n\}$ and $\{w_n\}$ converges to α and β respectively in $F(X)$. Thus, there exist u and v in $F(X)$ such that

$$\lim_{n\to\infty} z_n = \alpha = Fu \text{ and } \lim_{n\to\infty} w_n = \beta = Fv. \tag{8}$$

Now, we have to prove that $\alpha = f(u, v)$ and $\beta = f(v, u)$. Using (2.1.4) and Lemma

(1.10), we obtain that

$$
\begin{aligned}
&\frac{1}{2b} S_b(f(u,v), f(u,v), \alpha) \\
&\quad \leqslant \lim_{n\to\infty} \sup\, S_b(f(u,v), f(u,v), g(x_{2n+1}, y_{2n+1})) \\
&\quad \leqslant \lim_{n\to\infty} \sup\, k \max \left\{ \begin{array}{l} S_b(Fu, Fu, Gx_{2n+1}), S_b(Fv, Fv, Gy_{2n+1}), \\ S_b(f(u,v), f(u,v), Fu), S_b(f(v,u), f(v,u), Fv), \\ S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Gx_{2n+1}), \\ S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Gy_{2n+1}), \\ \frac{1}{4b^2} \left[\begin{array}{l} S_b(f(u,v), f(u,v), Gx_{2n+1}) \\ + S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Fu) \end{array} \right], \\ \frac{1}{4b^2} \left[\begin{array}{l} S_b(f(v,u), f(v,u), Gy_{2n+1}) \\ + S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Fv) \end{array} \right] \end{array} \right\} \\
&\quad = \lim_{n\to\infty} \sup\, k \max \left\{ \begin{array}{l} S_b(\alpha, \alpha, z_{2n}), S_b(\beta, \beta, w_{2n}), S_b(f(u,v), f(u,v), \alpha), \\ S_b(f(v,u), f(v,u), \beta), S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{1}{4b^2} \left[S_b(f(u,v), f(u,v), z_{2n}) + S_b(z_{2n+1}, z_{2n+1}, \alpha) \right], \\ \frac{1}{4b^2} \left[S_b(f(v,u), f(v,u), w_{2n}) + S_b(w_{2n+1}, w_{2n+1}, \beta) \right] \end{array} \right\} \\
&\quad \le k \max \left\{ \begin{array}{l} 0, 0, S_b(f(u,v), f(u,v), \alpha), S_b(f(v,u), f(v,u), \beta), 0, 0, \\ \frac{1}{4b^2}[2b\ S_b(f(u,v), f(u,v), \alpha) + 0], \frac{1}{4b^2}[2b\ S_b(f(v,u), f(v,u), \beta) + 0] \end{array} \right\} \\
&\quad = k \max \left\{ S_b(f(u,v), f(u,v), \alpha), S_b(f(v,u), f(v,u), \beta) \right\}.
\end{aligned}
$$

Similarly,

$$
\frac{1}{2b} S_b(f(v,u), f(v,u), \beta) \le k \max \left\{ S_b(f(u,v), f(u,v), \alpha), S_b(f(v,u), f(v,u), \beta) \right\}.
$$

Thus, we have

$$
\frac{1}{2b} \max \left\{ \begin{array}{c} S_b(f(u,v), f(u,v), \alpha), \\ S_b(f(v,u), f(v,u), \beta) \end{array} \right\} \le k \max \left\{ \begin{array}{c} S_b(f(u,v), f(u,v), \alpha), \\ S_b(f(v,u), f(v,u), \beta) \end{array} \right\}.
$$

It follows that $f(u,v) = \alpha$ and $f(v,u) = \beta$. Thus, (α, β) is a coupled coincidence point of f and F. Since $\{f, F\}$ is a w-compatible pair, we have $F\alpha = f(\alpha, \beta)$ and $F\beta = f(\beta, \alpha)$.

