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Error estimation for nonlinear pseudoparabolic equations with nonlocal boundary conditions in reproducing kernel space

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Abstract. In this paper, we discuss about nonlinear pseudoparabolic equations with nonlocal boundary conditions. An effective error estimation for this method has not yet been discussed. The aim of this paper is to fill this gap.

Keywords: Reproducing kernel method, error estimation, nonlinear pseudoparabolic equation.

1. Introduction

The nonlinear problems are very important. All kinds of boundary value conditions arise in the problems which make them more difficult to be solved. Nonlocal boundary conditions arise naturally in various engineering models and physical phenomena. Pseudoparabolic equation with nonlocal boundary conditions is one of the nonlinear problems. The precise statement of the problem is given as follows:
Let $T > 0$ and

$$\mathcal{D} = \Big\{(x,t) \in \mathbb{R}^2 : \quad \alpha < x < \beta, \quad 0 < t < T\Big\}.$$

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Determine a function $u : \mathcal{D} \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\big(a(x,t)\frac{\partial u}{\partial x}\big) - \eta\frac{\partial^2}{\partial t}\big(a(x,t)\frac{\partial u}{\partial x}\big) = f\big(x,t,u,\frac{\partial u}{\partial}\big) \\ u(x,0) = u_0(x) \\ u(\alpha,t) = \mu(t) \\ \int_\alpha^\beta u(x,t)dx = E(t). \end{cases} \tag{1}$$

We shall assume:

H_1. $0 < C_0 \leqslant a \leqslant C_1, \quad C_2 \leqslant \dfrac{\partial a}{\partial t} \leqslant C_3, \quad |\dfrac{\partial a}{\partial t}| \leqslant C_4, \text{For all}(x,t) \in \mathcal{D}.$

H_2. There exists a positive constant L such that

$$|f(x,t,p_1,p_2) - f(x,t,q_1,q_2)| \leqslant L(|p_1 - p_2| + |q_1 - q_2|), \quad (x,t) \in \mathcal{D}.$$

For simplicity, we always take $\alpha = 0$, $\beta = 1$. Equations (1) are the conditions for determining solution of this, that is, the solution of (1). If we could construct a function space, in which each function satisfies (1), then we can solve (1) in the function space. In order to construct such a function space, we need homogenize the conditions. Put

$$\tilde{u}(x,0) = u_0(x) - U(x,0) - u_0(x) + U_0(x),$$

where

$$U(x,t) = (1 - 3x^2 + 2x)E(t) + (1 - 2x)(\mu(t) - E(t)), \qquad U_0(x) = U(x,0).$$

Then we can obtain

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} - \frac{\partial}{\partial x}\big(a(x,t)\dfrac{\partial \tilde{u}}{\partial x}\big) - \eta\frac{\partial^2}{\partial t \partial x}\big(a(x,t)\dfrac{\partial \tilde{u}}{\partial x}\big) = g(x,t) + \tilde{f}\big(x,t,\tilde{u},\dfrac{\partial \tilde{u}}{\partial x}\big), \\ \tilde{u}(x,0) = 0, \\ \tilde{u}(\alpha,t) = 0, \\ \int_\alpha^\beta \tilde{u}(x,t)\,dx = 0. \end{cases} \tag{2}$$

Following we will always replace $\tilde{u}$ and $\tilde{f}$ with u and f in 1, respectively.

In [2], the existence and uniqueness of the solution for problem (1) are proved. In [13], It has been proved a very simple numerical algorithm for the approximations of problem (1) based on the reproducing kernel space.

The rest of the paper is organized as follows. In the next section, the reproducing kernel method for solving(1) is recalled. The error estimation is presented in Section 3. Numerical examples are provided in Section 4. Finally, the concussion of the paper will be dramn in Section 5.

2. Reproducing kernel method

In this section, we discuss about reproducing kernel space and solving nonlinear pseudoparabolic equations with nonlocal boundary conditions in reproducing kernel space. To solve (1), first we construct several reproducing kernel spaces.

