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On categories of merotopic, nearness, and filter algebras

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Abstract. We study algebraic properties of categories of Merotopic, Nearness, and Filter Algebras. We show that the category of filter torsion free abelian groups is an epireflective subcategory of the category of filter abelian groups. The forgetful functor from the category of filter rings to filter monoids is essentially algebraic and the forgetful functor from the category of filter groups to the category of filters has a left adjoint.

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1. Introduction

We first describe three categories which contain ***Top***, the category of topological spaces (sometimes with a separation axiom). They are ***Mer***, the category of the merotopic spaces of Katetov [8], ***Near***, the category of nearness spaces of Herrlich [4], and ***Fil***, the category of filter spaces of Katetov [8].

Let X be a set and $\mathbf{P^2(X)}$ be the set whose members are all collections of subsets of X. For any member $\mathcal{A}$ of $\mathbf{P^2(X)}$, we write

$$\sec \mathcal{A} := \{B \subseteq X : A \cap B \neq \phi \text{ for all } A \in \mathcal{A}\}$$

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and

$$\text{stack}_X\mathcal{A} := \{E \subseteq X : A \subseteq E \text{ for some member A of } \mathcal{A}\}.$$

A member $\mathcal{A}$ of $\mathbf{P^2(X)}$ is called a **filter** iff $X \in \mathcal{A}$, $\phi \notin \mathcal{A}$, intersection of any two sets in $\mathcal{A}$ is in $\mathcal{A}$, and all supersets of members of $\mathcal{A}$ are in $\mathcal{A}$.

For members $\mathcal{A}$ and $\mathcal{B}$ of $\mathbf{P^2(X)}$, the **join** of $\mathcal{A}$ and $\mathcal{B}$, denoted by the symbol $\mathcal{A} \vee \mathcal{B}$, is the collection of all subsets of X of the form $A \cup B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and the **meet** of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \wedge \mathcal{B}$, is the collection of all subsets of the form $A \cap B$ with A and B belonging to $\mathcal{A}$ and $\mathcal{B}$ respectively. We say $\mathcal{A}$ **refines** $\mathcal{B}$ iff for each member A of $\mathcal{A}$ there exists a member B of $\mathcal{B}$ containing A. $\mathcal{A}$ **corefines** $\mathcal{B}$ iff for each $A \in \mathcal{A}$ corresponds a set $B \in \mathcal{B}$ contained in A.

A pair (X, ξ) of a set X and a subset ξ of $\mathbf{P^2(X)}$ is said to be a **prenearness space** iff ξ is a nonempty proper subset of $\mathbf{P^2(X)}$ containing all members of $\mathbf{P^2(X)}$ with nonempty intersection and all corefinements of each of its members.

The prenearness space (X, ξ) is called a **merotopic space** iff ξ containing the join of two members of $\mathbf{P^2(X)}$ means one of them belongs to ξ, in other words, for any $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{P^2(X)}$ whose join $\mathcal{A} \vee \mathcal{B}$ is an element of ξ, either $\mathcal{A}$ or $\mathcal{B}$ is already in ξ.

The merotopic space (X, ξ) is called a **nearness space** iff ξ contains all members $\mathcal{A}$ in $\mathbf{P^2(X)}$ with the property that the associated closure collection

$$cl_\xi\mathcal{A} := \{cl_\xi A : A \in \mathcal{A}\}$$

belongs to ξ, where

$$cl_\xi A := \{x \in X : \{A, \{x\}\} \in \xi\}.$$

If (X, ξ) and (Y, η) are two merotopic, prenearness, or nearness spaces, then a mapping $f : X \rightarrow Y$ is called **uniformly continuous** iff $f[\mathcal{A}] := \{f(A) : A \in \mathcal{A}\}$ is a member of η for each $\mathcal{A} \in \xi$.

P − Near, ***Mer***, and ***Near*** are the categories with objects which are prenearness, merotopic, and nearness spaces respectively with uniformly continuous mappings as morphisms. ***Mer*** is a bicoreflective full subcategory of ***P − Near*** and ***Near*** is a bireflective full subcategory of ***Mer*** (see [5]).

