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Fixed point theory in generalized orthogonal metric space

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Abstract. In this paper, among the other things, we prove the existence and uniqueness theorem of fixed point for mappings on a generalized orthogonal metric space. As a consequence of this, we obtain the existence and uniqueness of fixed point of Cauchy problem for the first order differential equation.

Keywords: Fixed point, orthogonal set, solution, generalized metric space, Cauchy problem.

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1. Introduction

Concept of generalized metric space has been introduced in [8]. Extension of Banach fixed point theorem in generalized metric space has been established in [8]. This has been studied by many authors and important results has been obtained [3, 6, 7].
Recently, Eshaghi and et. [4] introduced the notion of orthogonal sets and orthogonal metric spaces. They also proved a real extension of Banach contractive principle [4]. Generalizations of this theorem has been considered in [1, 2, 9].
Let us consider Cauchy problem for the first order differential equation

$$\begin{cases} \acute{x}(t) = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

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where the function f is defined in the domain

$$|t - t_0| \leqslant a, \ |x - x_0| \leqslant b,$$

and satisfies the condition

$$|f(t, x_1) - f(t, x_2)| \leqslant \frac{K}{|t - t_0|}|x_1 - x_2|, \ 0 < K < 1.$$

In this paper, we are interested to define a new concept of generalized orthogonal metric space. Conditions under them a function in a generalized orthogonal metric space has a unique fixed point will be obtained. Furthermore, we apply the obtained results to show existence and uniqueness of solution of Cauchy problem for the first order differential equation (1). The solution of differential equation will be expressed as a fixed point of a suitable integral operator. The paper is organized as follows:

In section 2, we state some definitions and recall extension of Banach fixed point theorem in orthogonal metric space. In section 3, we prove main result and we show the existence and uniqueness of fixed point for mappings on generalized orthogonal metric space. In section 4, applying the result of section 3, we prove the existence and uniqueness of solution of Cauchy problem for the first order differential equation (1).

2. Preliminaries

In this section, some preliminaries and notations which are necessary for later are recalled.

The extended line is the ordered space $[-\infty, +\infty]$, considering of all points of the number line $\mathbb{R}$ and two points, denoted by $-\infty$, $+\infty$ with the usual order relation for points of $\mathbb{R}$. A map $d : X \times X \to [0, \infty]$ is called a generalized metric on the set X, if the following conditions are satisfied:

1. $d(x, y) = d(y, x)$ for any points $x, y \in X$.
2. $d(x, y) = 0 \iff x = y$ for any points $x, y \in X$.
3. $d(x, z) \leqslant d(x, y) + d(y, z)$ for any points $x, y \in X$ considering that if $d(x, y) = \infty$ or $d(y, z) = \infty$ then $d(x, y) + d(y, z) = \infty$.

A set X is called a generalized metric space with a defined metric on it and denoted by (X, d).

Definition 2.1 [4] Let $X \neq \phi$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0; \ ((\forall y; y \perp x_0) \ \ or \ \ (\forall y; x_0 \perp y)),$$

it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

In the following, we give some examples of orthogonal sets.

***Example* 2.2** Let $X = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set..

By the following example, we can see that x_0 is not necessarily unique.

***Example* 2.3** Let $X = [0, \infty)$, we define $x \perp y$ if $xy \in \{x, y\}$, then by setting $x_0 = 0$ or $x_0 = 1$, $(X, \perp)$ is an O-set.

Let $(X, \perp)$ be an O-set. We consider the notion of O-sequence.

Definition 2.4 [4] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if

$$((\forall n; x_n \perp x_{n+1}) \quad or \quad (\forall n; x_{n+1} \perp x_n)).$$

Let $(X, d, \perp)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). Now, we consider following definitions.

Definition 2.5 [4] The space X is orthogonally complete (briefly O-complete) if every cauchy O-sequence is convergent.

Let $(X, d, \perp)$ be an orthogonal metric space and $0 < \lambda < 1$.

Definition 2.6 [4] *i)* A mapping $f : X \to X$ is said to be orthogonal contraction ($\perp-$contraction) with Lipchitz constant λ if

$$d(fx, fy) \leqslant \lambda d(x, y) \qquad if x \perp y. \tag{2}$$

ii) A mapping $f : X \to X$ is called orthogonal preserving ($\perp-$preserving) if $f(x) \perp f(y)$ if $x \perp y$.

iii) A mapping $f : X \to X$ is orthogonal continuous ($\perp-$continuous) in $a \in X$ if for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X such that $a_n \to a$ then $f(a_n) \to f(a)$. Also f is $\perp-$continuous on X if f is $\perp-$continuous in each $a \in X$.