Definition 2.1 The function space $W^3[0,1]$ is defined by

$$W^3[0,1] = \Big\{u|\ u'' \text{ is absolutly continuous on } [0,1],\ u''' \in L^2[0,1],\ u(0) = \int_0^1 u(x)dx = 0\Big\}.$$

Inner product and norm of $W^3[0,1]$ are respectively defined by

$$< u, v >_{W^3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{'''}(x)v^{'''}(x)dx,$$

$$\| u \|_{W^3} = \sqrt{< u, u >_{W^3}}, \qquad u, v \in W^3[0,1],$$

$W^3[0,1]$ is a reproducing kernel space and it's reproducing kernel $R_y(x)$ can be obtained by

$$R_y(x) = \begin{cases} a_1 + a_2x^+ a_3x^2 + \cdots + a_6x^5 + \frac{cx^6}{6!}, \\ b_1 + b_2x^+ b_3x^2 + \cdots + b_6x^5 + \frac{cx^6}{6!}. \end{cases} \tag{3}$$

We can obtain the coefficients a_i, b_i by using mathematica. For more details refer [13].

Definition 2.2 The function space $W^2[0,T]$ is defined by

$$W^2[0,T] = \Big\{u(x)|u'(x) \text{is absolutly continuous in } [0,T],$$
$$u''(x) \in L^2[0,T], u(0) = 0\Big\}.$$

The inner product and norm of $W^2[0,T]$ are respectively defined by

$$< u, v >_{W^2} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^T u''(t)v''(t)dt,$$

$$\| u \|_{W^2} = \sqrt{< u, u >_{W^2}}.$$

Definition 2.3 Let $\mathcal{D} = [0,1]\times[0,T]$, then the reproducing kernel space $W(\mathcal{D})$ is defined by

$$W(\mathcal{D}) = \Big\{u(x,t)|\frac{\partial^3 u}{\partial x^2 t}\text{is completely continuous in } \mathcal{D},$$
$$\frac{\partial^5 u}{\partial x^3 t^2} \in L^2(\mathcal{D}), u(0,t) = \int_0^1 u dx = 0\Big\}.$$

The inner product and norm of $W(\mathcal{D})$ is defined by

$$\begin{aligned} < u(x,t), v(x,t) >_W = & \sum_{i=0}^{2} \int_0^T \Big[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}u(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}v(0,t)\Big]dt \\ & + \sum_{j=0}^{1}\langle\frac{\partial^j}{\partial t^j}u(x,0), \frac{\partial^j}{\partial t^j}v(x,0)\rangle_W \\ & + \int\int_D \frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}u(x,t)\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}v(x,t)dxdt, \end{aligned}$$

and

$$\| u \|_W = \sqrt{< u, u >_W}.$$

In Equation (1), put

$$(\mathbb{L}u)(x,t) = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\big(a(x,t)\frac{\partial u}{\partial x}\big) - \eta\frac{\partial^2}{\partial t \partial x}\big(a(x,t)\frac{\partial u}{\partial x}\big).$$

It is clear that $\mathbb{L} : W(\mathcal{D}) \longrightarrow L^2(\mathcal{D})$ is a bounded linear operator. Put $\varphi_i(x,t) = R_{(y,s)}(x,t)$ and $\psi_i(x,t) = \mathbb{L}^*\varphi_i(x,t)$, where $R_{(y,s)}(x,t)$ is the reproducing kernel of $W(\mathcal{D})$, $\mathbb{L}^*$ is the adjoint operator of $\mathbb{L}$. The orthonormal system $\{\tilde{\psi}_i(x,t)\}_{i=1}^{\infty}$ of $W(\mathcal{D})$ can be derived using Gram-Schmidt orthogonalization process of $\{\psi_i(x,t)\}_{i=1}^{\infty}$ as follows,

$$\tilde{\psi}_i(x,t) = \sum_{k=1}^{i}\beta_{ik}\psi_k(x,t), \tag{4}$$

where β_{ik} are orthogonal coefficients.

According to [13], we have the following theorems and lemmas:

Lemma 2.4 $\psi_i(x,t) \in W(\mathcal{D})$.

Lemma 2.5 The function system $\{\psi_i(x,t)\}_{i=1}^{\infty}$ is a complete system in $W(\mathcal{D})$.

Theorem 2.6 If $u(x,t)$ is the solution of (1), then

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}|g(x_k,t_k) + \alpha_k|\tilde{\psi}_i(x,t),$$

where

$$\alpha_k = f(x_k, t_k, u(x_k, t_k), \partial_x u(x_k, t_k)|_{x=x_k}), \quad k = 1, 2, \dots,$$

and

$$u_n(x,t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} |g(x_k, t_k) + \alpha_k| \tilde{\psi}_i(x,t). \tag{5}$$

Lemma 2.7 If $u(x,t) \in W(\mathcal{D})$, then there exists a constant c such that

$$|u(x,t)| \leqslant c \parallel u(x,t) \parallel_m, |u^{(k)}(x,t)| \leqslant c \parallel u(x,t) \parallel_m, \qquad 1 \leqslant k \leqslant m-1$$

.