If ***X*** denotes any one of these categories, any ***X***-object is simply denoted by X if the ***X***-structure ξ on X is understood. Moreover, a collection $\mathcal{A}$ of subsets of X is said to be

(1) **near** in X provided $\mathcal{A}$ is a member of ξ,

(2) **micromeric** in X provided the collection $\sec\mathcal{A}$ is near in X,

(3) **far** in X provided $\mathcal{A}$ is not near in X,

(4) **uniform cover** of X provided the collection $\{X \backslash A : A \in \mathcal{A}\}$ is far in X.

$\mathcal{A}$ is called a **stack** on X iff $\mathcal{A} = \text{stack}_X\mathcal{A}$. The structure of a merotopic space is determined by the set of merotopic stacks because a collection is micromeric iff its stack in X is micromeric.

A filter on X is said to be a **Cauchy filter** iff it is micromeric on X. X is called a **filter-merotopic** (or just a **filter**) **space** iff every micromeric stack contains a Cauchy filter. The full subcategory of ***Mer*** whose objects are all filter spaces will be denoted by ***Fil***. This category ***Fil*** is cartesian closed (see [9]) and is bicoreflective and hereditary in ***Mer***.

A family $\Omega = (n_j)_{j\in J}$ of natural numbers indexed by some set J is called a **type**. The index set J is called the **order** of Ω. In the following, we let a type $\Omega = (n_j)_{j\in J}$ be fixed. A pair $(|A|, (\omega_j)_{j\in J})$ of a set $|A|$ and a family $\omega_j : |A|^{n_j} \longrightarrow |A|$ $(j \in J)$ of mappings is called an Ω-**algebra** (see, for example, [2]). For the sake of simplicity, we write A instead of the pair $(|A|, (\omega_j)_{j\in J})$ and $\omega_{j,A}$ for the $\mathbf{n}_j$**-ary operation** ω_j on A. If the Ω-algebra A is clear from the context, we drop the suffix A in denoting its n_j-ary $(j \in J)$ operation. If A and B are Ω-algebras, then a mapping $f : |A| \longrightarrow |B|$ is said to be an Ω-**morphism** $f : A \longrightarrow B$ iff for each $j \in J$, $f \circ \omega_{j,A} = \omega_{j,B} \circ f^n$ where $n = n_j$ and $f^n : |A|^n \longrightarrow |B|^n$ is the mapping with the obvious definition $(a_1, \ldots, a_n) \longrightarrow (fa_1, \ldots, fa_n)$.

The symbol $\boldsymbol{Alg(\Omega)}$ denotes the category whose objects are Ω-algebras and whose morphisms are Ω-morphisms.

Let $\boldsymbol{X}$ be a construct with finite concrete powers and $\boldsymbol{A}$ be a subcategory of $\boldsymbol{Alg(\Omega)}$. By a **paired object** (from $\boldsymbol{X}$ and $\boldsymbol{A}$) is meant an ordered pair (X, A) where X and A are objects in $\boldsymbol{X}$ and $\boldsymbol{A}$ respectively with the same underlying set such that, for each $j \in J$, the $n(= n_j)$-ary operation $\omega_{j,A} : |A|^n \longrightarrow |A|$ on A is an $\boldsymbol{X}$-morphism

$$\omega_{j,A} : X^n \longrightarrow X.$$

In this case, we write $\omega_{j,X}$ for the $\boldsymbol{X}$-morphism from X^n to X whose underlying function is $\omega_{j,A}$. If (X, A) and (X', A') are two paired objects (from $\boldsymbol{X}$ and $\boldsymbol{A}$), then an $\boldsymbol{X}$-morphism $f : X \longrightarrow X'$ that is also an $\boldsymbol{A}$-morphism $f : A \longrightarrow A'$ is called a **paired morphism** (from $\boldsymbol{X}$ and $\boldsymbol{A}$) and is denoted by $f : (X, A) \longrightarrow (X', A')$. The category of all paired objects (from $\boldsymbol{X}$ and $\boldsymbol{A}$) together with paired morphisms (from $\boldsymbol{X}$ and $\boldsymbol{A}$) is called the **paired category** (from $\boldsymbol{X}$ and $\boldsymbol{A}$). We denote this category by $\boldsymbol{X \diamond A}$.

