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An introduction to fixed-circle problem on soft metric spaces

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Abstract. Recently, soft set theory has been extensively studied both theoretically and practically with different approaches. On the other hand, fixed-circle problem has been investigated as a geometric generalization of fixed-point theory and this problem can be applied to some applicable areas. With these two perspectives, in this paper, we obtain some soft fixed-circle results using different auxiliary functions on a soft metric space. To do this, we are inspired various contractive conditions. The obtained results can be considered as an existence or uniqueness theorem. The proved theorems are supported by some illustrative examples. Finally, we give a list of geometric consequences of these results.

Keywords: Soft circle, soft fixed circle, soft metric space.

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1. Introduction and motivation

"*Soft set theory*" was introduced as a general mathematical tool for coping with encountered difficulties and uncertain problems in different fields such as engineering, medical science etc. [21]. After then, this theory has been extensively studied for both theoretic and applicable studies with various approaches (for example, see [8, 9, 13, 14, 16, 18, 23, 33–35] and the references therein). For example, some basic operations related to soft set theory were defined in [19]. Also, the notion of a soft topological space was introduced by Shabir and Naz [29]. After then, some basic topological notions such as soft interior, soft continuity, soft compactness etc. were given in [38].

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If we want to give an example for the application area, in [14], an application to food engineering was obtained and the obtained results were optimized the results by using appropriate parameters and degree of membership functions. On the other hand, recently various decision making problems have been studied using the notion of soft set (for example, see [7, 27, 28] and the references therein). After these examples, the advantages of studying soft set theory can be briefly listed as follows:

- Using a parametrization point of view, soft set theory presented a new mathematical model.
- Thanks to soft set theory, the constructed model can allow to deal with uncertainties.
- Using a soft set, complicated objects can become more understandable by means of parametrization.
- Soft set theory is related to soft computing models such as fuzzy set theory, rough set theory etc.
- Soft set theory has been used in the various applications such as topology, decision making problem, algebraic structure etc.

Recently, "*fixed-point theory*" has been extensively studied with different aspects. One of these aspects is to investigate the geometric properties of the fixed point set $Fix(T)$ of a self-mapping $T : X \to X$ when T has more than one fixed point. For this purpose, "*fixed-circle problem*" was introduced on a metric space by Özgür and Taş [26]. This problem can be considered as a geometric generalization to the fixed-point theory. In many studies, some solutions was investigated using diverse approaches and contractions (see, for example [20, 22, 25, 30, 31] and the references therein). For example, in [22], some fixed-disc results were obtained using the set of simulation functions on metric spaces. In [30], bilateral-type solutions to the fixed-circle problem were proved and an application to rectified linear units application was given. Also, this problem came to be thought of more generally as the fixed figure problem such as fixed-ellipse problem, fixed-Cassini curve problem etc. (see, [10, 15, 24, 32] for more details). What kind of advantages does working with a fixed-circle or fixed-figure results give us?

- In cases where the number of fixed points is more than one, a geometric meaning can be attributed to the set of fixed points.
- Thanks to the increase in the number of fixed points, the applicability of the obtained theory increases.
- Non-unique fixed points plays an important role a real life problem such as discontinuous activation functions etc.
- The set of non-unique fixed points may form a geometric shape such as circle, ellipse, hyperbola, Apollonius circle etc.
- By means of the fixed-circle problem, some theoretical results in the literature can be generalized.

From the above motivation, the main purpose is to obtain new solution to the "*Fixed-circle problem*" using the notion of a soft set. To do this, at first, we introduce the notion of a soft circle with some nice examples. After then, we prove six existence fixed-circle theorems and three uniqueness fixed-circle theorems on a soft metric space. Also, we prove a theorem which exclude the identity soft mapping. To show the validity of our obtained results, we give one necessary examples. Finally, we construct a list of geometric consequences of these results. The obtained results are important in generalizing the known results in the literature.

To summarize, the paper is organized as follows: After we give fundamental objectives and advantages of this work in Section 1, we recall some basic notions related to soft set theory in Section 2. We define a concept of a fixed circle and prove some existence and uniqueness theorems on a soft metric space in Section 3. In the last section, we mention the importance of this paper and propose some open problems as the future work.

2. Preliminaries

In this section, we recall some basic concepts related to soft set theory.

Definition 2.1 [21] Let X be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over X if and only if F is a mapping from E into the set of all subsets of the universe set X, that is, $F : E \to P(X)$, where $P(X)$ is the set of all subsets of the set X.

Definition 2.2 [19] Let (F, E) be a soft set over a universe set X.

(1) (F, E) is said to be a null soft set denoted by $\widetilde{\emptyset}$ if $F(e) = \emptyset$ for all $e \in E$.
(2) (F, E) is said to be an absolute soft set denoted by $\widetilde{X}$ if $F(e) = X$ for all $e \in E$.

Definition 2.3 [12] Let $A, B \subset E$ be nonempty subsets.

(1) For two soft sets (F, A) and (G, B) over a common universe X, (F, A) is said to be a soft subset of (G, B) if $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$. Then we write $(F, A) \,\widetilde{\subset}\, (G, B)$.
(2) Two soft sets (F, A) and (G, B) over a common universe X are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 [19] Let $A, B \subset E$ be nonempty subsets. The union of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e) & ; e \in A - B \\ G(e) & ; e \in B - A \\ F(e) \cup G(e) & ; e \in A \cap B \end{cases},$$

for all $e \in C$. It is denoted by $(F, A) \,\widetilde{\cup}\, (G, B) = (H, C)$.

Definition 2.5 [11] Let $A, B \subset E$ be nonempty subsets. The intersection of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$. It is denoted by $(F, A) \,\widetilde{\cap}\, (G, B) = (H, C)$.

Definition 2.6 [6] Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all nonempty bounded subsets of $\mathbb{R}$ and E be a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. It is denoted by (F, E). If specifically (F, E) is a singleton soft set then identifying (F, E) with the corresponding soft element, it will be called a soft real number and denoted by $\widetilde{r}$, $\widetilde{s}$, $\widetilde{t}$ etc.

$\overline{0}$ and $\overline{1}$ are the soft real numbers where $\overline{0}(e) = 0$ and $\overline{1}(e) = 1$ for all $e \in E$, respectively.

