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On quasi-Baer modules

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Abstract. Let R be a ring, σ be an endomorphism of R and M_R be a σ-rigid module. A module M_R is called *quasi-Baer* if the right annihilator of a principal submodule of R is generated by an idempotent. It is shown that an R-module M_R is a quasi-Baer module if and only if $M[[x]]$ is a quasi-Baer module over the skew power series ring $R[[x;\sigma]]$.

Keywords: Quasi-Baer modules, σ-rigid modules, quasi-Armendariz modules.

1. Introduction

Throughout the paper R always denotes an associative ring with unity and M_R will stand for a right R-module. Recall from [15] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent.

In [15] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings. The class of Baer rings includes the von Neumann algebras. In [10] Clark defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. Then he used the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Every prime ring is a quasi-Baer ring. In 1974, Armendariz considered the behavior of a polynomial ring over a Baer ring by obtaining the following result: Let R be a *reduced* ring (i.e., R has no nonzero nilpotent elements). Then $R[x]$ is a Baer ring if and only if R is a Baer ring ([3], Theorem B). Armendariz provided an example to show that the reduced condition

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is not superfluous. In [6] Birkenmeier, Gary F.; Kim, J.Y.; Park, J.K. showed that the quasi-Baer condition is preserved by many polynomial extension.

From now on, we always denote the skew power series ring by $R[[x;\sigma]]$, where $\sigma : R \longrightarrow R$ is an endomorphism. The skew power series ring $R[[x;\sigma]]$ is then the ring consisting of all power series of the form $\sum_{i=0}^{\infty} a_i x^i$ $(a_i \in R)$, which are multiplied using the distributive law and the Ore commutation rule $xa = \sigma(a)x$, for all $a \in R$. Given a right R-module M_R, we can make $M[[x]]$ into a right $R[[x;\sigma]]$-module by allowing power series from $R[[x;\sigma]]$ to act on power series in $M[[x]]$ in the obvious way, and applying the above "twist" whenever necessary. The verification that this defines a valid $R[[x;\sigma]]$-module structure on $M[[x]]$ is almost identical to the verification that $R[[x;\sigma]]$ is a ring, and it is straightforward.

For a nonempty subset X of M, put $ann_R(X) = \{a \in R \mid Xa = 0\}$. In [20], Lee-Zhou introduced Baer, quasi-Baer and p.p.-modules as follows:

(1) M_R is called *Baer* if, for any subet X of M, $ann_R(X) = eR$ where $e^2 = e \in R$.

(2) M_R is called *quasi-Baer*, if for any submodule $X \subseteq M$, $ann_R(X) = eR$ where $e^2 = e \in R$.

(3) M_R is called *p.p.* if, for any element $m \in M$, $ann_R(M) = eR$ where $e^2 = e \in R$.

Clearly, a ring R is Baer (resp. p.p. or quasi-Baer) if and only if R_R is Baer (resp. p.p. or quasi-Baer) module. If R is a Baer (resp. p.p. or quasi-Baer) ring, then for any right ideal I of R, I_R is Baer (resp. p.p. or quasi-Baer) module.

A module M_R is called *principally quasi-Baer (or simply p.q.-Baer)* if, for any $m \in M$, $ann_R(mR) = eR$ where $e^2 = e \in R$. It is clear that R is a right p.q.-Baer ring if and only if R_R is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

We use $I(R)$, $S_\ell(R)$ and $C(R)$ to denote the set of idempotents, the set of left semi-central idempotents and the center of R, respectively.

In this paper we show that, if M_R is a σ-rigid module, then $M[[x]]$ is quasi-Baer over $R[[x;\sigma]]$ if and only if M_R is quasi-Baer. As a corollary, we show that if R is σ-rigid then $R[[x;\sigma]]$ is quasi-Baer if and only if R is quasi-Baer.

2. Quasi-Baer modules

According to Kim et al. [16], a ring R is called *power-serieswise Armendariz* if whenever $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$, we have $a_i b_j = 0$ for all i, j. Let $\sigma \in End(R)$ and M be an R-module.

According to Lee and Zhou [20], a module M_R is called *σ-Armendariz of power series type* if the following conditions are satisfied:

(1) For $m \in M$ and $a \in R, ma = 0$ if and only if $m\sigma(a) = 0$.