In this work, we assume that all subcategories are isomorphism closed. The fact that the most of the natural subcategories fall into this class justifies our assumption. Unless otherwise stated, $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote arbitrary constructs with finite concrete powers, and $\boldsymbol{A}$ represents any subcategory of $\boldsymbol{Alg(\Omega)}$. We write $|X|$ for the underlying set of an object X in a construct. For the sake of simplicity, we will denote an object (X, A) in the paired category $\boldsymbol{X \diamond A}$ (from $\boldsymbol{X}$ and $\boldsymbol{A}$) either by X or by A. We will use a similar identification for morphisms in the paired category.

2. Essentially algebraic and algebraic subcategories

We explore algebraic properties of paired categories with the following lemma whose proof can be found in [3].

Lemma 2.1 Suppose that $\boldsymbol{X}$ is monotopological, $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg}(\Omega')$, $\boldsymbol{B}$ is an essentially algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, and $H : \boldsymbol{B} \longrightarrow \boldsymbol{A}$ is a concrete functor such that the association

$$\tilde{H}(X, B) := (X, HB)$$

for any $\boldsymbol{X \diamond B}$-object (X, B) and $\tilde{H}f := (f, Hf)$ for any paired morphism

$$f : (X, B) \longrightarrow (X', B'),$$

is a concrete functor $\tilde{H} : \boldsymbol{X \diamond B} \longrightarrow \boldsymbol{X \diamond A}$ in the commutative square diagram

$$\begin{array}{ccc} \boldsymbol{X}\diamond\boldsymbol{B} & \xrightarrow{\tilde{H}} & \boldsymbol{X}\diamond\boldsymbol{A} \\ T \downarrow & & \downarrow T' \\ \boldsymbol{B} & \xrightarrow[H]{} & \boldsymbol{A} \end{array},$$

where T and T' are forgetful functors.

Then the following hold.

(a) If H has a left adjoint, then $\tilde{H}$ also has a left adjoint and $\tilde{H}$ is (generating, monosource) - factorizable.

(b) If H reflects isomorphisms, then $\tilde{H}$ reflects isomorphisms.

(c) If H is essentially algebraic, then $\tilde{H}$ is essentially algebraic.

Proposition 2.2 If $\boldsymbol{X}$ is any one of the categories $\boldsymbol{Mer}$, $\boldsymbol{P-Near}$, $\boldsymbol{Near}$, or $\boldsymbol{Fil}$, then the forgetful functors $\boldsymbol{X}\diamond\boldsymbol{Rng}\rightarrow\ \boldsymbol{X}\diamond\boldsymbol{Ab}$ and $\boldsymbol{X}\diamond\boldsymbol{Rng}\rightarrow\ \boldsymbol{X}\diamond\boldsymbol{Mon}$ ($\boldsymbol{Rng}$, $\boldsymbol{Ab}$, and $\boldsymbol{Mon}$ are the categories of rings, abelian groups, or monoids repectively) are essentially algebraic.

Proof. Since the forgetful functors $\boldsymbol{Rng} \rightarrow\ \boldsymbol{Ab}$ and $\boldsymbol{Rng} \rightarrow\ \boldsymbol{Mon}$ are essentially algebraic (see [1, 23.18]), the associated forgetful functors are essentially algebraic by Lemma 2.1 (part (c)). ■

Lemma 2.3 Suppose that $\boldsymbol{X}$ is monotopological, $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg}(\Omega')$, and $\boldsymbol{B}$ is an essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$.

