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Existence and multiplicity of positive solutions for a class of semilinear elliptic system with nonlinear boundary conditions

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Abstract. This study concerns the existence and multiplicity of positive weak solutions for a class of semilinear elliptic systems with nonlinear boundary conditions. Our results is depending on the local minimization method on the Nehari manifold and some variational techniques. Also by using Mountain Pass Lemma, we establish the existence of at least one solution with positive energy.

Keywords: Critical point, semilinear elliptic system, nonlinear boundary value problem, Fibering map, Nehari manifold.

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1. Introduction

The class of elliptic partial differential problems involving the nonlinear boundary conditions arise from many branches of science, for example in mechanics, geometry and other sciences. The attention of many authors have been attracted toward the existence and multiplicity of solutions for semilinear elliptic problems with nonlinear boundary conditions. The aim of this paper is to prove existence and multiplicity results of nontrivial

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nonnegative solutions for the semilinear elliptic system:

$$\begin{cases} \Delta u + m_1(x)u = \dfrac{1}{p}\lambda f_u(x,u,v) + g_u(x,u,v) & x \in \Omega, \\ \Delta v + m_2(x)v = \dfrac{1}{p}\lambda f_v(x,u,v) + g_v(x,u,v) & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{q}\mu h_u(x,u,v) + \dfrac{1}{r} j_u(x,u,v) & x \in \partial\Omega, \\ \dfrac{\partial v}{\partial n} = \dfrac{1}{q}\mu h_v(x,u,v) + \dfrac{1}{r} j_v(x,u,v) & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\lambda, \mu > 0$, $1 < q, r < 2 < p < 2^*$, where 2^* is the critical Sobolev exponent ($2^* = \frac{2N}{N-2}$ if $N > 2$, $2^* = \infty$ if $N \leqslant 2$), $\frac{\partial}{\partial n}$ is the outer normal derivative, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $m_1, m_2 \in C(\bar{\Omega})$ are positive bounded functions. Also f, g, h and j are C^1-positively homogenous functions of degrees p, 1, q and r respectively such that $f(x,0,0) = g(x,0,0) = h(x,0,0) = j(x,0,0) = 0$.

We say $\psi(x,u,v)$ is a positively homogeneous function of degree α whenever $\psi(x,tu,tv) = t^\alpha \psi(x,u,v)$ for every $t > 0$. It is clear that if $\alpha \geqslant 0$ and $\psi(x,u,v)$ is an $\alpha-$homogeneous C^1-function, then $\psi(x,u,v) \leqslant K_\psi(|u|^\alpha + |v|^\alpha)$, where $K_\psi = \max\{\psi(x,u,v) : (x,u,v) \in \overline{\Omega} \times \mathbb{R}^2, \ |u|^\alpha + |v|^\alpha = 1\}$. So from assumptions over f, g, h and j we conclude that there exist positive constants K_f, K_g, K_h and K_j such that

$$\begin{cases} f(x,u,v) \leqslant K_f(|u|^p + |v|^p), & g(x,u,v) \leqslant K_g(|u| + |v|), \\ h(x,u,v) \leqslant K_h(|u|^q + |v|^q), & j(x,u,v) \leqslant K_j(|u|^r + |v|^r). \end{cases} \tag{2}$$

Over the last years, many authors have studied the existence of solutions for the following elliptic system

$$\begin{cases} -\Delta u + m_1(x)|u|^{p-2}u = F_u(x,u,v) + G_u(x,u,v) & x \in \Omega, \\ -\Delta v + m_2(x)|v|^{p-2}v = F_v(x,u,v) + G_v(x,u,v) & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = H_u(x,u,v) + J_u(x,u,v), \quad \dfrac{\partial v}{\partial n} = H_v(x,u,v) + J_v(x,u,v) & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded region in $\mathbb{R}^N (N > 2)$ with smooth boundary $\partial\Omega$ and F, G, H and J are positively homogenous functions of different degrees. For instance, Brown and Wu [6] considered the case $m_1 = m_2 = 0, G(x,u,v) = J(x,u,v) = 0$ and

$$\begin{cases} F(x,u,v) = \dfrac{1}{\alpha+\beta} f(x) u^\alpha v^\beta \\ H(x,u,v) = \lambda \dfrac{1}{q} g(x) u^q + \mu \dfrac{1}{q} h(x) v^q \end{cases}$$

where $\alpha > 1, \beta > 1, 2 < \alpha + \beta < 2^*$ and the weight functions f, g, h satisfy the following conditions:
- $f \in C(\overline{\Omega})$ with $\|f\|_\infty = 1$ and $f^+ = max\{f, 0\} \not\equiv 0$.
- g, $h \in C(\partial\Omega)$ with $\|g\|_\infty = \|h\|_\infty = 1$ and $g^\pm = max\{\pm g, 0\} \not\equiv 0$ and $h^\pm = max\{\pm h, 0\} \not\equiv 0$.