(2) For any $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x]]$ and $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\sigma]]$, $m(x)f(x) = 0$ implies $m_i \sigma^i(a_j) = 0$ for all i, j.

Definition 2.1 Given a module M_R and an endomorphism $\sigma : R \to R$. Then M_R is called a *σ-rigid* module if for each $m \in M, a \in R$, $ma = 0$ if and only if $m\sigma(Ra) = 0$.

Theorem 2.2 Let M_R be a *σ-rigid* module.

(1) If $M[[x]]_{R[[x;\sigma]]}$ is p.q.-Baer, then M_R is p.q.-Baer.

(2) If M_R is p.q.-Baer, then M is *power-serieswise σ-quasi-Armendariz.*

Proof. (1) Let $m \in M$. Since $M[[x]]_R$ is p.q.-Baer, there exists an idempotent $e(x) = e_0 + e_1(x) + ... \in R[[x;\sigma]]$, such that $ann_{R[[x;\sigma]]}(mR[[x;\sigma]]) = e(x)R[[x;\sigma]]$.

Since $mRe(x) = 0$, $mRe_0 = 0$. Thus $e_0R \subseteq ann_R(mR)$. Let $b \in ann_R(mR)$. Then $b \in ann_{R[[x;\sigma]]}(mR[[x;\sigma]])$, since M is a σ-rigid module. Thus $b = e(x)b$ and $b = e_0b \in e_0R$. Therefore $ann_R(mR) = e_0R$ and M_R is p.q.-Baer.

(2) Assume that $(\sum_{i=0}^{\infty} m_ix^i)R[[x;\sigma]](\sum_{j=0}^{\infty} b_jx^j) = 0$ with $m_i \in M, b_j \in R$. Let c be an arbitrary element of R. Then we have the following equation:

$$\sum_{k=0}^{\infty}(\sum_{i+j=k} m_ix^icb_jx^j) = \sum_{k=0}^{\infty}(\sum_{i+j=k} m_i\sigma^i(cb_j))x^k = 0$$

and hence

$$\sum_{i+j=k} m_i\sigma^i(cb_j) = 0 \quad forall \quad k \geqslant 0. \tag{1}$$

We show that $m_ix^iRb_jx^j = 0$ for all i, j. We proceed by induction on $i+j$. From equation 1, we obtain $m_0Rb_0 = 0$. This proves the case $i+j = 0$. Now suppose that $m_ix^iRb_jx^j = 0$ for $i + j \leqslant n-1$. Then $b_j \in ann_R(m_iR)$ for $j = 0, ..., n-1$ and $i = 0, ..., n-1-j$, since M is σ-rigid. Now $ann_R(m_iR) = e_iR$ for some idempotent $e_i \in R$. Thus $e_ib_j = b_j$ for $j = 0, ..., n-1$ and $i = 0, ..., n-1-j$. If we put $f_j = e_0...e_{n-1-j}$ for $j = 0, ..., n-1$, then $f_jb_j = b_j$ and $f_j \in ann_R(m_0R) \cap ... \cap ann_R(m_{m-n-j}R)$. For $k = n$ replacing c by cf_0 in 1 and using σ-rigid property of M, we obtain $m_0cb_n = m_0cf_0b_n = 0$. Hence $m_0Rb_n = 0$. Continuing this process (replacing c by cf_j in 1, for $j = 1, ..., n-1$ and using σ-rigid property of M), we obtain $m_iRb_j = 0$ and so $m_ix^iRb_jx^j = 0$ for $i + j = n$. Therefore M_R is power-serieswise σ-quasi-Armendariz. ■

Lemma 2.3 Let M_R be an σ-rigid module and M_R be a p.q.-Baer module. Let $ann_T(m(x)T) = e(x)T$ for some idempotent $e(x) = e_0 + e_1x + ... \in T$. Then $ann_T(m(x)T) = e_0T$ and e_0 is an idempotent of R.