Theorem 2.8 The approximate solution $u_N(x,t)$ and it's derivatives $\partial_{xt}^{i+j} u_N(x,t)$, $i = 0,1,2$, $j = 0,1$ uniformly converge to exact solution $u(x,t)$ and it's derivatives $\partial_{xt}^{i+j} u(x,t)$, $i = 0,1,2$, $j = 0,1$ respectively.

3. Error estimation

In this section, we give the error estimation for nonlinear pseudoparabolic equations with nonlocal boundary conditions in reproducing kernel.

Theorem 3.1 Let $u_N(x,t)$ be the approximate solution of (1) in space $W(\mathcal{D})$ and $u(x,t)$ be the exact solution of (1). If $0 = x_1 < x_2 < \cdots < x_N = 1$, and $0 = t_1 < t_2 < \cdots < t_M = T$, and if $a(x,t), f(x,t) \in c^2[0,1]$, then

$$\parallel u(x,t) - u_N(x,t) \parallel \leqslant d_1 h, \quad \parallel u^{(k)}(x,t) - u_N^{(k)}(x,t) \parallel \leqslant d_1 h, \quad 1 \leqslant k \leqslant 2.$$

Proof. Note here that

$$Lu_N(x,t) = \sum_{i=1}^{N} A_i L\tilde{\psi}_i(x,t),$$

and

$$\begin{aligned}
(Lu_N)(x_n, t_n) &= \sum_{i=1}^{N} A_i (L\tilde{\psi}_i(x,t), \varphi_n(x,t)) \\
&= \sum_{i=1}^{N} A_i (\tilde{\psi}_i(x,t), L^*\varphi_n(x,t)) \\
&= \sum_{i=1}^{N} A_i (\tilde{\psi}_i(x,t), \psi_n(x,t)).
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n}\beta_{nj}(Lu_N)(x_j,t_j) &= \sum_{i=1}^{N}A_i(\tilde{\psi}_i(x,t),\sum_{j=1}^{n}\beta_{nj}\psi_j(x,t)) \\ &= \sum_{i=1}^{N}A_i(\tilde{\psi}_i(x,t),\tilde{\psi}_n(x,t)) \\ &= A_n.\end{aligned}$$

By induction, we have

$$Lu_N(x_j,t_j) = f(x_j,t_j), \qquad j = 1,2,\cdots,N,$$

By replacement

$$R_N(x,t) = f(x,t) - Lu_N(x,t),$$

Obviously, $R_N(x,t) = 0$, $j = 1,2,\cdots,N$ for $0 = x_1 < x_2 < \cdots < x_N = 1$, and $0 = t_1 < \cdots < t_M = T$. Suppose that $l(x,t)$ is a polynomial of degree 1 that interpolates the function $R_N(x,t)$. It is clear that $l(x,t) = 0$. Also, for $t \in [t_i,t_{i+1}]$ and $\forall x \in [x_i,x_{i+1}]$ we have

$$R_N(x,t) = R_N(x,t) - l(x,t) = \frac{\partial^4_{xt}R_N(\xi_i,\eta_i)}{2!}(x-x_i)(x-x_{i+1})(t-t_i)(t-t_{i+1}), \quad (6)$$

Hence, for $\xi_i \in [x_i,x_{i+1}] \times [t+i,t_{i+1}]$,

$$|R_N(x,t)| \leqslant \frac{|\partial^4_{xt}R_N(\xi_i,\eta_i)|}{32}h^4 = c_ih_i^4, \qquad c_i = \frac{|\partial^4_{xt}R_N(\xi_i,\eta_i)|}{32}, \qquad h_i = |x_{i+1}-x_i|\times|t_{i+1}-t_i|.$$

Putting $c = max_{1\leqslant i\leqslant N-1}c_i$ and $h = max_{1\leqslant i\leqslant N-1}h_i$, we have

$$\parallel R_N(x,t) \parallel_\infty = \max_{x\in[0,1],t\in[0,T]}|R_N(x,t)| \leqslant ch^4.$$