From (2.1.4) and Lemma (1.10), we obtain that

$$
\begin{aligned}
&\frac{1}{2b} S_b(f(\alpha,\beta), f(\alpha,\beta), \alpha) \\
&\leqslant \lim_{n\to\infty} \sup\, S_b(f(\alpha,\beta), f(\alpha,\beta), g(x_{2n+1}, y_{2n+1})) \\
&\leq \lim_{n\to\infty} \sup\, k\ \max \left\{ \begin{array}{c} S_b(F\alpha, F\alpha, Gx_{2n+1}), S_b(F\beta, F\beta, Gy_{2n+1}), \\ S_b(f(\alpha,\beta), f(\alpha,\beta), F\alpha), S_b(f(\beta,\alpha), f(\beta,\alpha), F\beta), \\ S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), Gx_{2n+1}), \\ S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), Gy_{2n+1}), \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(\alpha,\beta), f(\alpha,\beta), Gx_{2n+1}) \\ +S_b(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), F\alpha) \end{array} \right], \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(\beta,\alpha), f(\beta,\alpha), Gy_{2n+1}) \\ +S_b(g(y_{2n+1}, x_{2n+1}), g(y_{2n+1}, x_{2n+1}), F\beta) \end{array} \right] \end{array} \right\} \\
&= \lim_{n\to\infty} \sup\, k\ \max \left\{ \begin{array}{c} S_b(F\alpha, F\alpha, z_{2n}), S_b(F\beta, F\beta, w_{2n}), \\ S_b(f(\alpha,\beta), f(\alpha,\beta), F\alpha), S_b(f(\beta,\alpha), f(\beta,\alpha), F\beta), \\ S_b(z_{2n+1}, z_{2n+1}, z_{2n}), S_b(w_{2n+1}, w_{2n+1}, w_{2n}), \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(\alpha,\beta), f(\alpha,\beta), z_{2n}) \\ +S_b(z_{2n+1}, z_{2n+1}, f(\alpha,\beta)) \end{array} \right], \\ \frac{1}{4b^2}\left[\begin{array}{l} S_b(f(\beta,\alpha), f(\beta,\alpha), w_{2n}) \\ +S_b(w_{2n+1}, w_{2n+1}, f(\beta,\alpha)) \end{array} \right] \end{array} \right\} \\
&\leq k\ \max \left\{ \begin{array}{c} 2bS_b(f(\alpha,\beta), f(\alpha,\beta), \alpha), 2bS_b(f(\beta,\alpha), f(\beta,\alpha), \beta), 0, 0, 0, 0, \\ \frac{1}{4b^2}[2bS_b(f(\alpha,\beta), f(\alpha,\beta), \alpha) + b^2 S_b(\alpha, \alpha, f(\alpha,\beta))], \\ \frac{1}{4b^2}[2bS_b(f(\beta,\alpha), f(\beta,\alpha), \beta) + b^2 S_b(\beta, \beta, f(\beta,\alpha))] \end{array} \right\} \\
&= 2b\ k\ \max \left\{ S_b(f(\alpha,\beta), f(\alpha,\beta), \alpha), S_b(f(\beta,\alpha), f(\beta,\alpha), \beta) \right\}.
\end{aligned}
$$

Similarly,

$$
\frac{1}{2b} S_b(f(\beta,\alpha), f(\beta,\alpha), \beta) \leq 2b\ k\ \max \left\{ \begin{array}{l} S_b(f(\alpha,\beta), f(\alpha,\beta), \alpha), \\ S_b(f(\beta,\alpha), f(\beta,\alpha), \beta) \end{array} \right\}.
$$

Hence,

$$
\frac{1}{2b} \max \left\{ \begin{array}{l} S_b(f(\alpha,\beta), f(\alpha,\beta), \alpha), \\ S_b(f(\beta,\alpha), f(\beta,\alpha), \beta) \end{array} \right\} \leq 2b\ k\ \max \left\{ \begin{array}{l} S_b(f(\alpha,\beta), f(\alpha,\beta), \alpha), \\ S_b(f(\beta,\alpha), f(\beta,\alpha), \beta) \end{array} \right\}.
$$