Proof. Let $m(x) = m_0 + m_1x + ...$ By Theorem 2.2, M is power serieswise σ-quasi-Armendaris. Since $m(x)Te(x) = 0$, so $m_iRe_0 = 0$, for each $i \geqslant 0$. Hence $e_0 \in ann_T(m(x)T)$, and $e_0T \subseteq e(x)T$. Now let $f(x) = b_0 + b_1x + ... \in ann_T(m(x)T)$. Then $m_iRb_j = 0$, for all i, j, since M is power serieswise σ-quasi-Armendariz and σ-rigid. Thus $b_j \in ann_T(m(x)T) = e(x)T$, since M_R is σ-rigid. Hence $b_j = e(x)b_j$ and $b_j = e_0b_j$ for each j. Therefore $f(x) = e_0f(x) \in e_0T$. ■

Theorem 2.4 Let M be an σ-rigid module and $T = R[[x;\sigma]]$. Then $M[[x]]_T$ is p.q.-Baer if and only if M_R is p.q.-Baer and the right annihilator of any countably-generated submodule of M is generated by an idempotent.

Proof. If $M[[x]]_T$ is p.q.-Baer, then by Theorem 2.2, M_R is p.q.-Baer. Let $X = \{a_0, a_1, ...\}$ be a countable subset of M and $\langle X \rangle$ be the right submodule of M generated by X. Let $m(x) = a_0 + a_1x + ...$ $M[[x]]_T$ is p.q.-Baer, so by Lemma 2.3, there exists an idempotent $e \in R$ such that $ann_T(m(x)T) = eT$. Clearly, $eR \subseteq ann_R(\langle X \rangle)$. Let $b \in ann_R(\langle X \rangle)$. Then $a_iRb = 0$ for each i. Hence $m(x)Tb = 0$, Since M_R is σ-rigid, $b = eb \in eR$. Consequently, $ann_R(\langle X \rangle) = eR$.

Now assume M_R is p.q.-Baer and the right annihilator of any countably-generated submodule of M generated by an idempotent. Let $m(x) = \sum_{i=0}^{\infty} m_ix^i \in M[[x]]$. Let N be the submodule of M generated by the coefficients $\{m_0, m_1, ...\}$. Then $ann_R(N) = eR$

for some idempotent $e \in R$. Since $m_iRe = 0$ for each i and M_R is σ-rigid, so by σ-rigid property of M_R, $m(x)Te = 0$ and that $eT \subseteq ann_T(m(x)T)$. Now let $f(x) = \sum_{j=0}^{\infty} a_j x^j \in ann_T(m(x)T)$. Then $m_iRa_j = 0$, for each i, j, since M is power serieswise σ-quasi-Armendariz. Then $a_j \in eR$, for each j, and $a_j = ea_j$. Therefore $f(x) = ef(x) \in eT$. Consequently, $M[[x]]_T$ is p.q.-Baer. ■

Corollary 2.5 Let M be a right R-module. Then $M[[x]]_{R[[x]]}$ is right p.q.-Baer if and only if M_R i right p.q.-Baer and for any countably-generated submodule N of M, $ann_R(N) = eR$ for an idempotent $e \in R$.

Remark 1 *In [20], it was proved that, if M_R is σ-Armendariz of power series type, then $M[[x]]_T$ is p.p. if and only if for any countable subset X of M, $ann_R(X) = eR$ where $e^2 = e \in R$. By Zalesskii and Neroslavskii [9], there is a simple Notherian ring R which is not a domain and in which 0 and 1 are the only idempotents. Thus R_R is p.q.-Baer ring which is not right p.p. Therefore our corollary 2.4, is not implied from [20].*

The following example show that the condition "for any countably-generated submodule N of M, $ann_R(N) = eR$ for an idempotent $e \in R$" in corollary 2.5 is not superfluous. On the other hand there is p.q.-Baer module M_R such that $M[[x]]_{R[[x]]}$ is not p.q.-Baer.

***Example* 2.6** Let M_1 be a right p.q.-Baer R_1-module. Let

$$M = \{(m_n) \in \prod_{n=1}^{\infty} M_n \mid m_n \text{ is eventually contsant } \}$$

where $M_n = M_1$ for $n > 1$ and let

$$R = \{(a_n) \in \prod_{n=1}^{\infty} R_n \mid a_n \text{ is eventually constant } \}$$

where $R_n = R_1$ for $n > 1$. Clearly M is a right R module. Clearly M is right p.q.-Baer. Let m be a nonzero element of M_1. Let $m_1 = (m, 0, 0, ...)$, $m_2 = (m, 0, m, 0, 0, ...)$, $m_3 = (m, 0, m, 0, m, 0, 0, ...)$, ... Let $\langle X\rangle$ be the submodule of M generated by $X = \{m_1, m_2, ...\}$. One can show that $ann_R(\langle X\rangle)$ is not generated by any idempotent, hence by Theorem 2.4, $M[[x]]_{R[[x]]}$ is not right p.q.-Baer.