Definition 2.7 [6] Let (F, E) and (G, E) be two soft real numbers.

(1) $(F, E) = (G, E)$ if $F(e) = G(e)$ for each $e \in E$.
(2) $(F + G)(e) = \{x + y : x \in F(e), y \in G(e)\}$ for each $e \in E$.
(3) $(F - G)(e) = \{x - y : x \in F(e), y \in G(e)\}$ for each $e \in E$.
(4) $(F.G)(e) = \{x.y : x \in F(e), y \in G(e)\}$ for each $e \in E$.

(5) $(F/G)(e) = \{x/y : x \in F(e), y \in G(e) - \{0\}\}$ for each $e \in E$.

Definition 2.8 [6] For two soft real numbers

(1) $\widetilde{r} \,\widetilde{\leqslant}\, \widetilde{s}$ if $\widetilde{r}(e) \leqslant \widetilde{s}(e)$ for all $e \in E$,
(2) $\widetilde{r} \,\widetilde{\geqslant}\, \widetilde{s}$ if $\widetilde{r}(e) \geqslant \widetilde{s}(e)$ for all $e \in E$,
(3) $\widetilde{r} \,\widetilde{<}\, \widetilde{s}$ if $\widetilde{r}(e) < \widetilde{s}(e)$ for all $e \in E$,
(4) $\widetilde{r} \,\widetilde{>}\, \widetilde{s}$ if $\widetilde{r}(e) > \widetilde{s}(e)$ for all $e \in E$.

Definition 2.9 [5] A soft set (P, E) over X is said to be a soft point if there is exactly one $e \in E$ such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \emptyset$ for all $e' \in E - \{e\}$. It will be denoted by P_e^x.

Definition 2.10 [5] A soft point P_e^x is said to be belongs to a soft set (F, E) if $e \in E$ and $P(e) = \{x\} \subset F(e)$. It is written by $P_e^x \,\widetilde{\in}\, (F, E)$.

Definition 2.11 [5] Two soft points P_e^x, $P_{e'}^y$ are said to be equal if $e = e'$ and $P(e) = P(e')$, that is, $x = y$. Thus,

$$P_e^x \neq P_{e'}^y \Longleftrightarrow x \neq y \text{ or } e \neq e'.$$

Proposition 2.12 [5] The union of any collection of soft point can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it, that is,

$$(F, E) = \widetilde{\bigcup_{P_e^x \widetilde{\in} (F,E)}} P_e^x.$$

Remark 1 *[1, 36] Let $A, B \subset E$ be nonempty subsets. If f is a soft mapping from a soft set (F, A) to a soft set (G, B), which is denoted by $f : (F, A) \widetilde{\longrightarrow} (G, B)$, then for each soft point $P_{e_1}^x \,\widetilde{\in}\, (F, A)$, there exists only one soft point $P_{e_2}^y \,\widetilde{\in}\, (G, B)$ such that $f\left(P_{e_1}^x\right) = P_{e_2}^y$.*

Let $SP(\widetilde{X})$ be the collection of all soft points of $\widetilde{X}$ and $\mathbb{R}(E)^*$ be the set of all non-negative soft real numbers.

Definition 2.13 [5] A mapping $\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft metric on the soft set $\widetilde{X}$ if $\widetilde{d}$ satisfies the following conditions:
($\widetilde{d}1$) $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) \,\widetilde{\geqslant}\, \overline{0}$ for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{X})$.
($\widetilde{d}2$) $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ if and only if $P_{e_1}^x = P_{e_2}^y$.
($\widetilde{d}3$) $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \widetilde{d}(P_{e_2}^y, P_{e_1}^x)$ for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{X})$.
($\widetilde{d}4$) $\widetilde{d}(P_{e_1}^x, P_{e_3}^z) \,\widetilde{\leqslant}\, \widetilde{d}(P_{e_1}^x, P_{e_2}^y) + \widetilde{d}(P_{e_2}^y, P_{e_3}^z)$ for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{X})$.

The soft set $\widetilde{X}$ with a soft metric $\widetilde{d}$ on $\widetilde{X}$ is called a soft metric space and denoted by $\left(\widetilde{X}, \widetilde{d}, E\right)$.

Example 2.14 [5] Let $X \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the nonempty set of parameters. Let $\widetilde{X}$ be the absolute soft set and $\overline{x}$ denotes the soft real number such that $\overline{x}(e) = x$ for all $e \in E$. Then the function $\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$ defined by

$$\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = |\overline{x} - \overline{y}| + |\overline{e}_1 - \overline{e}_2|,$$

for all $P_{e_1}^{x}, P_{e_2}^{y} \in SP(\widetilde{X})$, where "$|.|$" denotes the modulus of soft real numbers, is a soft metric on $\widetilde{X}$.

***Example* 2.15** [5] Let $X \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the nonempty set of parameters. Then the function $\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^{*}$ defined by

$$\widetilde{d}(P_{e_1}^{x}, P_{e_2}^{y}) = \begin{cases} \overline{0} \,;\, P_{e_1}^{x} = P_{e_2}^{y} \\ \overline{1} \,;\, P_{e_1}^{x} \neq P_{e_2}^{y} \end{cases},$$

for all $P_{e_1}^{x}, P_{e_2}^{y} \in SP(\widetilde{X})$, is a soft metric on $\widetilde{X}$. $\widetilde{d}$ is called the discrete soft metric on the soft set $\widetilde{X}$ and $\left(\widetilde{X}, \widetilde{d}, E\right)$ is said to be the discrete soft metric space.

***Example* 2.16** [37] Let $E = \mathbb{R}$ be a parameter set and $X = \mathbb{R}^2$. Let us consider a metric d on this sets. Then the function $\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^{*}$ defined by

$$\widetilde{d}(P_{e_1}^{x}, P_{e_2}^{y}) = d(x, y) + |\overline{e}_1 - \overline{e}_2|\,,$$

for all $P_{e_1}^{x}, P_{e_2}^{y} \in SP(\widetilde{X})$, is a soft metric on $\widetilde{X}$. $\widetilde{d}$ is a soft metric on the soft set $\widetilde{X}$ and $\left(\widetilde{X}, \widetilde{d}, E\right)$ is a soft metric space.