Example **2.7** Let $X = [0, 1)$ and let the metric on X be the Euclidian metric. Define $x \perp y$ if $xy \leqslant \max\{\frac{x}{2}, \frac{y}{2}\}$. X is not complete but it is O-complete. Let $x \perp y$ and $xy \leqslant \frac{x}{2}$. If $\{x_k\}$ is an arbitrary Cauchy O-sequence in X, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all n or there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} \leqslant \frac{1}{2}$ for all n. It follows that $\{x_{k_n}\}$ converges to a $x \in [0, 1)$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent. Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x}{2} & , \quad x \leqslant \frac{1}{2}, \\ 0 & , \quad x > \frac{1}{2}. \end{cases}$$

Also, $x \perp y$ and $xy \leqslant \frac{x}{2}$. So $x = 0$ or $y \leqslant \frac{1}{2}$. We have the following cases:

case 1) $x = 0$ and $y \leqslant \frac{1}{2}$. Then $f(x) = 0$ and $f(y) = \frac{y}{2}$.

case 2) $x = 0$ and $y > \frac{1}{2}$. Then $f(x) = f(y) = 0$.

case 3) $y \leqslant \frac{1}{2}$ and $x \leqslant \frac{1}{2}$. Then $f(y) = \frac{y}{2}$ and $f(x) = \frac{x}{2}$.

case 4) $y \leqslant \frac{1}{2}$ and $x > \frac{1}{2}$. Then $x - y > y$, $f(y) = \frac{y}{2}$ and $f(x) = 0$.

These cases implies that $f(x)f(y) \leqslant \frac{f(x)}{2}$. Hence f is $\perp$-preserving.
Also, one can see that $|f(x) - f(y)| \leqslant \frac{1}{2}|x - y|$. Hence f is $\perp-$contraction. But f is not a contraction. Otherwise, for two points $\frac{1}{2}$ and $\frac{2}{3}$ and for all $0 < c < 1$ we have $|f(\frac{1}{2}) - f(\frac{2}{3})| \leqslant c|\frac{1}{2} - \frac{2}{3}|$ and one can conclude that, it's a contradiction.
Let $\{x_n\}$ be an arbitrary O-sequence in X such that $\{x_n\}$ converges to $x \in X$. Since f is $\perp-$contraction, for each $n \in \mathbb{N}$ we have

$$|f(x_n) - f(x)| \leqslant \frac{1}{2}|x_n - x|.$$

As n goes to infinity, f is $\perp$-continuous. But as it can be easily seen that f is not continuous.

Now, we can state the main theoretical result of [4]. Sufficient conditions under them will any mapping on an orthogonal metric space have a unique fixed point are given in the following theorem.

Theorem 2.8 Let $(X, d, \perp)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp-$continuous, $\perp-$contraction (with Lipschitz constant λ) and $\perp-$preserving. Then f has a unique fixed point x^* in X and is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.

On the other hand, one has the following assertion.

Theorem 2.9 Given a point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and consider the differential equation (1). Let P be a Picard mapping defined by

$$(Px)(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau,\ t \in \mathbb{R}. \tag{3}$$

Note that $(Px)(t_0) = x_0$ for any x. The mapping $x : I \to \mathbb{R}^n$ is a solution to $\acute{x} = f(t, x)$ with the initial condition $x(t_0) = x_0$ if and only if $x = Px$.

Proof. Assuming $x = Px$,

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau.$$

meaning $\acute{x} = f(t, x(t))$, $x(t_0) = x_0$. Conversely, assuming x is a solution to $\acute{x} = f(t, x)$ with the initial condition $x(t_0) = x_0$,

$$\acute{x} = f(t, x(t)),\ x(t_0) = x_0,$$

meaning

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau,$$

and so $x = Px$. ■

3. Main Results

In this section, we state and prove our existence and uniqueness result. We begin with the following definition.