(1) If $\boldsymbol{B}$ is also a reflective subcategory of $\boldsymbol{A}$, then $\boldsymbol{X}\diamond\boldsymbol{B}$ is a reflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

(2) If $\boldsymbol{B}$ is also an epireflective subcategory of $\boldsymbol{A}$, then $\boldsymbol{X}\diamond\boldsymbol{B}$ is an epireflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

Proof. The inclusion map $H : \boldsymbol{B} \rightarrow\ \boldsymbol{A}$ is essentially algebraic, and hence $\tilde{H}$, defined as in Lemma 2.1, is essentially algebraic (by Lemma 2.1). Thus $\boldsymbol{X}\diamond\boldsymbol{B}$ is a reflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

For the second part, if $\boldsymbol{B}$ is also an epireflective subcategory of $\boldsymbol{A}$, then $H : \boldsymbol{B} \rightarrow\ \boldsymbol{A}$ has a left adjoint, and hence $\tilde{H}$ has left adjoint (by Lemma 2.1), which in turn implies that $\boldsymbol{X}\diamond\boldsymbol{B}$ is an epireflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

■

Proposition 2.4 Suppose that $\boldsymbol{X}$ is any one of the categories $\boldsymbol{Mer}$, $\boldsymbol{P-Near}$, $\boldsymbol{Near}$, or $\boldsymbol{Fil}$. Then

(a) $\boldsymbol{X}\diamond\boldsymbol{Ab}$ is an epireflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{Grp}$, where $\boldsymbol{Grp}$ is the category of groups.

(b) If $\boldsymbol{B}$ is the category $\boldsymbol{TfAb}$ of Torsion free abelian groups or the category $\boldsymbol{Ab}_n$ of abelian groups annihilated by a fixed integer n, then $\boldsymbol{X}\diamond\boldsymbol{B}$ is an epireflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{Ab}$.

and

(c) $\boldsymbol{X}\diamond\boldsymbol{B}$ is an epireflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{Alg(\Omega)}$ where $\boldsymbol{B}$ is any SP-class of Ω-algebras.

Proof. This proposition is a consequence of the second part of Lemma 2.3 because $\boldsymbol{Ab}$ is an epireflective subcategory of $\boldsymbol{Grp}$; $\boldsymbol{TfAb}$ and $\boldsymbol{Ab}_n$ are epireflective subcategories of

$\boldsymbol{Ab}$ (see [6, 26.2 (2) (b)]); and any SP-class of Ω-algebras is an epireflective subcategory of $\boldsymbol{Alg(\Omega)}$ ([7]). ■

The following lemma, whose proof can be found in [3], shows that algebraic subcategories of $\boldsymbol{Alg(\Omega)}$ can be paired with well-fibred topological categories to yield algebraic subcategories of paired categories.

Lemma 2.5 Suppose that $\boldsymbol{X}$ is a well-fibred topological category with finitely productive quotients, $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg}(\Omega')$, $\boldsymbol{B}$ is an algebraic subcategory of $\boldsymbol{Alg}(\Omega)$, and $H : \boldsymbol{B} \longrightarrow \boldsymbol{A}$ is a concretely algebraic functor that induces a concrete functor $\tilde{H} : \boldsymbol{X}\diamond\boldsymbol{B} \longrightarrow \boldsymbol{X}\diamond\boldsymbol{A}$ as defined in the commutative square diagram in Lemma 2.1. Then $\tilde{H}$ is algebraic.

Proposition 2.6 The functors $\boldsymbol{Mer}\diamond\boldsymbol{Rng} \longrightarrow \boldsymbol{Mer}\diamond\boldsymbol{Ab}$ and $\boldsymbol{Mer}\diamond\boldsymbol{Vec} \longrightarrow \boldsymbol{Mer}\diamond\boldsymbol{Ab}$ are algebraic, where $\boldsymbol{Vec}$ is the category of vector spaces.

Proof. Since the forgetful functors $\boldsymbol{Rng} \longrightarrow \boldsymbol{Ab}$ and $\boldsymbol{Vec} \longrightarrow \boldsymbol{Ab}$ are algebraic [6, 32.20], it follows from Lemma 2.5 that $\tilde{H}$, described in the commutative diagram in Lemma 2.1, has left adjoint, where $\boldsymbol{B}$ is $\boldsymbol{Rng}$ or $\boldsymbol{Vec}$. ■

Lemma 2.7 If $\boldsymbol{Y}$ is a coreflective subcategory of $\boldsymbol{X}$ and the pair (X, A) is an object in $\boldsymbol{X}\diamond\boldsymbol{A}$ such that X is an object in $\boldsymbol{Y}$ then (X, A) is also an object in $\boldsymbol{Y}\diamond\boldsymbol{A}$.