It follows that $\alpha = F\alpha = f(\alpha,\beta)$, and $\beta = F\beta = f(\beta,\alpha)$. Therefore (α,β) is common coupled fixed point of (f, F). Since $f(X \times X) \subseteq G(X)$, there exist $a, b \in X$ such that

$f(\alpha,\beta)=\alpha=Ga$ and $f(\beta,\alpha)=\beta=Gb$. From (2.1.4), by Lemma 1.7 and $b\geqslant 1$ we have

$$\begin{aligned}
S_b(\alpha,\alpha,g(a,b)) &= S_b(f(\alpha,\beta),f(\alpha,\beta),g(a,b))\\
&\leqslant k\ \max\left\{\begin{array}{l} S_b(F\alpha,F\alpha,Ga),S_b(F\beta,F\beta,Gb),S_b(f(\alpha,\beta),f(\alpha,\beta),F\alpha),\\ S_b(f(\beta,\alpha),f(\beta,\alpha),F\beta),S_b(g(a,b),g(a,b),Ga),S_b(g(b,a),g(b,a),Gb),\\ \frac{1}{4b^2}[S_b(f(\alpha,\beta),f(\alpha,\beta),Ga)+S_b(g(a,b),g(a,b),F\alpha)],\\ \frac{1}{4b^2}[S_b(f(\beta,\alpha),f(\beta,\alpha),Gb)+S_b(g(b,a),g(b,a),F\beta)]\end{array}\right\}\\
&= k\ \max\left\{\begin{array}{l} 0,0,0,0,S_b(g(a,b),g(a,b),\alpha),S_b(g(b,a),g(b,a),\beta),\\ \frac{1}{4b^2}[0+S_b(g(a,b),g(a,b),F\alpha)],\frac{1}{4b^2}[0+S_b(g(b,a),g(b,a),F\beta)]\end{array}\right\}\\
&\leq\ b\ k\ \max\left\{S_b(\alpha,\alpha,g(a,b)),S_b(\beta,\beta,g(b,a))\right\}.
\end{aligned}$$

Similarly,

$$S_b(\beta,\beta,g(b,a))\leq b\ k\ \max\left\{S_b(\alpha,\alpha,g(a,b)),S_b(\beta,\beta,g(b,a))\right\}.$$

Thus,

$$\max\left\{S_b(\alpha,\alpha,g(a,b)),S_b(\beta,\beta,g(b,a))\right\}\leq bk\max\left\{S_b(\alpha,\alpha,g(a,b)),S_b(\beta,\beta,g(b,a))\right\}.$$

It follows that $g(a,b)=\alpha=Ga$ and $g(b,a)=\beta=Gb$. Since the pair $\{g,G\}$ is w-compatible, we have $G\alpha=g(\alpha,\beta)$ and $G\beta=g(\beta,\alpha)$. Using (2.1.4), we obtain

$$\begin{aligned}
S_b(\alpha,\alpha,g(\alpha,\beta)) &= S_b(f(\alpha,\beta),f(\alpha,\beta),g(\alpha,\beta))\\
&\leqslant k\max\left\{\begin{array}{c} S_b(F\alpha,F\alpha,G\alpha),S_b(F\beta,F\beta,G\beta),\\ S_b(f(\alpha,\beta),f(\alpha,\beta),F\alpha),S_b(f(\beta,\alpha),f(\beta,\alpha),F\beta),\\ S_b(g(\alpha,\beta),g(\alpha,\beta),G\alpha),S_b(g(\beta,\alpha),g(\beta,\alpha),G\beta),\\ \frac{1}{4b^2}[S_b(f(\alpha,\beta),f(\alpha,\beta),G\alpha)+S_b(g(\alpha,\beta),g(\alpha,\beta),F\alpha)],\\ \frac{1}{4b^2}[S_b(f(\beta,\alpha),f(\beta,\alpha),G\beta)+S_b(g(\beta,\alpha),g(\beta,\alpha),F\beta)]\end{array}\right\}\\
&\leq k\max\left\{\begin{array}{c} S_b(\alpha,\alpha,g(\alpha,\beta)),S_b(\beta,\beta,g(\beta,\alpha)),0,0,0,0,\\ \frac{1}{4b^2}[S_b(\alpha,\alpha,g(\alpha,\beta))+S_b(g(\alpha,\beta),g(\alpha,\beta),\alpha)],\\ \frac{1}{4b^2}[S_b(\beta,\beta,g(\beta,\alpha))+S_b(g(\beta,\alpha),g(\beta,\alpha),\beta)]\end{array}\right\}\\
&= k\max\left\{S_b(\alpha,\alpha,g(\alpha,\beta)),S_b(\beta,\beta,g(\beta,\alpha))\right\}.
\end{aligned}$$