3. Main results

In this section, at first, we present the notions of a soft circle and a soft fixed circle with some illustrative examples. We prove some soft fixed-circle theorems on soft metric spaces with different aspects. The importance of working with a fixed-circle problem with the help of soft set is to increase the number of fixed points of a mapping by means of parametrization. In this way, these theorems can be used in various fields of application such as neural networks, activation functions etc.

Definition 3.1 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space and $\widetilde{r}$ be a soft real number with $\widetilde{r} \,\widetilde{>}\, \overline{0}$. The soft circle is defined by

$$C\left(P_{e_1}^{x_0}, \widetilde{r}\right) = \widetilde{\bigcup} \left\{ P_{e_2}^{x} \,\widetilde{\in}\, \widetilde{X} : \widetilde{d}\left(P_{e_2}^{x}, P_{e_1}^{x_0}\right) = \widetilde{r} \right\},$$

with the center $P_{e_1}^{x_0}$ and the radius $\widetilde{r}$.

***Example* 3.2** Let us consider the discrete soft metric space $\left(\widetilde{X}, \widetilde{d}, E\right)$ as in Example 2.15. Then we get

$$C\left(P_{e_1}^{x_0}, \widetilde{r}\right) = \begin{cases} SP(\widetilde{X}) \,\widetilde{-}\, \left\{P_{e_1}^{x_0}\right\} \,;\, \widetilde{r} = 1 \\ \widetilde{\emptyset} \qquad\qquad ;\, \widetilde{r} \neq 1 \end{cases}.$$

***Example* 3.3** Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space as in Example 2.14. Suppose that $X \subset \mathbb{R}$ is a nonempty set and $E \subset \mathbb{R}$ is the set of parameters. Then we get the circle

$C\left(P_1^0, \overline{1}\right)$ with the center P_1^0 and the radius $\overline{1}$ as follows:

$$C\left(P_1^0, \overline{1}\right) = \widetilde{\bigcup}\left\{P_\lambda^x : \widetilde{d}\left(P_1^0, P_\lambda^x\right) = \overline{1}\right\}$$

and

$$\widetilde{d}\left(P_1^0, P_\lambda^x\right) = \left|\overline{x} - \overline{0}\right| + \left|\overline{\lambda} - \overline{1}\right| \Longrightarrow |x| + |\lambda - 1| = 1.$$

To determine the point x and λ, we investigate the following cases:

Case 1: $x \geqslant 0$ and $\lambda \geqslant 1 \Longrightarrow x + \lambda - 1 = 1 \Longrightarrow x + \lambda = 2$.

Case 2: $x \geqslant 0$ and $\lambda < 1 \Longrightarrow x + 1 - \lambda = 1 \Longrightarrow x = \lambda$.

Case 3: $x < 0$ and $\lambda \geqslant 1 \Longrightarrow -x + \lambda - 1 = 1 \Longrightarrow \lambda - x = 2$.

Case 4: $x < 0$ and $\lambda < 1 \Longrightarrow -x + 1 - \lambda = 1 \Longrightarrow x = -\lambda$.

Then the points x and λ can be seen in the following figure.

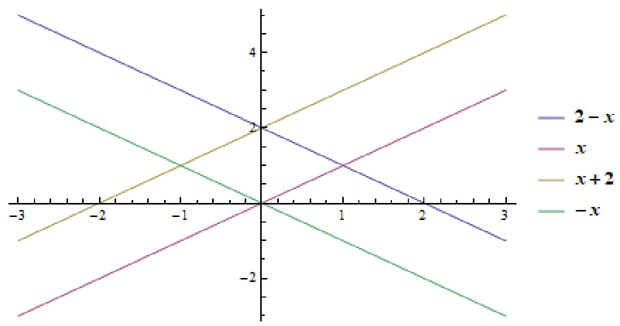

Figure 1. The set of the points x and λ.

Therefore, using the points x and λ lies on the above figure. We write infinite number of soft points. Consequently, the circle $C\left(P_1^0, \overline{1}\right)$ has infinite number of soft points.

***Example* 3.4** Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space as in Example 2.16. Let $X = \mathbb{R}$, $E = \{1, 2, 3\}$ and the metric d defined as $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$. Then we get the soft metric $\widetilde{d}$ as $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = |e^x - e^y| + |\overline{e}_1 - \overline{e}_2|$. Then we get the circle $C\left(P_2^1, \overline{1}\right)$ with the center P_2^1 and the radius $\overline{1}$ as follows:

$$C\left(P_2^1, \overline{1}\right) = \widetilde{\bigcup}\left\{P_\lambda^x : \widetilde{d}\left(P_2^1, P_\lambda^x\right) = \overline{1}\right\} \text{ and } \widetilde{d}\left(P_2^1, P_\lambda^x\right) = |e^x - e| + |\lambda - 2| = 1.$$

To determine the point x and λ, we investigate the following cases:

Case 1: If $\lambda - 2 \geqslant 0$ and $e^x - e \geqslant 0$, then we get $P_\lambda^x = P_3^1$.

Case 2: If $\lambda - 2 \geqslant 0$ and $e^x - e < 0$, then we get $P_\lambda^x = P_3^1$ and $P_\lambda^x = P_2^{\ln(e-1)}$.

Case 3: If $\lambda - 2 < 0$ and $e^x - e \geqslant 0$, then we get $P_\lambda^x = P_1^{\ln(e-2)}$.

Case 4: If $\lambda - 2 < 0$ and $e^x - e < 0$, then we get $P_\lambda^x = P_1^1$.

Consequently, we have

$$C\left(P_2^1, \overline{1}\right) = \left\{P_3^1, P_2^{\ln(e-1)}, P_1^{\ln(e-2)}, P_1^1\right\}.$$

Definition 3.5 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping and $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle. If $T\left(P_{e_2}^{x}\right) = P_{e_2}^{x}$ for all $P_{e_2}^{x} \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, then $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is called as a soft fixed circle of T.

In the following subsection, we give some existence soft fixed-circle theorems.

3.1 *Existence theorems for a soft fixed-circle*

In this subsection, we prove new existence theorems for a soft fixed circle and give some necessary illustrative examples. To do this, we inspire the Caristi type contraction [3]. For this purpose, let us consider the following soft mapping $\varphi : \widetilde{X} \to \mathbb{R}(E)^*$ defined as

$$\varphi(P_e^x) = \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right), \tag{1}$$

for all $P_e^x \widetilde{\in} \widetilde{X}$, where $P_{e_1}^{x_0}$ is the center of a soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$.