Proof. We need to prove that each algebraic operation on A is a $\boldsymbol{Y}$-morphism. Let $j \in J$ and $n = n_j$. To avoid ambiguity, let us use the symbol Y to indicate X regarded as a $\boldsymbol{Y}$-object and write Y^n and X^n for the products of X to itself n times in the categories $\boldsymbol{Y}$ and $\boldsymbol{X}$ respectively. Since $\boldsymbol{Y}$ is coreflective in $\boldsymbol{X}$, the $\boldsymbol{Y}$-product Y^n is the $\boldsymbol{Y}$-coreflection of the $\boldsymbol{X}$-product X^n. Therefore, any $\boldsymbol{X}$-morphism $X^n \longrightarrow X$ is also a $\boldsymbol{Y}$-morphism

$$Y^n \longrightarrow Y.$$

In particular, the n_j-ary operation on A being an $\boldsymbol{X}$-morphism $\omega_{j,X} : X^n \longrightarrow X$ is indeed a $\boldsymbol{Y}$-morphism $\omega_{j,Y} : Y^n \longrightarrow Y$. ■

Proposition 2.8 Let $\boldsymbol{A}$ be a subcategory of $\boldsymbol{Alg(\Omega)}$. A merotopic space which is an object of $\boldsymbol{P-Near}\diamond\boldsymbol{A}$ is also an object of $\boldsymbol{Mer}\diamond\boldsymbol{A}$.

Proof. The result follows from Lemma 2.7, because $\boldsymbol{Mer}$ is a bicoreflective subcategory of $\boldsymbol{P-Near}$. ■

Lemma 2.9 If $\boldsymbol{Y}$ is a subcategory of $\boldsymbol{X}$ such that concrete powers in $\boldsymbol{Y}$ agree with concrete powers in $\boldsymbol{X}$ then $\boldsymbol{Y}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

In particular, if $\boldsymbol{Y}$ is an epireflective subcategory of $\boldsymbol{X}$, then $\boldsymbol{Y}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

Proof. Let (Y, A) be any object in $\boldsymbol{Y}\diamond\boldsymbol{A}$. For each $j \in J$, the n_j-th product Y^{n_j} of Y in the category $\boldsymbol{Y}$ is the same as the n_j-th product of Y in the category $\boldsymbol{X}$ and the n_j-ary operation $\omega_{j,Y} : Y^{n_j} \longrightarrow Y$, being a morphism in $\boldsymbol{Y}$, must be a morphism in $\boldsymbol{X}$. Thus (Y, A) is also an $\boldsymbol{X}\diamond\boldsymbol{A}$-object. Obviously $\boldsymbol{Y}\diamond\boldsymbol{A}$-morphisms are also morphisms in $\boldsymbol{X}\diamond\boldsymbol{A}$.

If $\boldsymbol{Y}$ is an epireflective subcategory of $\boldsymbol{X}$, then the products in $\boldsymbol{Y}$ do agree with those in $\boldsymbol{X}$ so that $\boldsymbol{Y}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$ by what was proved above. ■

Proposition 2.10 Let $\boldsymbol{A}$ be a subcategory of $\boldsymbol{Alg(\Omega)}$. A nearness space is an object of $\boldsymbol{Near}\diamond\boldsymbol{A}$ iff it is an object of $\boldsymbol{Mer}\diamond\boldsymbol{A}$.

In particular, $\boldsymbol{Near}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{Mer}\diamond\boldsymbol{A}$.