They found that the above problem has at least two nonnegative solutions when the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. Also in [17], Wu considered the case $F(x,u,v) = \frac{1}{q}\lambda f(x)u^q + \frac{1}{q}\mu g(x)v^q, G(x,u,v) = J(x,u,v) = 0$ and $H(x,u,v) = \frac{2}{\alpha+\beta}h(x)u^\alpha v^\beta$, where $1 < q < 2, \alpha > 1, \beta > 1$ satisfy $2 < \alpha+\beta < 2^*$ and the weights f, g, h satisfy some suitable conditions. The author showed this problem has at least two solutions when (λ, μ) belongs to a certain subset of R^2.
In [14], Feng-Yun Lu proved the existence at least two nontrivial nonnegative solutions in the case $m_1 = m_2 = 1, F(x,u,v) = \frac{1}{q}\lambda f(x)u^q + \frac{1}{q}\mu g(x)v^q, G(x,u,v) = J(x,u,v) = 0$ and $H(x,u,v) = \frac{1}{\alpha+\beta}h(x)u^\alpha v^\beta$, where $\alpha > 1, \beta > 1$ and $2 < \alpha+\beta < 2^*$. Recently, Fan [11] studied the case $F(x,u,v) = \frac{1}{r}\lambda u^r + \frac{1}{r}\mu v^r, H(x,u,v) = J(x,u,v) = 0$ and $G(x,u,v) = \frac{2}{\alpha+\beta}u^\alpha v^\beta$ for $1 < r < p < 2^*$. By using the Nehari manifold and the Lusternik-Schnirelman category, the author proved the problem admits at least $\text{cat}(\Omega)+1$ positive solutions. Moreover, equations involving positively homogeneous functions have been considered in many papers, such as [2–4, 8, 12, 13, 15] and the references cited therein.

In this paper, at first, by exploiting the relationship between the Nehari manifold, fibering maps and extraction of the palais-smale sequences in the Nehari manifold and using the Rellich-Kondrachov Theorem [5] we establish the existence of local minimizers for Euler functional associated with the equation and so we prove the existence of nonnegative solutions of system (1). Then by using Mountain Pass Lemma [16], we establish the existence of at least one solution with positive energy.

Problem (1) is posed in the framework of the Sobolev space $W = W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ with the norm

$$\|(u,v)\|_W = \left(\int_\Omega (|\nabla u|^2 + m_1(x)|u|^2)dx + \int_\Omega (|\nabla v|^2 + m_2(x)|v|^2)dx \right)^{\frac{1}{2}},$$

which is equivalent to the standard one and we use the standard $\mathrm{L}^r(\Omega)$ spaces whose norms denoted by $\|u\|_r$. Throughout this paper, we denote by S_r and $\bar{S}_r$ the best Sobolev and the best Sobolev trace constant for the embeddings of $W^{1,2}(\Omega)$ into $\mathrm{L}^r(\Omega)$ and $W^{1,2}(\Omega)$ into $\mathrm{L}^r(\partial\Omega)$, respectively. So we have

$$\frac{(\|(u,v)\|_W^2)^r}{(\int_{\partial\Omega}(|u|^r + |v|^r)dx)^2} \geqslant \frac{1}{4\bar{S}_r^{2r}} \quad \text{and} \quad \frac{(\|(u,v)\|_W^2)^r}{(\int_\Omega(|u|^r + |v|^r)dx)^2} \geqslant \frac{1}{4S_r^{2r}}. \tag{3}$$

Now we will show the existence and multiplicity results of nontrivial solutions of system (1) by looking for critical points of the associated Euler functional

$$\ell_{\lambda,\mu}(u,v) = \frac{1}{2}M(u,v) - \frac{1}{p}\lambda F(u,v) - G(u,v) - \frac{1}{q}\mu H(,u,v) - \frac{1}{r}J(u,v), \tag{4}$$

where

$$M(u,v) = \int_\Omega (|\nabla u|^2 + m_1(x)|u|^2)dx + \int_\Omega (|\nabla v|^2 + m_2(x)|v|^2)dx$$

and

$$\begin{cases} F(u,v) = \displaystyle\int_\Omega f(x,|u|,|v|)dx, \quad G(u,v) = \int_\Omega g(x,|u|,|v|)dx, \\ H(u,v) = \displaystyle\int_{\partial\Omega} h(x,|u|,|v|)d\sigma, \quad J(u,v) = \int_{\partial\Omega} j(x,|u|,|v|)d\sigma. \end{cases} \tag{5}$$