Now we obtain the existence theorem for a soft fixed circle.

Theorem 3.6 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle and a soft mapping $\varphi : \widetilde{X} \to \mathbb{R}(E)^*$ be defined as in (1). If T satisfies the following conditions

(*i*) $\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leqslant} \varphi(P_e^x) - \varphi(T(P_e^x))$,

(*ii*) $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{\geqslant} \widetilde{r}$,

for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, then $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T.

Proof. Let $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be an arbitrary soft point. Now we show $T(P_e^x) = P_e^x$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. By the condition (*i*) and the definition of φ, we get

$$\begin{aligned} \widetilde{d}(P_e^x, T(P_e^x)) \ \widetilde{\leqslant} \ \varphi(P_e^x) - \varphi(T(P_e^x)) &= \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \\ &= \widetilde{r} - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right). \end{aligned} \tag{2}$$

From the condition (*ii*), we get the following cases:

Case 1: If $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{>} \widetilde{r}$, then we get a contradiction with the inequality (2).

Case2: If $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) = \widetilde{r}$, then using the inequality (2), we obtain

$$\widetilde{d}(P_e^x, T(P_e^x)) \ \widetilde{\leqslant} \ \widetilde{r} - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) = \widetilde{r} - \widetilde{r} = \overline{0}$$

and so we get $T(P_e^x) = P_e^x$. Consequently, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T. ■

Remark 2 *Theorem 3.6 is a generalization of Theorem* 2.1 *given in [26]. If we take the parameter set E with only one element, then Theorem 3.6 coincide Theorem* 2.1 *given in [26].*

We give a following illustrative example.

***Example* 3.7** Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space and $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle. Let us define a soft mapping $T : \widetilde{X} \to \widetilde{X}$ as

$$T(P_e^x) = \begin{cases} P_e^x \ ; & P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \\ P_{e_2}^{\beta} \ ; & \text{otherwise} \end{cases},$$

for all $P_e^x \widetilde{\in} \widetilde{X}$, where $P_{e_2}^{\beta}$ is a constant soft point such that $\widetilde{d}\left(P_{e_2}^{\beta}, P_{e_1}^{x_0}\right) \widetilde{>} \widetilde{r}$, that is, $P_{e_2}^{\beta}$ is exterior of the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. Then T satisfies the conditions of Theorem 3.6. Consequently, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T.

Theorem 3.8 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle and a soft mapping $\varphi : \widetilde{X} \to \mathbb{R}(E)^*$ be defined as in (1). If T satisfies the following conditions

(i) $\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leqslant} \varphi(P_e^x) + \varphi(T(P_e^x)) - \overline{2}\widetilde{r}$,

(ii) $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{\leqslant} \widetilde{r}$,

for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, then $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T.

Proof. Let $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be an arbitrary soft point. Using the condition (i), we obtain

$$\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leqslant} \varphi(P_e^x) + \varphi(T(P_e^x)) - \overline{2}\widetilde{r}$$

and by the definition of φ, we get

$$\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leqslant} \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) + \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) - \overline{2}\widetilde{r} = \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \widetilde{r}. \tag{3}$$

From the condition (ii), we get the following cases:

Case 1: If $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{<} \widetilde{r}$, then we get a contradiction with the inequality (3).

Case2: If $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) = \widetilde{r}$, then using the inequality (3), we obtain

$$\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leqslant} \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \widetilde{r} = \widetilde{r} - \widetilde{r} = \overline{0},$$

that is, $T(P_e^x) = P_e^x$. Consequently, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T. ■

Remark 3 *Theorem 3.8 is a generalization of Theorem* 2.2 *given in [26]. If we take the parameter set E with only one element, then Theorem 3.8 coincide Theorem* 2.2 *given in [26].*

***Example* 3.9** Let us consider Example 3.7. Then T satisfies the conditions of Theorem 3.8. Consequently, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T.

Theorem 3.10 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle and a soft mapping $\varphi : \widetilde{X} \to \mathbb{R}(E)^*$ be defined as in (1). If T satisfies the following conditions

(i) $\widetilde{d}(P_e^x, T(P_e^x)) \widetilde{\leqslant} \varphi(P_e^x) - \varphi(T(P_e^x))$,

(ii) $\overline{h}\widetilde{d}(P_e^x, T(P_e^x)) + \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{\leqslant} \widetilde{r}$,

for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, where $\overline{0} \widetilde{\leqslant} \overline{h} \widetilde{<} \overline{1}$, then $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T.

Proof. We suppose that $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ such that $T(P_e^x) \neq P_e^x$. Using the conditions

(i), (ii) and the definition of φ, we have

$$\begin{aligned}\widetilde{d}\left(P_e^x, T(P_e^x)\right) &\widetilde{\leqslant} \varphi(P_e^x) - \varphi(T(P_e^x)) \\ &= \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) = \widetilde{r} - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \\ &\widetilde{\leqslant} \overline{h}\widetilde{d}\left(P_e^x, T(P_e^x)\right) + \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \\ &= \overline{h}\widetilde{d}\left(P_e^x, T(P_e^x)\right),\end{aligned}$$

a contradiction with $\overline{0} \widetilde{\leqslant} \overline{h} \widetilde{<} \overline{1}$. So it should be $T(P_e^x) = P_e^x$. Consequently, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T. ■

Remark 4 *Theorem 3.10 is a generalization of Theorem* 2.3 *given in [26]. If we take the parameter set E with only one element, then Theorem 3.10 coincide Theorem* 2.3 *given in [26].*

***Example* 3.11** Let us consider Example 3.7. Then T satisfies the conditions of Theorem 3.10. Consequently, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is a soft fixed circle of T.