Definition 3.1 A map $d : X \times X \to [0, \infty]$ is called a generalized metric on the orthogonal set $(X, \perp)$. If the following condition are satisfied:

1. $d(x, y) = d(y, x)$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;
2. $d(x, y) = 0 \Longleftrightarrow x = y$ for any points $x, y \in X$ such that $x \perp y$ and $y \perp x$;
3. $d(x, z) \leqslant d(x, y) + d(y, z)$ for any points $x, y, z \in X$ such that $x \perp y$, $y \perp z$ and $x \perp z$ considering that if $d(x, y) = \infty$ or $d(y, z) = \infty$ then $d(x, y) + d(y, z) = \infty$.

In this case the orthogonal set X is called generalized orthogonal metric space and is denoted by $(X, d, \perp)$. The concept of completeness of a generalized orthogonal metric space is defined in the usual way. Let $(X, d, \perp)$ be a generalized orthogonal complete metric space. We have the following fixed point theorem.

Theorem 3.2 Let $f : X \to X$ be a $\perp$-preserving and $\perp$-continuous map such that

1) $d(fx, fy) \leqslant \lambda d(x, y)$ for any points x and y in X such that $x \perp y$ and $0 \leqslant \lambda < 1$;

2) For any point $x \in X$ there exists n_0 such that for $(f, \perp)$-orbit $\{f^n x\}_{n=0}^{\infty}$ we have $d(f^{n_0} x, f^{n_0+1} x) < \infty$;

3) If $x \perp y$, $fx = x$ and $fy = y$ then $d(x, y) < \infty$;

Then there exists a unique fixed point x^* of the map f and $\lim_{n\to\infty} f^n x = x^*$ for any point $x \in X$.

Proof. We consider the $(f, \perp)$-orbit $\{f^n x\}_{n=0}^{\infty}$ of an arbitrary point $x \in X$. Suppose that

$$x \perp fx,\ fx \perp f^2 x,\ f^2 x \perp f^3 x,\ \cdots, f^n x \perp f^{n+1} x, \cdots.$$

By virtue of 2, one can find an n_0 such that $d(f^{n_0} x, f^{n_0+1} x) < \infty$. Then for $n \geqslant n_0$;

$$\begin{aligned} d(f^n x, f^{n+1} x) &\leqslant \lambda d(f^{n-1} x, f^n x) \\ &\leqslant \lambda^2 d(f^{n-2} x, f^{n-1} x) \\ &\leqslant \lambda^3 d(f^{n-3} x, f^{n-2} x) \\ &\leqslant \vdots \\ &\leqslant \lambda^{n-n_0} d(f^{n_0} x, f^{n_0+1} x), \end{aligned}$$

and

$$\begin{aligned} d(f^n x, f^{n+m} x) &\leqslant d(f^n x, f^{n+1} x) + d(f^{n+1} x, f^{n+2} x) + \cdots + d(f^{n+m-1} x, f^{n+m} x) \\ &\leqslant \lambda^{n-n_0} d(f^{n_0} x, f^{n_0+1} x) + \cdots + \lambda^{n+m-1-n_0} d(f^{n_0} x, f^{n_0+1} x) \\ &= [\lambda^{n-n_0} + \lambda^{n+1-n_0} + \cdots + \lambda^{n+m-1-n_0}] d(f^{n_0} x, f^{n_0+1} x) \\ &\leqslant \frac{\lambda^{n-n_0}}{1-\lambda} d(f^{n_0} x, f^{n_0+1} x). \end{aligned}$$

So, the $(f, \perp)$-orbit $\{f^n x\}_{n=0}^{\infty}$ is Cauchy and by completeness of X, it converges in $(X, d, \perp)$ to some point $x^* \in X$. By virtue of the $\perp$-continuity of f, x^* is a fixed point of the map f. Suppose $x \perp y$, $fx = x$ and $fy = y$ then by 3 we have $d(x, y) < \infty$ and 1 says that

$$d(x, y) = d(fx, fy) \leqslant \lambda d(x, y).$$

It's a contradiction. So the fixed point is unique and the theorem is proved. ■

Now we show how the fixed point theorem on generalized metric space [8] is a consequence of previous theorem.

Theorem 3.3 Let (X, d) be a generalized complete metric space, $f : X \to X$ be a map such that

1) $d(fx, fy) \leqslant \lambda d(x, y)$ for any points x and y in X and $0 \leqslant \lambda < 1$;
2) For any point $x \in X$ there exists n_0 such that $d(f^{n_0}x, f^{n_0+1}x) < \infty$;
3) If $fx = x$ and $fy = y$ then $d(x, y) < \infty$;
Then there exists a unique fixed point x^* of the map f and $\lim_{n\to\infty} f^n x = x^*$ for any point $x \in X$.