Moreover, a pair of functions $(u,v) \in W$ is said to be a weak solution of the problem (1), if $\langle \ell'_{\lambda,\mu}(u,v), (\varphi_1,\varphi_2)\rangle = 0$, i.e.

$$\begin{aligned} \int_\Omega \big(\nabla u.\nabla\varphi_1 + m_1(x)u\varphi_1\big)dx + \int_\Omega \big(\nabla v.\nabla\varphi_2 + m_2(x)v\varphi_2\big)dx \\ = \frac{1}{p}\lambda\int_\Omega (f_u\varphi_1 + f_v\varphi_2)dx + \int_\Omega (g_u\varphi_1 + g_v\varphi_2)dx \\ + \frac{1}{q}\mu\int_{\partial\Omega}(h_u\varphi_1 + h_v\varphi_2)d\sigma + \frac{1}{r}\int_{\partial\Omega}(j_u\varphi_1 + j_v\varphi_2)d\sigma, \end{aligned}$$

for all $(\varphi_1,\varphi_2) \in W$.

To get the solutions of system (1) we look for minimizers of the energy functional $\ell_{\lambda,\mu}$. But $\ell_{\lambda,\mu}$ is not bounded neither above nor below on W, so we introduce the Nehari manifold

$$\mathcal{N}_{\lambda,\mu}(\Omega) = \{(u,v) \in W \setminus \{(0,0)\} : \ \langle \ell'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality between W and W^{-1}, where W^{-1} is the dual space of the Sobolev space W. We recall that any nonzero solution of problem (1) belongs to $\mathcal{N}_{\lambda,\mu}(\Omega)$. Moreover, by definition, we have that $(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$ if and only if

$$M(u,v) - \lambda F(u,v) - G(u,v) - \mu H(u,v) - J(u,v) = 0. \tag{6}$$

The following result concerns the behavior of $\ell_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}(\Omega)$.

Lemma 1.1 $\ell_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$.

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$ be an arbitrary. Then by (2)–(6) we get

$$\begin{aligned} \ell_{\lambda,\mu}(u,v) &= \frac{p-2}{2p}M(u,v) - \mu\frac{q-p}{qq}H(u,v) - \frac{p-1}{p}G(u,v) - \frac{p-r}{rp}J(u,v) \\ &\geqslant \frac{p-2}{2p}M(u,v) - \mu\frac{p-q}{pq}K_h\int_{\partial\Omega}(|u|^q + |v|^q)d\sigma \\ &- \frac{p-1}{p}K_g\int_\Omega(|u|+|v|)dx - \frac{p-r}{pr}K_j\int_{\partial\Omega}(|u|^r + |v|^r)d\sigma \\ &\geqslant \frac{p-2}{2p}M(u,v) - 2\mu\bar{S}_q^q\frac{p-q}{pq}K_h(M(u,v))^{q/2} \\ &- 2S_1\frac{p-1}{p}K_g(M(u,v))^{\frac{1}{2}} - 2\bar{S}_r^r\frac{p-r}{pr}K_j(M(u,v))^{r/2}. \end{aligned}$$

Thus, $\ell_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$.

The Nehari manifold is closely linked to the behavior of functions of the form $\phi_{u,v}: t \mapsto \ell_{\lambda,\mu}(tu,tv)$ $(t > 0)$. Such maps are known as fibering maps. They were introduced by Drabek and Pohozaev in [9] and also were discussed in Brown and Zhang [7]. So for $(u,v) \in W$, we have

$$\begin{aligned}
\phi_{u,v}(t) &= \ell_{\lambda,\mu}(tu,tv) \\
&= \frac{t^2}{2}M(u,v) - \frac{t^p}{p}\lambda F(u,v) - tG(u,v) - \frac{t^q}{q}\mu H(u,v) - \frac{t^r}{r}J(u,v), \\
\phi'_{u,v}(t) &= \langle \ell'_{\lambda,\mu}(tu,tv),(u,v)\rangle \\
&= tM(u,v) - \lambda t^{p-1}F(u,v) - G(u,v) - \mu t^{q-1}H(u,v) - t^{r-1}J(u,v).
\end{aligned} \tag{7}$$