Theorem 3.12 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle and a soft mapping $\varphi_{\widetilde{r}} : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be defined by

$$\varphi_{\widetilde{r}}(\widetilde{u}) = \begin{cases} \widetilde{r} & ; \widetilde{u} = \widetilde{r} \\ \widetilde{u} + \widetilde{r} & ; \widetilde{u} \neq \widetilde{r} \end{cases},$$

for all $\widetilde{u} \,\widetilde{\in}\, \mathbb{R}(E)^*$ and $\widetilde{r} \widetilde{>} \overline{0}$. Assume that

(i) $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{\leqslant} \varphi_{\widetilde{r}}\left(\widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right)\right) + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right)$ for some $\overline{L} \widetilde{\leqslant} \overline{0}$ and each $P_e^x \,\widetilde{\in}\, \widetilde{X}$,

(ii) $\widetilde{r} \widetilde{\leqslant} \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right)$ for each $P_e^x \,\widetilde{\in}\, C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$,

(iii) $\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \widetilde{\geqslant} \overline{2}\widetilde{r}$ for each $P_e^x, P_{e_2}^y \,\widetilde{\in}\, C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $P_e^x \neq P_{e_2}^y$,

(iv) $\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \widetilde{<} \widetilde{r} + \widetilde{d}\left(P_{e_2}^y, T(P_e^x)\right)$ for each $P_e^x, P_{e_2}^y \,\widetilde{\in}\, C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $P_e^x \neq P_{e_2}^y$.

Then T fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$.

Proof. Let $P_e^x \,\widetilde{\in}\, C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be an arbitrary soft point. Using the conditions (i) and (ii), we get

$$\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{\leqslant} \varphi_{\widetilde{r}}\left(\widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right)\right) + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right) = \widetilde{r} + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right)$$

and so

$$\widetilde{r} \widetilde{\leqslant} \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \widetilde{\leqslant} \widetilde{r} + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right). \tag{4}$$

If $\overline{L} = \overline{0}$, then we have $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) = \widetilde{r}$ by the inequality (4), that is, $T(P_e^x) \,\widetilde{\in}\, C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. Suppose that $\widetilde{d}\left(P_e^x, T(P_e^x)\right) \neq \overline{0}$, that is, $T(P_e^x) \neq P_e^x$ for each $P_e^x \,\widetilde{\in}\, C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. By the condition (iii), we get

$$\widetilde{d}\left(T(P_e^x), T(T(P_e^x))\right) \widetilde{\geqslant} \overline{2}\widetilde{r} \tag{5}$$

and using the condition (iv), we obtain

$$\widetilde{d}\left(T(P_e^x), T(T(P_e^x))\right) \;\widetilde{<}\; \widetilde{r} + \widetilde{d}\left(T(P_e^x), T(P_e^x)\right) = \widetilde{r},$$

which contradicts with the inequality (5). So it should be $\widetilde{d}\left(P_e^x, T(P_e^x)\right) = \overline{0}$.

If $\overline{L} \;\widetilde{<}\; \overline{0}$ and $\widetilde{d}\left(P_e^x, T(P_e^x)\right) \neq \overline{0}$, then we a get a contradiction with the inequality (4). Hence it should be $\widetilde{d}\left(P_e^x, T(P_e^x)\right) = \overline{0}$; that is, $T(P_e^x) = P_e^x$. Consequently, T fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. ■

Remark 5 *Theorem 3.12 is a generalization of Theorem* 2.1 *given in [20]. If we take the parameter set E with only one element, then Theorem 3.12 coincide Theorem* 2.1 *given in [20].*

***Example* 3.13** Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space defined as in Example 2.14. Assume that $X \subset \mathbb{R}$ is a nonempty set, $E = \{1, 2\}$ is the set of parameters and $C(P_1^0, 1)$ is a circle with the center P_1^0 and the radius $\overline{1}$. Let us define the soft mapping $T : \widetilde{X} \to \widetilde{X}$ as

$$T(P_e^x) = \begin{cases} P_e^x \;;\; P_e^x \;\widetilde{\in}\; C(P_1^0, 1) \\ \overline{0} \qquad \text{otherwise} \end{cases},$$

for all $P_e^x \;\widetilde{\in}\; \widetilde{X}$. Then T satisfies the conditions of Theorem 3.12 with $\overline{L} = \overline{0}$. Consequently, $C(P_1^0, 1) = \left\{P_1^{-1}, P_1^1, P_2^0\right\}$ is a soft fixed circle of T.

Theorem 3.14 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle and a soft mapping $\varphi_{\widetilde{r}} : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be defined as in Theorem 3.12. Assume that

(i) $\overline{2}\widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \;\widetilde{\leqslant}\; \varphi_{\widetilde{r}}\left(\widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right)\right) + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right)$ for some $\overline{L} \;\widetilde{\leqslant}\; \overline{0}$ and each $P_e^x \;\widetilde{\in}\; \widetilde{X}$,

(ii) $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \;\widetilde{\leqslant}\; \widetilde{r}$ for each $P_e^x \;\widetilde{\in}\; C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$,

(iii) $\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \;\widetilde{\geqslant}\; \overline{2}\widetilde{r}$ for each $P_e^x, P_{e_2}^y \;\widetilde{\in}\; C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $P_e^x \neq P_{e_2}^y$,

(iv) $\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \;\widetilde{<}\; \widetilde{r} + \widetilde{d}\left(P_{e_2}^y, T(P_e^x)\right)$ for each $P_e^x, P_{e_2}^y \;\widetilde{\in}\; C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $P_e^x \neq P_{e_2}^y$.

Then T fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$.

Proof. Let $P_e^x \;\widetilde{\in}\; C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be an arbitrary soft point. By the conditions (i) and (ii), we obtain

$$\begin{aligned} &\overline{2}\widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \;\widetilde{\leqslant}\; \widetilde{d}\left(P_e^x, P_{e_1}^{x_0}\right) + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right) \\ \Longrightarrow\; &\overline{2}\widetilde{r} - \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) \;\widetilde{\leqslant}\; \widetilde{r} + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right) \\ \Longrightarrow\; &\widetilde{r} \;\widetilde{\leqslant}\; \widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right) \;\widetilde{\leqslant}\; \widetilde{r} + \overline{L}\widetilde{d}\left(P_e^x, T(P_e^x)\right). \end{aligned} \tag{6}$$

By the inequality (6) and the similar arguments used in the proof of Theorem 3.12, it can be easily seen that T fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. ■

Remark 6 *Theorem 3.14 is a generalization of Theorem* 2.6 *given in [20]. If we take the parameter set E with only one element, then Theorem 3.14 coincide Theorem* 2.6 *given in [20].*

***Example* 3.15** Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space defined as in Example 2.14. Assume that $X \subset \mathbb{R}$ is a nonempty set, $E = \{1, 2\}$ is the set of parameters and $C(P_1^0, 1)$ is a circle with the center P_1^0 and the radius $\overline{1}$. Let us define the soft mapping $T : \widetilde{X} \to \widetilde{X}$ as

$$T(P_e^x) = \begin{cases} P_e^x & ; P_e^x \widetilde{\in} C(P_1^0, 1) \\ P_e^{2x} & \text{otherwise} \end{cases},$$

for all $P_e^x \widetilde{\in} \widetilde{X}$. Then T satisfies the conditions of Theorem 3.14 with $\overline{L} = \overline{0}$. Consequently, $C(P_1^0, 1) = \left\{P_1^{-1}, P_1^1, P_2^0\right\}$ is a soft fixed circle of T.