Proof. For $x, y \in X$ define $x \perp y$ if $d(fx, fy) \leqslant d(x, y)$. Fix $x_0 \in X$. Since f satisfies the condition 1 then for each $y \in X$, $d(fx_0, fy) \leqslant \lambda d(x_0, y) \leqslant d(x_0, y)$. So $x_0 \perp y$. This means that $\perp$ is an orthogonal relation. Hence $(X, \perp)$ is an O-set and $(X, d, \perp)$ is a generalized orthogonal complete metric space. Mapping f satisfies condition 1 so for $x \perp y$ can be easily seen that $f(x) \perp f(y)$. Hence f is $\perp$-preserving. Let $\{x_n\}$ be an arbitrary O-sequence in X such that $\{x_n\}$ converges to $x \in X$. Since f satisfies 1, for each $n \in \mathbb{N}$ we have

$$d(fx_n, fx) \leqslant \lambda d(x_n, x).$$

As n goes to infinity, f is $\perp$-continuous. It is obvious that f satisfies all the hypotheses of the previous theorem. Applying previous theorem, there exists a unique fixed point x^* of the map f and $\lim_{n\to\infty} f^n x = x^*$ for any point $x \in X$. ■

4. An application to differential equation

In this section, we apply results in the previous section to show the existence and uniqueness of solution of Cauchy problem for the first order differential equation (1), where the function f is defined in the domain $D = \{(t, x); |t - t_0| \leqslant a, |x - x_0| \leqslant b\}$ and satisfied the condition

$$|f(t, x_1) - f(t, x_2)| \leqslant \frac{K}{|t - t_0|}|x_1 - x_2|, \qquad 0 < K < 1. \tag{4}$$

Let $M = \max_{(t,x)\in D} |f(t, x)|$. There exists $c = \min\{a, \frac{b}{M}\}$ such that

$$D_0 = \{(t, x); |t - t_0| \leqslant c, |x - x_0| \leqslant M|t - t_0|\},$$

lies in D. Since $|x - x_0| \leqslant M|t - t_0| \leqslant Mc$ then $c = \min\{a, \frac{b}{M}\}$ exists.
We are trying to find a solution ϕ_x for the differential equation (1) with initial condition $\phi_x(t_0) = x_0$ expressed in the form $\phi_x(t) = x_0 + h(t, x)$. Then the mapping ϕ defined by $\phi(t, x) = \phi_x(t)$ on the segment

$$\{(t, x); |t - t_0| \leqslant c, |x - x_0| \leqslant b\}, \tag{5}$$

is the general solution of (1). One can easily verify the following lemma:

Lemma 4.1 For any solution ϕ_x, the point $(t, \phi_x(t))$ lies in D_0 for all t such that $|t - t_0| \leqslant c$.

We are interested to obtain a mapping that satisfies the properties of Theorem 3.2 and fixed point of this mapping is the solution to differential equation (1). Let us consider the orthogonal metric space. This space should include all the mappings which could possibly be solutions. The space of all mappings $h(t, x)$ which added to x_0 could give us a solution ϕ_x with initial condition $\phi_x(t_0) = x_0$ will be considered. Denote this space by X. Since ϕ is defined on the segment (5) so h is defined on the segment (5), too.

Note that $h(t_0, x) = 0$ for any $h \in X$ and any solution ϕ_x where 0 is the zero vector in $\mathbb{R}^n$. In space X, we define a relation $\perp$ by

$$h_1 \perp h_2 \iff \|h_1\|\|h_2\| \leqslant b(\|h_1\| \vee \|h_2\|), \tag{6}$$

where $\|h_1\| \vee \|h_2\| = \|h_1\| or \|h_2\|$ which is an orthogonality relation on X. It shows that the space X is an orthogonal space.
Let $d : X \times X \to [0, \infty]$ be given by