We have,

$$\parallel R_N(x,t) \parallel_1 = \sqrt{\langle R_N(x,t),R_N(x,t)\rangle} = \langle R_N(x,t),R_N(x,t)\rangle^{\frac{1}{2}}$$

$$
\begin{aligned}
&=\Big[\sum_{i=0}^{2}\int_0^T\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}R_N(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}R_N(0,t)\right]dt\\
&\quad+\sum_{j=0}^{1}\langle\frac{\partial^j}{\partial t^j}R_N(x,0),\frac{\partial^j}{\partial t^j}R_N(x,0)\rangle_{W_1}\\
&\quad+\int\int_D\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}R_N(x,t)\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}R_N(x,t)dxdt\Big]^{\frac{1}{2}}\\
&=\Big[\sum_{i=0}^{2}\int_0^T\big[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}R_N(0,t)\big]^2dt\\
&\quad+\|\frac{\partial^j}{\partial t^j}R_N(x,0)\|^2\\
&\quad+\int_0^1\int_0^T\big[\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}R_N(x,t)\big]^2dxdt\Big]^{\frac{1}{2}}
\end{aligned}
$$

Obviously,

$$\sum_{i=0}^{2}\int_0^T\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}R_N(0,t)\right]^2dt\leqslant c^2h^8, \tag{7}$$

$$\|\frac{\partial^j}{\partial t^j}R_N(x,0)\|^2\leqslant ch^4, \tag{8}$$

and

$$
\begin{aligned}
\int_0^1\int_0^T\big[\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}R_N(x,t)\big]^2dxdt&=\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\int_{x_i}^{x_{i+1}}\int_{t_i}^{t_{i+1}}\big[\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}R_N(x,t)\big]^2dxdt\\
&\leqslant\tilde{c}h^2\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\int_{x_i}^{x_{i+1}}\int_{t_i}^{t_{i+1}}dxdt\\
&\leqslant\tilde{c}\bar{c}h^2\leqslant cch^2,
\end{aligned} \tag{9}
$$

where cc is constant. In view of (7), (8) and (9), there axists a constant C such that

$$
\begin{aligned}
\|R_N(x,t)\|_1=&\Big[\sum_{i=0}^{2}\int_0^T\big[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}R_N(0,t)\big]^2dt\\
&+\|\frac{\partial^j}{\partial t^j}R_N(x,0)\|^2\\
&+\int_0^1\int_0^T\big[\frac{\partial^3}{\partial x^3}\frac{\partial^2}{\partial t^2}R_N(x,t)\big]^2dxdt\Big]^{\frac{1}{2}}\\
&\leqslant Ch,
\end{aligned}
$$

Noting that

$$u(x,t) - u_N(x,t) = L^{-1}R_N(x,t),$$

there exists a constant d_1 such that

$$\|u(x,t) - u_N(x,t)\|_3 = \|L^{-1}R_N(x,t)\|_3 \leqslant \| L^{-1} \| \| R_N(x,t) \|_1 \leqslant d_1 h,$$

According to Lemma (2.7), it is easy to see that

$$\|u(x,t) - u_N(x,t)\|_\infty \leqslant d_1 h, \qquad \|u^{(k)}(x,t) - u_N^{(k)}(x,t)\|_\infty \leqslant d_1 h.$$

■

4. Numerical example

Example We consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x} - \frac{\partial^2 u}{\partial x^2 \partial t} = f\big(x,t,u,\frac{\partial u}{\partial x}\big), & 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant 1, \\ u(x,0) = u_0(x), & 0 \leqslant x \leqslant 1, \\ u(\alpha,t) = \mu(t), & 0 \leqslant t \leqslant 1, \\ \int_\alpha^\beta u(x,t)dx = E(t), & 0 \leqslant t \leqslant 1, \end{cases} \tag{10}$$

where $\mu(t) = 4 + \cos 2t$, $E(t) = 4 + \cos 2t + 8e^t \sin^2 0.5$, $u_0(x) = 5 + 4\sin x$, $f(x,t,p,q) = -2\sin 2t + 12e^t \sin x - \sin p - \cos q$. The true solution is $u(x,t) = \cos 2t + 4(e^t \sin x + 1)$. By using Mathematica 5.0 and the presented method in this paper, we calculate the approximate solution $u_N(x,t)$. Table 1 gives the true solution, approximate solution and their relative errors.