Theorem 3.16 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space, $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping, $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be a soft circle and a soft mapping $\varphi_{\widetilde{r}}^* : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ be defined by

$$\varphi_{\widetilde{r}}^*(\widetilde{u}) = \begin{cases} \overline{0} & ; \widetilde{u} = \overline{0} \\ \widetilde{u} - \widetilde{r} & ; \widetilde{u} \widetilde{<} \overline{0} \end{cases},$$

for all $\widetilde{u} \widetilde{\in} \mathbb{R}(E)^*$ and $\widetilde{r} \widetilde{>} \overline{0}$. If a soft mapping T satisfies the following conditions

(i) $\widetilde{d}\left(T(P_e^x), P_{e_1}^{x_0}\right) = \widetilde{r}$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$,

(ii) $\widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \widetilde{>} \widetilde{r}$ for each $P_e^x, P_{e_2}^y \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $P_e^x \neq P_{e_2}^y$,

(iii) $\widetilde{d}(T(P_e^x), T(P_{e_2}^y)) \widetilde{\leqslant} \widetilde{d}(P_e^x, P_{e_2}^y) - \varphi_{\widetilde{r}}^*\left(\widetilde{d}(P_e^x, T(P_e^x))\right)$ for each $P_e^x, P_{e_2}^y \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$,

then T fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$.

Proof. Let $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ be an arbitrary soft point. By the conditions (i), we have $T(P_e^x) \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, for all $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. To show $T(P_e^x) = P_e^x$, we assume that $T(P_e^x) \neq P_e^x$. By the condition (ii), we obtain

$$\widetilde{d}(T(P_e^x), T(T(P_e^x))) \;\widetilde{>}\; \widetilde{r} \tag{7}$$

and using the condition (ii), we get

$$\begin{aligned} \widetilde{d}(T(P_e^x), T(T(P_e^x))) \;&\widetilde{\leqslant}\; \widetilde{d}(P_e^x, T(P_e^x)) - \varphi_{\widetilde{r}}^*\left(\widetilde{d}(P_e^x, T(P_e^x))\right) \\ &= \widetilde{d}(P_e^x, T(P_e^x)) - \widetilde{d}(P_e^x, T(P_e^x)) + \widetilde{r} = \widetilde{r}, \end{aligned}$$

which a contradiction with the inequality (7). Therefore, it should be $T(P_e^x) = P_e^x$ and so T fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. ■

Remark 7 *Theorem 3.16 is a generalization of Theorem* 3 *given in [25]. If we take the parameter set E with only one element, then Theorem 3.16 coincide Theorem* 2.1 *given in [25].*

***Example* 3.17** Let us consider Example 3.13. Then T satisfies the conditions of Theorem 3.16 with $\overline{L} = \overline{0}$. Consequently, $C(P_1^0, 1) = \left\{P_1^{-1}, P_1^1, P_2^0\right\}$ is a soft fixed circle of T.

Now, we give the following theorem to exclude the identity soft mapping.

Theorem 3.18 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space. Let us consider a soft mapping $T : \widetilde{X} \to \widetilde{X}$ which has a soft fixed circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and the soft mapping $\varphi : \widetilde{X} \to \mathbb{R}(E)^*$ defined as in (1). Then T satisfies the condition

$$\widetilde{d}\left(P_{e_2}^y, T(P_{e_2}^y)\right) \;\widetilde{\leqslant}\; \overline{h}\left[\varphi\left(P_{e_2}^y\right) - \varphi\left(T(P_{e_2}^y)\right)\right], \tag{8}$$

for every $P_{e_2}^y \widetilde{\in} \widetilde{X}$ and $\overline{0} \widetilde{<} \overline{h} \widetilde{<} \overline{1}$ if and only if $T = I_{\widetilde{X}}$, where the identity soft mapping $I_{\widetilde{X}} : \widetilde{X} \to \widetilde{X}$ defined by $I_{\widetilde{X}}(P_e^x) = P_e^x$ for all $P_e^x \widetilde{\in} \widetilde{X}$.

Proof. Let $P_{e_2}^y \widetilde{\in} \widetilde{X}$ and $T(P_{e_2}^y) \neq P_{e_2}^y$. Then using the inequality (8), we get

$$\begin{aligned}
\widetilde{d}\left(P_{e_2}^y, T(P_{e_2}^y)\right) &\widetilde{\leqslant} \ \overline{h}\left[\varphi\left(P_{e_2}^y\right) - \varphi\left(T(P_{e_2}^y)\right)\right] \\
&= \overline{h}\left[\widetilde{d}\left(P_{e_2}^y, P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_{e_2}^y), P_{e_1}^{x_0}\right)\right] \\
&\widetilde{\leqslant} \ \overline{h}\left[\widetilde{d}\left(P_{e_2}^y, T(P_{e_2}^y)\right) + \widetilde{d}\left(T(P_{e_2}^y), P_{e_1}^{x_0}\right) - \widetilde{d}\left(T(P_{e_2}^y), P_{e_1}^{x_0}\right)\right] \\
&= \overline{h}\widetilde{d}\left(P_{e_2}^y, T(P_{e_2}^y)\right),
\end{aligned}$$

a contradiction. Hence it should be $T(P_{e_2}^y) = P_{e_2}^y$, that is, $T = I_{\widetilde{X}}$. The converse statement is clear. ■

Corollary 3.19 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space and $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping. If T satisfies the conditions of Theorem 3.6 (resp. Theorem 3.8, Theorem 3.10, Theorem 3.12, Theorem 3.14 and Theorem 3.16) but the condition (8) is not satisfied by T, then $T \neq I_{\widetilde{X}}$.