Proof. A near Ω-algebra is a merotopic Ω-algebra by Lemma 2.9 because $\boldsymbol{Near}$ is a bireflective subcategory of $\boldsymbol{Mer}$. The converse is trivial since the products in $\boldsymbol{Near}$ agree with the products in $\boldsymbol{Mer}$. ■

Let $\boldsymbol{A}$ be a subcategory of $\boldsymbol{Alg(\Omega)}$ and $\boldsymbol{X}$ be any one of the categories $\boldsymbol{P-Near}$, $\boldsymbol{Mer}$, or $\boldsymbol{Near}$. If X is an $\boldsymbol{A}$-object, $(X_i)_{i\in I}$ is a family of $\boldsymbol{X\diamond A}$-objects and

$$f_i : X \longrightarrow X_i$$

is an $\boldsymbol{A}$-homomorphism for each $i \in I$, then the initial structure with respect to $(f_i)_{i\in I}$ in $\boldsymbol{X}$ makes X an $\boldsymbol{X\diamond A}$-object. A subspace of an $\boldsymbol{X\diamond A}$-object which is also a sub-$\boldsymbol{A}$-object is itself an $\boldsymbol{X\diamond A}$-object. Thus we can say that a sub-$\boldsymbol{X\diamond A}$-object of an $\boldsymbol{X\diamond A}$-object is a subspace of an $\boldsymbol{X}$-space which is also a sub-$\boldsymbol{A}$-object. For instance, a submerotopic group Y of a merotopic group X is a merotopic subspace of X which is a subgroup of X. If concrete products exist in $\boldsymbol{A}$ and if $(X_i)_{i\in I}$ is a family of $\boldsymbol{X\diamond A}$-objects, then the cartesian product ΠX_i is also an $\boldsymbol{X\diamond A}$-object with the initial $\boldsymbol{X}$-structure with respect to the natural projections.

The initial $\boldsymbol{X}$-structures in $\boldsymbol{P-Near}$ and $\boldsymbol{Mer}$ are different, while those in $\boldsymbol{Near}$ coincide with the construction in $\boldsymbol{Mer}$.

Let X be a set, $((X_i, \xi_i))_{i\in I}$ be a family of $\boldsymbol{X}$-spaces and $f_i : X \longrightarrow X_i$ be a map for each $i \in I$. Define

$$\xi := \{\mathcal{A} \in \boldsymbol{P}^2(X) : f_i[\mathcal{A}] \in \xi_i \text{ for all } i \in I\}.$$

Then ξ is a prenearness structure on X, initial with respect to the family $(f_i : X \longrightarrow X_i)_{i\in I}$. If each X_i is a merotopic (or nearness) space, then the merotopic reflection of ξ, defined as the collection of all members A of $\mathbf{P^2}(\mathbf{\Pi}X_i)$ such that there is no finite join of elements in $\mathbf{P^2}(\mathbf{\Pi}X_i)\backslash\xi$ which corefines $\mathcal{A}$, is the merotopic (or nearness, respectively) structure on X, that is initial with respect to the family $(f_i : X \longrightarrow X_i)_{i\in I}$.

We say that **final epi sinks are finitely productive** in $\boldsymbol{X}$ iff the product

$$(f_i \times g_k : X_i \times Y_k \longrightarrow X \times Y, X \times Y)_{i\in I,k\in K}$$

of any two final epi sinks $(f_i : X_i \longrightarrow X, X)_{i\in I}$ and $(g_k : Y_k \longrightarrow Y, Y)_{k\in K}$ in $\boldsymbol{X}$ is final in $\boldsymbol{X}$.

Lemma 2.11 Suppose that final epi sinks are finitely productive in $\boldsymbol{X}$, A is an $\boldsymbol{A}$-object, $(X_i)_{i\in I}$ is a family of $\boldsymbol{X}$-objects, and

$$(f_i :\mid X_i \mid\longrightarrow \mid A \mid)_{i\in I}$$

is a class of functions Ω-admissible to A. If X is an $\boldsymbol{X}$-object with the same underlying set as A such that $(f_i : X_i \longrightarrow X, X)_{i\in I}$ is a final epi sink in $\boldsymbol{X}$, then (X, A) is an $\boldsymbol{X\diamond A}$-object.