Similarly,

$$S_b(\beta,\beta,g(\beta,\alpha))\leqslant k\max\left\{S_b(\alpha,\alpha,g(\alpha,\beta)),S_b(\beta,\beta,g(\beta,\alpha))\right\}.$$

Thus,

$$\max\left\{S_b(\alpha,\alpha,g(\alpha,\beta)),S_b(\beta,\beta,g(\beta,\alpha))\right\}\leqslant k\max\left\{S_b(\alpha,\alpha,g(\alpha,\beta)),S_b(\beta,\beta,g(\beta,\alpha))\right\}.$$

It follows that $\alpha=g(\alpha,\beta)$ and $\beta=g(\beta,\alpha)$. Thus, $\alpha=G\alpha=g(\alpha,\beta)$ and $\beta=G\beta=g(\beta,\alpha)$. Hence, (α,β) is a common coupled fixed point of f,g,F and G. To prove uniqueness let us suppose $(\alpha^1,\beta^1)\in X\times X$ is another common coupled fixed point of f,g,F

and G such that $\alpha \neq \alpha^1$ and $\beta \neq \beta^1$. From (2.1.4), we have that

$$\begin{aligned}
S_b(\alpha,\alpha,\alpha^1) &= S_b(f(\alpha,\beta), f(\alpha,\beta), g(\alpha^1,\beta^1)) \\
&\leqslant k \max \left\{ \begin{array}{c} S_b(F\alpha, F\alpha, G\alpha^1), S_b(F\beta, F\beta, G\beta^1), \\ S_b(f(\alpha,\beta), f(\alpha,\beta), F\alpha), S_b(f(\beta,\alpha), f(\beta,\alpha), F\beta), \\ S_b(g(\alpha^1,\beta^1), g(\alpha^1,\beta^1), G\alpha^1), S_b(g(\beta^1,\alpha^1), g(\beta^1,\alpha^1), G\beta^1), \\ \frac{1}{4b^2}\left[S_b(f(\alpha,\beta), f(\alpha,\beta), G\alpha^1) + S_b(g(\alpha^1,\beta^1), g(\alpha^1,\beta^1), F\alpha)\right], \\ \frac{1}{4b^2}\left[S_b(f(\beta,\alpha), f(\beta,\alpha), G\beta^1) + S_b(g(\beta^1,\alpha^1), g(\beta^1,\alpha^1), F\beta)\right] \end{array} \right\} \\
&= k \max \left\{ \begin{array}{c} S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1), 0, 0, 0, 0, \\ \frac{1}{4b^2}\left[S_b(\alpha,\alpha,\alpha^1) + S_b(\alpha^1,\alpha^1,\alpha)\right], \frac{1}{4b^2}\left[S_b(\beta,\beta,\beta^1) + S_b(\beta^1,\beta^1,\beta)\right] \end{array} \right\} \\
&= k \ \max\left\{ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1) \right\}.
\end{aligned}$$

Similarly,

$$S_b(\beta,\beta,\beta^1) \leqslant k \max\left\{ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1) \right\}.$$

Thus,

$$\max\left\{ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1) \right\} \leqslant k \max\left\{ S_b(\alpha,\alpha,\alpha^1), S_b(\beta,\beta,\beta^1) \right\}.$$

It is a contradiction. Hence, (α,β) is the unique common coupled fixed point of f, g, F and G. ■