3.2 *A uniqueness theorem for a soft fixed-circle*

In this subsection, we investigate a uniqueness condition for a soft fixed-circle of a soft mapping $T : \widetilde{X} \to \widetilde{X}$. At first, we give the following example to show a soft mapping which has two soft fixed-circles.

***Example* 3.20** Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space and $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, $C\left(P_{e_2}^y, \widetilde{\rho}\right)$ be any soft circles. Let us define a soft mapping $T : \widetilde{X} \to \widetilde{X}$ as

$$T(P_e^x) = \begin{cases} P_e^x \ ; & P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right) \widetilde{\cup} C\left(P_{e_2}^y, \widetilde{\rho}\right) \\ P_{e_3}^{\alpha} \ ; & \text{otherwise} \end{cases},$$

for all $P_e^x \widetilde{\in} \widetilde{X}$, where $P_{e_3}^{\alpha}$ is a constant soft point such that $\widetilde{d}\left(P_{e_3}^{\alpha}, P_{e_1}^{x_0}\right) \neq \widetilde{r}$ and $\widetilde{d}\left(P_{e_3}^{\alpha}, P_{e_2}^y\right) \neq \widetilde{\rho}$. Then T fixes soft circles both $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $C\left(P_{e_2}^y, \widetilde{\rho}\right)$.

If we consider Example 3.20, then we say that the studying a uniqueness theorem of a soft fixed circle gains an importance. According to the above example, we prove the following theorem using the Banach type contraction [2].

Theorem 3.21 Let $\left(\widetilde{X}, \widetilde{d}, E\right)$ be a soft metric space and $T : \widetilde{X} \to \widetilde{X}$ be a soft mapping which fixes the soft circle $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$. If the condition

$$\widetilde{d}\left(T(P_e^x), T(P_{e_2}^y)\right) \ \widetilde{\leqslant} \ \overline{h}\widetilde{d}\left(P_e^x, P_{e_2}^y\right), \tag{9}$$

is satisfied for all $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$, $P_{e_2}^y \widetilde{\in} \widetilde{X} \setminus C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and some $\overline{0} \ \widetilde{\leqslant} \ \overline{h} \ \widetilde{<} \ \overline{1}$ by T, then $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ is the unique soft fixed circle of T.

Proof. Let us suppose that there exist two soft fixed circles $C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $C\left(P_{e_2}^y, \widetilde{\rho}\right)$ of the soft mapping T. If we take $P_{e_3}^u \widetilde{\in} C\left(P_{e_1}^{x_0}, \widetilde{r}\right)$ and $P_{e_4}^v \widetilde{\in} C\left(P_{e_2}^y, \widetilde{\rho}\right)$ with $P_{e_3}^u \neq P_{e_4}^v$, then using the inequality (9), we get

$$\widetilde{d}\left(T(P_{e_3}^u),T(P_{e_4}^v)\right)=\widetilde{d}\left(P_{e_3}^u,P_{e_4}^v\right)\ \widetilde{\leqslant}\ \overline{h}\widetilde{d}\left(P_{e_3}^u,P_{e_4}^v\right),$$

a contradiction because of $\overline{0}\ \widetilde{\leqslant}\ \overline{h}\ \widetilde{<}\ \overline{1}$. Thereby, we have $P_{e_3}^u=P_{e_4}^v$ for all $P_{e_3}^u\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$, $P_{e_4}^v\ \widetilde{\in}\ C\left(P_{e_2}^y,\widetilde{\rho}\right)$ and so T has a unique soft fixed circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. ■

To obtain the uniqueness theorem, the choice of the contractive condition is not unique. For example, using the Kannan type contraction [17], we prove the following theorem.

Theorem 3.22 Let $\left(\widetilde{X},\widetilde{d},E\right)$ be a soft metric space and $T:\widetilde{X}\rightarrow\widetilde{X}$ be a soft mapping which fixes the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. If the condition

$$\widetilde{d}\left(T(P_e^x),T(P_{e_2}^y)\right)\ \widetilde{\leqslant}\ \overline{h}\left[\widetilde{d}(T(P_e^x),P_e^x)+\widetilde{d}\left(T(P_{e_2}^y),P_{e_2}^y\right)\right] \tag{10}$$

is satisfied for all $P_e^x\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$, $P_{e_2}^y\ \widetilde{\in}\ \widetilde{X}\setminus C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ and some $\overline{0}\ \widetilde{\leqslant}\ \overline{h}\ \widetilde{<}\ \overline{\tfrac{1}{2}}$ by T, then $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ is the unique soft fixed circle of T.

Proof. Let us suppose that there exist two soft fixed circles $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ and $C\left(P_{e_2}^y,\widetilde{\rho}\right)$ of the soft mapping T. If we take $P_{e_3}^u\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ and $P_{e_4}^v\ \widetilde{\in}\ C\left(P_{e_2}^y,\widetilde{\rho}\right)$ with $P_{e_3}^u\neq P_{e_4}^v$, then using the inequality (10), we get

$$\widetilde{d}\left(T(P_{e_3}^u),T(P_{e_4}^v)\right)=\widetilde{d}\left(P_{e_3}^u,P_{e_4}^v\right)\ \widetilde{\leqslant}\ \overline{h}\left[\widetilde{d}\left(T(P_{e_3}^u),P_{e_3}^u\right)+\widetilde{d}\left(T(P_{e_4}^v),P_{e_4}^v\right)\right]=\overline{0},$$

a contradiction. Thereby, we have $P_{e_3}^u=P_{e_4}^v$ for all $P_{e_3}^u\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$, $P_{e_4}^v\ \widetilde{\in}\ C\left(P_{e_2}^y,\widetilde{\rho}\right)$ and so T has a unique soft fixed circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. ■

Using Chatterjea type contraction [4], we obtain another uniqueness theorem.