Definition 2.7 (Z. Liu [21]). Let $\{e_0, e_1, ...\}$ be a countable family of idempotents of R. We say $\{e_0, e_1, ...\}$ has a generalized join in $I(R)$ if there exists an idempotent $e \in I(R)$ such that:
(1) $e_iR(1 - e) = 0$
(2) If $f \in I(R)$ is such that $e_iR(1 - f) = 0$, then $eR(1 - f) = 0$.

Lemma 2.8 Let R be a ring and $S_\ell(R) \subseteq C(R)$. Then the following are equivalent:

(1) R is right p.q.-Baer and any countable family of idempotents in R has a generalized join in $I(R)$.
(2) R is right p.q.-Baer and the right annihilator of any countably-generated right ideal of R is generated by an idempotent.

Proof. (1) ⇒ (2) Let $X = \{a_i\}_{i \in I}$ be a countable subset of R and $\langle X\rangle$ be the right ideal of R generated by X. Then for each $a_i \in X$, $ann_R(a_iR) = e_iR$ for some idempotent $e_i \in R$. Let h be a generalized join of the set $\{1 - e_i \mid i \in I\}$. Then $(1 - e_i)R(1 - h) = 0$. Hence $r(1 - h) = e_ir(1 - h)$ for all $r \in R$. Since $e_i \in S_\ell(R) \subseteq C(R)$, $a_ir(1 - h) = a_ie_ir(1 - h) = 0$

for all i and each $r \in R$. Hence $(1-h) \in ann_R(\langle X \rangle)$ and $(1-h)R \subseteq ann_R(\langle X \rangle)$. Suppose that $b \in ann_R(\langle X \rangle)$. Hence $b = e_i b$ for each i. Since $e_i \in S_\ell(R) \subseteq C(R)$, $bR(1-e_i) = 0$ for each i. Since R is right p.q.-Baer, so $ann_R(bR) = fR$, where f is a left semicentral idempotent of R. Thus $(1-e_i) \in ann_R(bR) = fR$, so $(1-e_i) = f(1-e_i)$ for each i. Hence from $(1-e_i) \in C(R)$, we have $(1-e_i)R(1-f) = 0$. Since h is a generalized join of the set $\{1-e_i \mid i \in I\}$, $hR(1-f) = 0$. Hence $b = b - bf = (1-f)b = (1-h)(1-f)b \in (1-h)R$. Therefore $ann_R(\langle X \rangle) = (1-h)R$.

(2) $\Rightarrow$ (1) Suppose that $\{e_i \mid i = 0, 1, ...\}$ is a countable family of idempotent of R. Let J be the right ideal of R generated by $\{e_i \mid i = 0, 1, ...\}$. then $ann_R(J) = eR$ for some left semicentral idempotent e. Let $h = 1-e$. Then $e_i r(1-h) = 0$ for each $r \in R$. Suppose that f is an idempotent of R such $e_i R(1-f) = 0$ for each i. Then $r(1-f) \in ann_R(J)$ for each $r \in R$. Thus $r(1-f) = er(1-f)$ and $hr(1-f) = (1-e)r(1-f) = 0$. Hence h is a generalized join of the set $\{e_i \mid i = 0, 1, ...\}$. ■

Theorem 2.9 Let R be a ring with $S_\ell(R) \subseteq C(R)$ and σ be an endomorphism of R. Let R_R be a σ-rigid module. Then the following are equivalent:

(1) $R[[x;\sigma]]$ is right p.q.Baer.
(2) R is right p.q-Baer and any countable family of idempotents of R has a generalized join in $I(R)$.

Proof. This follows from Theorem 2.4 and Lemma 2.8. ■

Corollary 2.10 *(*Z. Liu [21]*)*. Let R be a ring with $S_\ell(R) \subseteq C(R)$. Then the following conditions are quivalent:

(1) $S = R[[x]]$ is right p.q.-Baer.
(2) R is right p.q.-Baer and any countable family of idempotents in R has a generalized join in $I(R)$.

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