It is easy to see that $\phi'_{u,v}(t) = 0$ if and only if $(tu,tv) \in \mathcal{N}_{\lambda,\mu}(\Omega)$. In particular, $(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$ if and only if $\phi'_{u,v}(1) = 0$, i.e. elements in $\mathcal{N}_{\lambda,\mu}(\Omega)$ correspond to stationary points of fibering maps. Thus, it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflection and so we define

$$\begin{aligned}
\mathcal{N}^{+}_{\lambda,\mu} &= \{(tu,tv) \in W : \phi'_{u,v}(t) = 0, \phi''_{u,v}(t) > 0\}, \\
\mathcal{N}^{-}_{\lambda,\mu} &= \{(tu,tv) \in W : \phi'_{u,v}(t) = 0, \phi''_{u,v}(t) < 0\}, \\
\mathcal{N}^{0}_{\lambda,\mu} &= \{(tu,tv) \in W : \phi'_{u,v}(t) = 0, \phi''_{u,v}(t) = 0\}.
\end{aligned} \tag{8}$$

The following lemma shows that minimizers for $\ell_{\lambda,\mu}(u,v)$ on $N_{\lambda,\mu}(\Omega)$ are usually critical points for $\ell_{\lambda,\mu}$, as proved by Brown and Zhang in [7] or in Aghajani et al. [1].

Lemma 1.2 Let (u_0,v_0) be a local minimizer for $\ell_{\lambda,\mu}(u,v)$ on $\mathcal{N}_{\lambda,\mu}(\Omega)$, if $(u_0,v_0) \notin \mathcal{N}^0_{\lambda,\mu}(\Omega)$, then (u_0,v_0) is a critical point of $\ell_{\lambda,\mu}$.

The purpose of this paper is to prove the following results.

Theorem 1.3 If $q \leqslant r$, $G(u,v) > 0$ and $J(u,v) \leqslant 0$, then there exists $\Lambda^* \subset (\mathbb{R}^+)^2$ such that for $(\lambda,\mu) \in \Lambda^*$, system (1) has at least two positive distinct solutions.

Theorem 1.4 If $G(u,v) > 0$ and $F(u,v) > 0$, then there exists Λ^{**} $\left(\Lambda^* \subseteq \Lambda^{**} \subseteq (\mathbb{R}^+)^2\right)$ such that for $(\lambda,\mu) \in \Lambda^{**}$, system (1) has at least one nontrivial solution with positive energy.

This paper is organized as follows. In section 2 we point out some notations and preliminaries and give a fairly complete description of the Nehari manifold and fibering map. Finally Theorem 1.3 and Theorem 1.4 are proved in section 3.

2. Preliminaries and auxiliary results

In this section some properties of the Nehari manifold and fibering map will be perused. First, motivated by Lemma 1.2, we will get conditions for $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Lemma 2.1 If $q \leqslant r$, $G(u,v) > 0$ and $J(u,v) \leqslant 0$, then there exists $\Lambda_0 \subset (\mathbb{R}^+)^2$ such that for $(\lambda,\mu) \in \Lambda_0$ and $q \leqslant r$, we have $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

proof. Suppose otherwise, let $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$ be an arbitrary, then by (7) and (8) we have

$$\phi''_{u,v}(1) = M(u,v) - \lambda(p-1)F(u,v) - \mu(q-1)H(u,v) - (r-1)J(u,v) = 0, \quad (9)$$

and

$$\phi'_{u,v}(1) = M(u,v) - \lambda F(u,v) - G(u,v) - \mu H(u,v) - J(u,v) = 0. \quad (10)$$

Using (2), (3), (5), (9) and (10) we obtain

$$\begin{aligned}(2-q)M(u,v) &= (p-q)\lambda F(u,v) + (1-q)G(u,v) + (r-q)J(u,v)\\ &< (p-q)\lambda K_f \int_\Omega (|u|^p + |v|^p)dx \leqslant 2(p-q)(S_p^p \lambda K_f)(M(u,v))^{\frac{p}{2}},\end{aligned}$$

which concludes

$$M(u,v) > \left(\frac{2-q}{2(p-q)S_p^p\lambda K_f}\right)^{\frac{2}{p-2}}. \quad (11)$$