Theorem 3.23 Let $\left(\widetilde{X},\widetilde{d},E\right)$ be a soft metric space and $T:\widetilde{X}\rightarrow\widetilde{X}$ be a soft mapping which fixes the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. If the condition

$$\widetilde{d}\left(T(P_e^x),T(P_{e_2}^y)\right)\ \widetilde{\leqslant}\ \overline{h}\left[\widetilde{d}\left(T(P_e^x),P_{e_2}^y\right)+\widetilde{d}\left(T(P_{e_2}^y),P_e^x\right)\right] \tag{11}$$

is satisfied for all $P_e^x\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$, $P_{e_2}^y\ \widetilde{\in}\ \widetilde{X}\setminus C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ and some $\overline{0}\ \widetilde{\leqslant}\ \overline{h}\ \widetilde{<}\ \overline{\tfrac{1}{2}}$ by T, then $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ is the unique soft fixed circle of T.

Proof. Let us suppose that there exist two soft fixed circles $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ and $C\left(P_{e_2}^y,\widetilde{\rho}\right)$ of the soft mapping T. If we take $P_{e_3}^u\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ and $P_{e_4}^v\ \widetilde{\in}\ C\left(P_{e_2}^y,\widetilde{\rho}\right)$ with $P_{e_3}^u\neq P_{e_4}^v$, then using the inequality (11), we get

$$\widetilde{d}\left(T(P_{e_3}^u),T(P_{e_4}^v)\right)=\widetilde{d}\left(P_{e_3}^u,P_{e_4}^v\right)\ \widetilde{\leqslant}\ \overline{h}\left[\widetilde{d}\left(T(P_{e_3}^u),P_{e_4}^v\right)+\widetilde{d}\left(T(P_{e_4}^v),P_{e_3}^u\right)\right]=\overline{2h}\widetilde{d}\left(P_{e_3}^u,P_{e_4}^v\right),$$

a contradiction because of $\overline{0}\ \widetilde{\leqslant}\ \overline{h}\ \widetilde{<}\ \overline{\tfrac{1}{2}}$. Thereby, we have $P_{e_3}^u=P_{e_4}^v$ for all $P_{e_3}^u\ \widetilde{\in}\ C\left(P_{e_1}^{x_0},\widetilde{r}\right)$, $P_{e_4}^v\ \widetilde{\in}\ C\left(P_{e_2}^y,\widetilde{\rho}\right)$ and so T has a unique soft fixed circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. ■

3.3 *Some consequences*

Now, we give the following remarks.

(1) Theorem 3.6 (resp. Theorem 3.8, Theorem 3.10, Theorem 3.12, Theorem 3.14 and Theorem 3.16) guarantees the existence of a soft fixed circle. Also, they can be considered as soft fixed-point theorems to show the existence of a soft fixed point in case the soft fixed circle has only one soft point.

(2) The condition (i) given in Theorem 3.6 guarantees that $T(P_e^x)$ is not in the exterior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. Similarly, the condition (ii) given in Theorem 3.6 guarantees that $T(P_e^x)$ is not in the interior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. Consequently, $T\left(C\left(P_{e_1}^{x_0},\widetilde{r}\right)\right) \widetilde{\subseteq} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$.

(3) The condition (i) given in Theorem 3.8 guarantees that $T(P_e^x)$ is not in the interior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. Similarly, the condition (ii) given in Theorem 3.8 guarantees that $T(P_e^x)$ is not in the exterior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. Consequently, $T\left(C\left(P_{e_1}^{x_0},\widetilde{r}\right)\right) \widetilde{\subseteq} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$.

(4) The condition (i) given in Theorem 3.10 guarantees that $T(P_e^x)$ is not in the exterior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$ for each $P_e^x \widetilde{\in} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. Also, the condition (ii) given in Theorem 3.10 implies that $T(P_e^x)$ can be lies on or exterior or interior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$. Hence, $T(P_e^x)$ should be lies on or interior of the soft circle $C\left(P_{e_1}^{x_0},\widetilde{r}\right)$.

(5) If we consider $\overline{L} = \overline{-1}$ in the condition (i) of Theorem 3.12, then we obtain

$$\widetilde{d}(P_e^x, T(P_e^x)) \ \widetilde{\leqslant}\ \varphi(P_e^x) - \varphi(T(P_e^x)).$$

Therefore, the condition (i) of Theorem 3.6 (resp. Theorem 3.10) is satisfied. On the other hand, the condition (ii) of Theorem 3.6 is the same as the condition (ii) of Theorem 3.12. Also, if the condition (ii) of Theorem 3.12 is satisfied, then the condition (ii) of Theorem 3.10 is satisfied.

(6) If we consider $\overline{L} = \overline{-1}$ in the condition (i) of Theorem 3.14, then we obtain

$$\widetilde{d}(P_e^x, T(P_e^x)) \ \widetilde{\leqslant}\ \varphi(P_e^x) + \varphi(T(P_e^x)) - \overline{2}\widetilde{r}.$$

Hence, the condition (i) of Theorem 3.8 is satisfied. Also, the condition (ii) of Theorem 3.8 is the same as the condition (ii) of Theorem 3.14.

(7) The condition (i) of Theorem 3.16 shows that $T\left(C\left(P_{e_1}^{x_0},\widetilde{r}\right)\right) \widetilde{\subseteq} C\left(P_{e_1}^{x_0},\widetilde{r}\right)$.

(8) Theorem 3.21 guarantees the uniqueness of a soft fixed circle. Also, it can be considered as a soft fixed-point result to show the uniqueness of a soft fixed point in case the soft fixed circle has only one soft point. On the other hand, to obtain a new uniqueness theorem of a soft fixed circle, different contractions should be used.

4. Conclusion and future work

In this paper, we introduced the notion of a soft fixed circle and proved some soft fixed-circle theorems on soft metric spaces with some illustrative examples. The importance of working with a fixed-circle problem with the help of soft set is to increase the number of fixed points of a mapping by means of parametrization. In this way, these theorems can be used in various fields of application such as neural networks, activation functions etc. Finally, we can leave the following open problem ideas:

Problem 1: Can the obtained theoretical results in this paper be applied to other areas such as engineering, activation functions, neural networks, discontinuity?

Problem 2: Can new solutions to the fixed-circle problem be found using the fuzzy soft set theory?

Problem 3: Is it possible to study other fixed-circle theorems using different contractive conditions on soft metric spaces?

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