Table 1. The numerical result of example

$(\mathbf{x},\mathbf{t})$	$\mathbf{u}(\mathbf{x},\mathbf{t})$	$\mathbf{u_n}(\mathbf{x},\mathbf{t})$	**Relative error**
$(\frac{1}{10}, \frac{1}{10})$	5.4214	5.42426	0.000527649
$(\frac{9}{10}, \frac{1}{5})$	5.72739	5.72966	0.000396222
$(\frac{7}{10}, \frac{2}{5})$	7.30674	7.30919	0.000335972
$(\frac{1}{2}, \frac{3}{5})$	8.26405	8.26508	0.000124337
$(\frac{3}{10}, \frac{4}{5})$	8.69865	8.6942	0.000512001
$(\frac{1}{10}, 1)$	8.69994	8.68389	0.00184534
$(1, \frac{1}{10})$	4.66935	4.67176	0.00051487
$(\frac{4}{5}, \frac{3}{10})$	6.60157	6.60397	0.000363313
$(\frac{3}{5}, \frac{1}{2})$	7.85664	7.85874	0.000267896
$(\frac{2}{5}, \frac{7}{10})$	8.54095	8.53987	0.000125427
$(\frac{1}{5}, \frac{9}{10})$	8.74809	8.73873	0.00106994
$(1, 1)$	12.7333	12.7289	0.000345507

5. Conclusion

In this paper, we discussed about nonlinear pseudoparabolic equatins with nonlocal boundary conditions. An effective error estimation for this method has not yet been discussed. In this paper, we give the error estimation for nonlinear pseudoparabolic equatins with nonlocal boundary conditions in reproducing kernel space.

References

[1] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. Ser. A, 272 (1220) (1972), 47-78.
[2] A. Bouziani, Solvability of nonlinear pseudoparabolic equation with a nonlocal boundary condition. Analysis 55 (2003), 883-904 .
[3] A. Bouziani, Initial-boundary value problems for a class of pseudoparabolic equations with integral boundary conditions. J. Math. Anal. Appl. 291 (2004), 371-386.
[4] P. J. Chen, M. E. Gurtin, On a theory of heat conduction involving two temperatures. Z. Angew. Math. Phys. 19 (1968), 614-627 .
[5] M. G. Cui, Y. Z. Lin, Nonlinear numerical analysis in the reproducing kernel space. Nova 2009.
[6] D. Q. Dai, H. Yu, Nonlocal boundary problems for a third-order one-dimensional nonlinear pseudoparabolic equation. SIAM J. Nonlinear Anal. 66 (2007), 179-191.
[7] G. V. Demidenko, S. V. Uspenskii, Equations and Systems that are not Solved with Respect to the Highest Derivative. Nauchnaya Kniga, Novosibirsk, 1998, 437-443.
[8] I. E. Egorov, S. G. Pyatkov and S. V. Popov, Nonclassical Operator Differential Equations. Nauka, Novosibirsk, 2000, 336-342.

[9] A. Favini, A. Yagi, Degenerate Differential Equations in Banach Spaces. vol. 215, Marcel Dekker, New York, 1999, 313-335.
[10] H. Gajewski, K. Grger, K. Zacharias, Nichtlineare Operatorgleichungen und Operatordif- ferentialgleichungen. Akademie, Berlin, 1974, 281-293.
[11] E. I. Kaikina, Nonlinear pseudoparabolic type equations on a half-line with large initial data. Nonlinear Anal. 67 (2007), 2839-2858 .
[12] A. I. Kozhanov, A.I, An initial-boundary value problem for equations of the generalized Boussi- nesq equation type with a nonlinear source. Mat. Zametki 65 (1) (1999), 70-75.
[13] Y. Z. Lin, Y. F. Zhou, Solving nonlinear pseudoparabolic equations with nonlocal boundary cnditions in reproducing kernel space. Numer Algor, 52 (2009), 173-189.
[14] X. Y. Li, B. Y. Wu, Error estimation for the reproducing kernel method to solve linear boundary value problems. App. Math, 243 (2013), 10-15.
[15] S. Mesloub, A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation. J. Math. Anal. Appl. 316 (2006), 189-209 .
[16] V. Padron, Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation. Trans. Am. Math. Soc. 356 (7) (2004), 2739-2756.
[17] R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. Amer. Math. Soc., Providence 49 (1997), 278-282.
[18] G. A. Sviridyuk, V. E. Fdorov, Analytic semigroups with kernels, and linear equations of Sobolev type. Sib. Mat. Z. 36 (5) (1995),1130-1145.