Proof. Since X has the final structure with respect to $f_i : X_i \longrightarrow X$, for any positive integer n, X^n has the final structure with respect to $f_{i_1} \times \ldots \times f_{i_n}, i_1 \in I, .., i_n \in I$, by hypothesis. Let $j \in J$ and $n = n_j$. We have to show that $\omega_{j,A}$ is an $\boldsymbol{X}$-morphism. However, $\omega_{j,A} \circ (f_{i_1} \times \ldots \times f_{i_n}), i_1 \in I, \ldots, i_n \in I$, being one of the $f_i's$ as (f_i) is Ω-admissible to A, is an $\boldsymbol{X}$-morphism. Consequently, the n_j-ary operation $\omega_{j,A}$ on A is an

X-morphism

$$\omega_{j,X} : X^n \longrightarrow X.$$

This being true for each $j \in J$, (X, A) is an ***X◇A***-object. ∎

Proposition 2.12 Let ***A*** be a subcategory of ***Alg(Ω)***. Suppose that $(X_i)_{i\in I}$ is a family of ***Fil◇A***-objects, X is an ***A***-object, and $f_i : X_i \longrightarrow X$ is a function for each $i \in$I. If (f_i) is Ω-admissible to X (that is, for each $j \in J, n = n_j, i_1, \ldots, i_n \in I$ there exists $j \in I$ such that $\omega_{j,A} \circ (f_{i_1} \times \ldots \times f_{i_n}) = f_j$), then X becomes a ***Fil◇A***-object with the final filter structure with respect to $(f_i)_{i\in I}$.

The quotient of a ***Fil◇A***-object is nothing but the quotient of the corresponding filter space.

Proof. Note that final epi sinks in the category ***Fil*** are finitely productive (see [8]). Since X has the final structure with respect to $f_i : X_i \longrightarrow X$, for any positive integer n, X^n has the final structure with respect to $f_{i_1} \times \ldots \times f_{i_n}, i_1 \in I, .., i_n \in I$, by hypothesis. Let $j \in J$ and $n = n_j$. We have to show that $\omega_{j,A}$ is a ***Fil***-morphism. However,

$$\omega_{j,A} \circ (f_{i_1} \times \ldots \times f_{i_n}), \;\; i_1 \in I, \ldots, i_n \in I,$$

being one of the $f_i's$ as (f_i) is Ω-admissible to A, is a ***Fil***-morphism. Consequently, the n_j-ary operation $\omega_{j,A}$ on A is a ***Fil***-morphism $\omega_{j,X} : X^n \longrightarrow X$. This being true for each $j \in J$, (X, A) is a ***Fil◇A***-object.

To prove the second part, assume that (X, A) is a ***Fil◇A***-object, $f : A \longrightarrow A'$ is an ***A***-morphism, and $f : X \longrightarrow X'$ is a quotient map in ***Fil***. We show that (X', A') is a ***Fil◇A***-object.

Let $j \in J$, $n = n_j$ and $\omega = \omega_{j,A}$, $\omega' = \omega_{j,A'}$ be n-ary operations on A and A' respectively. Since ***Fil*** has finitely productive quotients, X'^n is a quotient of X^n with respect to $f^n : X^n \longrightarrow X'^n$. Thus ω' is a ***Fil***-morphism iff $\omega' \circ f^n$ is a ***Fil***-morphism. However, because f is an Ω-homomorphism, we have the commutative diagram,

$$\begin{array}{ccc} |X^n| & \xrightarrow{\;\omega\;} & |X| \\ {\scriptstyle f^n}\downarrow & & \downarrow{\scriptstyle \omega} \\ |X'^n| & \xrightarrow[\;f\;]{} & |X'| \end{array},$$

which shows that $\omega' \circ f^n$, being equal to $f \circ \omega$, is a ***Fil***-morphism. This being true for each $j \in J$, (X', A') is a ***Fil◇A***-object.