$$d(h_1, h_2) = \|h_1 - h_2\| = \sup |h_1(t, x) - h_2(t, x)|, \tag{7}$$

for all $h_1, h_2 \in X$. Then d is a generalized metric on X and the generalized orthogonal metric space X will be denoted by $(X, d, \perp)$. Since every h is a function over a closed and bounded subset of Euclidean space, this supremum is actually attained in $(X, d, \perp)$. Hence the generalized orthogonal metric space $(X, d, \perp)$ is complete.
In generalized orthogonal metric space $(X, d, \perp)$, a mapping $A : (X, d, \perp) \to (X, d, \perp)$ can be defined by

$$(Ah)(t, x) = \int_{t_0}^{t} f(\tau, x_0 + h(\tau, x))d\tau, \tag{8}$$

for $|t - t_0| \leqslant c$ and $|x - x_0| \leqslant b$. Clearly $(\tau, x_0 + h(\tau, x))$ is in the domain of f for any (τ, x) in the appropriate region but we should be careful to check that Ah is in fact an element of $(X, d, \perp)$. To see this, take any $h \in X$. By construction of mapping A, Ah is a mapping defined on the segment $\{(t, x); |t - t_0| \leqslant c, |x - x_0| \leqslant b\}$ which added to x_0 could give a solution ϕ_x with initial condition $\phi_x(t_0) = x_0$, meaning $Ah \in X$.

We now discuss some properties of mapping A.
i) A is $\perp$-preserving mapping;
ii) $d(Ah_1, Ah_2) \leqslant \lambda d(h_1, h_2)$ for any h_1 and h_2 in X such that $h_1 \perp h_2$ and $0 \leqslant \lambda < 1$;
iii) A is $\perp$-continuous mapping;
iv) For any point $h \in X$ there exists n_0 such that for $(A, \perp)$-orbit $\{A^n h\}_{n=0}^{\infty}$ we have $d(A^{n_0} h, A^{n_0+1} h) < \infty$;
v) If $h_1 \perp h_2$, $Ah_1 = h_1$ and $Ah_2 = h_2$ then $d(h_1, h_2) < \infty$.

Proof. *i)* We recall that A is $\perp$-preserving if for $h_1, h_2 \in X$, $h_1 \perp h_2$, we have $Ah_1 \perp Ah_2$.

$$\begin{aligned}
|(Ah_1)(t, x)| &= \Big| \int_{t_0}^{t} f(\tau, x_0 + h_1(\tau, x))d\tau \Big| \\
&\leqslant \int_{t_0}^{t} |f(\tau, x_0 + h_1(\tau, x))| d\tau \\
&\leqslant \int_{t_0}^{t} M d\tau \\
&= M|t - t_0| \\
&\leqslant M \frac{b}{M} = b.
\end{aligned}$$

So,

$$\|Ah_1\|\|Ah_2\| \leqslant b\|Ah_2\|$$

Meaning that $Ah_1 \perp Ah_2$.

$ii)$ Let $h_1, h_2 \in X$ and $h_1 \perp h_2$ we have

$$\begin{aligned}
|Ah_1(t,x) - Ah_2(t,x)| &= \Big|\int_{t_0}^{t} f(\tau, x_0 + h_1(\tau,x))d\tau - \int_{t_0}^{t} f(\tau, x_0 + h_2(\tau,x))d\tau\Big| \\
&\leqslant \int_{t_0}^{t} |f(\tau, x_0 + h_1(\tau,x)) - f(\tau, x_0 + h_2(\tau,x))|d\tau \\
&\leqslant \int_{t_0}^{t} \frac{K}{|t-t_0|}|x_0 + h_1(\tau,x) - x_0 - h_2(\tau,x)|d\tau \\
&= \|h_1 - h_2\|\frac{K}{|t-t_0|}|t-t_0|.
\end{aligned}$$

Therefore

$$\|Ah_1 - Ah_2\| \leqslant K\|h_1 - h_2\|.$$

Hence for all $h_1, h_2 \in X$, $h_1 \perp h_2$ and $\lambda = K$ we have

$$d(Ah_1, Ah_2) \leqslant \lambda d(h_1, h_2).$$

$iii)$ Suppose $\{h_n\}$ is an O-sequence in X such that $\{h_n\}$ converging to $h \in X$. Because A is $\perp$-preserving, $\{Ah_n\}$ is an O-sequence. For each $n \in \mathbb{N}$, by ii, we have

$$\|Ah_n(t,x) - Ah(t,x)\| \leqslant K\|h_n - h\|.$$

As n goes to infinity, it follows that A is $\perp$-continuous.