***Example* 2.2** Let $X = [0,1]$ and

$$S_b : X \times X \times X \to \mathbb{R}^+ \quad \text{by} \quad S_b(x,y,z) = (|y+z-2x| + |y-z|)^2,$$

then S_b is S_b-metric space with $b = 4$. Define $f, g : X \times X \to X$ and $F, G : X \to X$ by

$$f(x,y) = \frac{x^2+y^2}{4^6}, \quad g(x,y) = \frac{x^2+y^2}{4^7}, \quad F(x) = \frac{x^2}{4} \text{ and } G(x) = \frac{x^2}{16}$$

also put $k = \frac{1}{4^7}$. Then

$$\begin{aligned}
&S_b(f(x,y), f(x,y), g(u,v))\\
&\quad = (|f(x,y) + g(u,v) - 2f(x,y)| + |f(x,y) - g(u,v)|)^2\\
&\quad = 4\ (|f(x,y) - g(u,v)|)^2\\
&\quad = 4\left|\frac{x^2+y^2}{4^6} - \frac{u^2+v^2}{4^7}\right|^2\\
&\quad = 4\ \left|\frac{4x^2-u^2}{4^7} + \frac{4y^2-v^2}{4^7}\right|^2\\
&\quad = 4\frac{1}{(4^4)^2}\left(\frac{1}{4}\left\{\left|\frac{4x^2-u^2}{16}\right| + \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2\\
&\quad \leqslant 4\frac{1}{(4^4)^2\,4}\left(\frac{1}{2}\left\{\left|\frac{4x^2-u^2}{16}\right| + \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2\\
&\quad \leqslant 4\frac{1}{4^9}\left(\max\left\{\left|\frac{4x^2-u^2}{16}\right|, \left|\frac{4y^2-v^2}{16}\right|\right\}\right)^2\\
&\quad = \frac{1}{4^9}\max\{S_b(Fx,Fx,Gu), S_b(Fy,Fy,Gv)\}\\
&\quad \leqslant k\max\left\{\begin{array}{c} S_b(Fx,Fx,Gu), S_b(Fy,Fy,Gv), S_b(f(x,y),f(x,y),Fx),\\ S_b(f(y,x),f(y,x),Fy), S_b(g(u,v),g(u,v),Gu), S_b(g(v,u),g(v,u),Gv),\\ \frac{1}{4b^2}\left[S_b(f(x,y),f(x,y),Gu) + S_b(g(u,v),g(u,v),Fx)\right],\\ \frac{1}{4b^2}\left[S_b(f(y,x),f(y,x),Gv) + S_b(g(v,u),g(v,u),Fy)\right] \end{array}\right\}.
\end{aligned}$$

It is clear that all conditions of Theorem 2.1 satisfied and $(0,0)$ is unique common coupled fixed point of f, g, F and G.

From Theorem 2.1, we have the following corollary.

Corollary 2.3 Let (X, S_b) be a S_b-metric space. Suppose that $f : X \times X \to X$ and $F : X \to X$ be satisfying

(2.3.1) $f(X \times X) \subseteq F(X)$,
(2.3.2) (f, F) are weakly compatible pairs,
(2.3.3) $F(X)$ is S_b-complete subspace of X,
(2.3.4) $S_b(f(x,y), f(x,y), f(u,v))$

$$\leq k\ \max\left\{\begin{array}{c} S_b(Fx,Fx,Fu), S_b(Fy,Fy,Fv), S_b(f(x,y),f(x,y),Fx),\\ S_b(f(y,x),f(y,x),Fy), S_b(f(u,v),f(u,v),Fu), S_b(f(v,u),f(v,u),Fv),\\ \frac{1}{4b^2}\left[S_b(f(x,y),f(x,y),Fu) + S_b(f(u,v),f(u,v),Fx)\right],\\ \frac{1}{4b^2}\left[S_b(f(y,x),f(y,x),Fv) + S_b(f(v,u),f(v,u),Fy)\right] \end{array}\right\}$$

for all $x, y, u, v \in X$, where $0 \leq k < \frac{1}{4b^2}$.

Then f and F have a unique common coupled fixed point in $X \times X$.

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