It remains to show that $f : (X, A) \longrightarrow (X', A')$ is a quotient map in ***Fil◇A***. Let (X'', A'') be any object in ***Fil◇A*** and $g : |X'| \longrightarrow |X''|$ be a function between the two sets such that $g \circ f$ is a ***Fil◇A***-morphism. Then g is a ***Fil***-morphism between X' and X'' because $g \circ f$ is one such and $f : X \longrightarrow X'$ is a quotient map in ***Fil***. Similarly g is also an Ω-homomorphism. Thus $g : (X', A') \longrightarrow (X'', A'')$ is a ***Fil◇A***-morphism. ∎

In [3], it is proved that, the forgetful functor U : ***X◇A*** $\longrightarrow$ ***X*** has a left adjoint whenever ***X*** is cartesian closed topological category and ***A*** is algebraic. Therefore, The forgetful functor U : ***Fil◇Grp*** $\longrightarrow$ ***Fil*** is algebraic. Here we give a constructive proof to show that U has a left adjoint.

Proposition 2.13 The forgetful functor $U : \boldsymbol{Fil\text{⊗}Grp} \longrightarrow \boldsymbol{Fil}$ has a left adjoint.

Proof. Suppose that X is an arbitrary filter space. Let A be the free group generated by X and $u : X \hookrightarrow A$ be the inclusion map. For each $n \in \mathbf{N}$ and for any subset L of $\mathbf{N}_n := \{1, 2, \ldots, n\}$, define $h_{n,L} : A^n \longrightarrow A$ by

$$h_{n,L}(a_1, \ldots, a_n) := \beta_{\chi_L(1)}(a_1) \ldots \beta_{\chi_L(n)}(a_n),$$

where χ_L is the characteristic function of L (i.e., $\chi_L(r) = 1$ if $r \in L$ and $\chi_L(r) = 0$ if $r \notin L$), $\beta_0 :=\mathrm{id}_A$, and β_1 is the inversion on A. Equip A with the final structure in the category $\boldsymbol{Fil}$ with respect to $(h_{n,L} \circ u^n : X^n \longrightarrow A)_{n\in\mathbf{N},L\subseteq\mathbf{N}_n}$.

We now show that A is a filter group. Noting that

$$\alpha \circ (h_{n,L} \circ u^n \times h_{m,K} \circ u^m) = h_{n+m,L\cup(n+K)} \circ u^{n+m}$$

is uniformly continuous and that $A \times A$ has the final structure (because $h_{n,L} \circ u^n$'s form an epi sink in $\boldsymbol{Fil}$ and $\boldsymbol{Fil}$ is cartesian closed) with respect to $h_{n,L} \circ u^n \times h_{m,K} \circ u^m$ ($n \in \mathbf{N}$, $m \in \mathbf{N}$, $L \subseteq \mathbf{N}_n$, $K \subseteq \mathbf{N}_m$), we conclude that the multiplication α on A is uniformly continuous. Since $\beta_1 \circ (h_{n,L} \circ u^n) = (h_{n,L'} \circ u^n) \circ r_n$, where r_n is the map from X^n to X^n which assigns $(x_n, \ldots, x_1)$ to $(x_1, \ldots, x_n)$ and L' is the complement of L in $\mathbf{N}_n, \beta_1$ is uniformly continuous. Hence A is a filter group. Finally we show that (u, A) is a universal map for X. Clearly u is uniformly continuous since $u = h_{1,\phi}\circ$u. Let A' be any filter group and $f : X \longrightarrow A'$ be uniformly continuous. Since A is a free group on X, there exists a unique group homomorphism $\bar{f} : A \longrightarrow A'$ which extends f, i.e., $\bar{f} \circ u =$f. It remains to show that $\bar{f}$ is uniformly continuous. It is enough to show, for each $n \in \mathbf{N}$ and for each subset L of $\mathbf{N}_n$, that the composition $\bar{f} \circ (h_{n,L} \circ u^n)$ is uniformly continuous. But $\bar{f} \circ (h_{n,L} \circ u^n) = h_{n,L} \circ f^n$ and $h_{n,L} \circ f^n$ is obviously uniformly continuous. This shows U has a left adjoint. ■

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