$iv)$ Let $h \in X$ and $\{A^n h\}_{n=0}^{\infty}$ be a $(A,\perp)$-orbit such that $h \perp Ah$ and $\|h\|\|Ah\| \leqslant b\|Ah\|$. we have

$$\begin{aligned}
|h(t,x) - Ah(t,x)| &= \Big|h(t,x) - \int_{t_0}^{t} f(\tau, x_0 + h(\tau,x))d\tau\Big| \\
&\leqslant |h(t,x)| + \Big|\int_{t_0}^{t} f(\tau, x_0 + h(\tau,x))d\tau\Big| \\
&\leqslant \|h\| + \int_{t_0}^{t} |f(\tau, x_0 + h(\tau,x))|d\tau \\
&\leqslant b + \int_{t_0}^{t} M d\tau \\
&\leqslant b + M|t-t_0| \\
&\leqslant b + M\frac{b}{M} \\
&= 2b < \infty.
\end{aligned}$$

Thus $\|h - Ah\| < \infty$. Therefore there exists $n_0 = 0$ such that $d(A^{n_0}h, A^{n_0+1}h) < \infty$.

$v)$ Suppose $h_1 \perp h_2$, $Ah_1 = h_1$ and $Ah_2 = h_2$. By part ii we have $d(h_1, h_2) = d(Ah_1, Ah_2) \leqslant \lambda d(h_1, h_2)$. So $d(h_1, h_2) < \infty$. ■

The mapping A defined above is $\perp$-preserving and $\perp$-continuous on the generalized orthogonal metric space $(X, d, \perp)$. Mapping A satisfies the hypotheses of Theorem 3.2. Thus, existence and uniqueness of its fixed point $h_0 \in X$ has been guaranteed by Theorem 3.2. The purpose is to incorporate this in a Picard mapping of potential solutions to differential equation (1). Existence and uniqueness of h_0 confirm existence and uniqueness of fixed point of the Picard mapping, which will in turn prove existence and uniqueness of solution of Cauchy problem for the differential equation (1).

We are looking for solutions expressed in the form $\phi_x(t) = x_0 + h(t, x)$. If h is a fixed point of A then $\phi_x(t) = x_0 + Ah(t, x)$ and when the solution ϕ_x is a fixed point of our Picard mapping, $\phi_x(t)$ will equal $(P\phi_x)(t)$. Hence,

$$\begin{aligned}(P\phi_x)(t) &= x_0 + (Ah)(t, x)\\ &= x_0 + \int_{t_0}^{t} f(\tau, x_0 + h(\tau, x))d\tau\\ &= x_0 + \int_{t_0}^{t} f(\tau, \phi_x(\tau))d\tau.\end{aligned}$$

By Theorem 2.9, ϕ_x is a solution of the differential equation $\acute{x} = f(t, x)$ with $\phi_x(t_0) = x_0$ if and only if $\phi_x = P\phi_x$. Now, let us define the function g given by

$$g(t, x) = x_0 + h_0(t, x).$$

Therefore, g is always well-defined in a neighborhood of (t_0, x_0). Applying the Picard mapping P, we have

$$(Pg)(t, x) = x_0 + (Ah_0)(t, x) = x_0 + h_0(t, x) = g(t, x),$$

which proves that, by Theorem 2.9, g is a solution of the differential equation (1) which satisfies the initial condition $g(t_0, x) = x_0$. Set $b = 0$, which restricts the initial x under our consideration to the specific point x_0. Find the solution $g(t, x_0) = x_0 + h_0(t, x_0)$. The uniqueness of the fixed point h_0 guarantees that this is the only solution with the initial condition x_0 that can be expressed in the form $x_0 + h_0(t, x_0)$. Now, consider any solution ϕ_{x_0} with $\phi_{x_0}(t_0) = x_0$. By Lemma 4.1 we have $\phi_{x_0}(t) \in D_0$ for all t such that $|t - t_0| \leqslant c$. Lable $\phi_{x_0}(t) - x_0$ by $h_\phi(t, x_0)$. So $h_\phi \in X$ and $\phi_{x_0}(t) = x_0 + h_\phi(t, x_0)$. Uniqueness of h_0 shows that all possible solutions to the differential equation with the given initial condition are expressed in the form $\phi_{x_0} = x_0 + h_0(t, x_0)$ for $h \in X$. Thus, as there is only one such function possible, the solution g is unique.

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