On the other hand, by relations (9), (10), (2), (3), (5) and Young inequality we get

$$\begin{aligned}(p-2)M(u,v) &= \mu(p-q)H(u,v) + (p-1)G(u,v) + (p-r)J(u,v)\\ &\leqslant \mu(p-q)K_h \int_{\partial\Omega} (|u|^q + |v|^q)d\sigma + (p-1)K_g \int_\Omega (|u| + |v|)dx\\ &\leqslant 2\mu \bar{S}_q^q (p-q)K_h(M(u,v))^{\frac{q}{2}} + 2S_1(p-1)K_g(M(u,v))^{\frac{1}{2}}\\ &\leqslant \frac{2(p-2)}{3q}\left(\frac{2-q}{2}\big(\frac{3q(p-q)}{(p-2)}\mu\bar{S}_q^q K_h\big)^{\frac{2}{2-q}} + \frac{q}{2}M(u,v)\right)\\ &+ \frac{2(p-2)}{3}\left(\frac{1}{2}\big(\frac{3(p-1)}{(p-2)}S_1K_g\big)^2 + \frac{1}{2}M(u,v)\right),\end{aligned}$$

so we have

$$M(u,v) \leqslant L + L', \quad (12)$$

where $L = \frac{(2-q)}{q}\big(\frac{3q(p-q)}{(p-2)}\mu\bar{S}_q^q K_h\big)^{\frac{2}{2-q}}$ and $L' = (\frac{3(p-1)}{(p-2)}S_1K_g)^2$. Now by (12) and (11) we must have

$$\left(\frac{2-q}{2(p-q)S_p^p\lambda K_f}\right)^{\frac{2}{p-2}} < L + L',$$

which is a contradiction for λ and μ sufficiently small. So there exists $\Lambda_0 \subset (\mathbb{R}^+)^2$ such that for $(\lambda,\mu) \in \Lambda_0,\ \mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Lemma 2.2 If $(u,v) \in N^-_{\lambda,\mu}$, $q \leqslant r$, $G(u,v) > 0$ and $J(u,v) \leqslant 0$, then $F(u,v) > 0$.

Proof. By using (7) and (8) for $(u,v) \in N^-_{\lambda,\mu}$ we have

$$F(u,v) \geqslant \frac{2-q}{\lambda(p-q)}M(u,v) + \frac{q-1}{\lambda(p-q)}G(u,v) + \frac{q-r}{\lambda(p-q)}J(u,v) > 0. \tag{13}$$

As it was mentioned in previous section we have, $\phi'_{u,v}(t) = 0$ if and only if $(tu, tv) \in \mathcal{N}_{\lambda,\mu}(\Omega)$. Therefore our purpose is to describe the nature of the derivative of the fibering maps for all possible signs of $F(u,v)$, to do this, at first we define the following functions

$$\begin{cases} R_\lambda(t) := \frac{1}{2}t^2 M(u,v) - \lambda\frac{1}{p}t^p F(u,v), \\ S_\mu(t) := \mu\frac{1}{q}t^q H(u,v) + tG(u,v) + \frac{1}{r}t^r J(u,v), \end{cases} \tag{14}$$

follows from (7) that, $\phi_{u,v}(t) = R_\lambda(t) - S_\mu(t)$ and in particular $\phi'_{u,v}(t) = 0$ if and only if $R'_\lambda(t) = S'_\mu(t)$, where

$$\begin{cases} R'_\lambda(t) = tM(u,v) - t^{p-1}\lambda F(u,v), \\ S'_\mu(t) = \mu t^{q-1}H(u,v) + G(u,v) + t^{r-1}J(u,v). \end{cases} \tag{15}$$

In the next result we see that, $\phi_{u,v}$ has positive values for all nonzero $(u,v) \in W$ whenever, λ and μ are sufficiently small.

Lemma 2.3 There exists $\Lambda_1 \subset (\mathbb{R}^+)^2$ such that $\phi_{u,v}(t) = R_\lambda(t) - S_\mu(t)$ takes on positive values for all non-zero $(u,v) \in W$, whenever $(\lambda,\mu) \in \Lambda_1$.

Proof. If $F(u,,v) \leqslant 0$, then $R_\lambda(t) > S_\mu(t)$ for t sufficiently large and so $\phi_{u,v}(t) > 0$. Otherwise, suppose there exists $(u,v) \in W$ such that $F(u,v) > 0$. By elementary calculus, we infer that $R_\lambda(t)$ takes a maximum at

$$t_{max} = \left(\frac{\|(u,v)\|^2_W}{\lambda F(u,v)}\right)^{\frac{1}{p-2}}, \tag{16}$$

then follow by (14), (16), (2), (3) and (5)

$$\begin{aligned} R_\lambda(t_{max}) &= \frac{p-2}{2p}\left(\frac{(\|(u,v)\|^2_W)^p}{\big(\lambda F(u,v)\big)^2}\right)^{\frac{1}{p-2}} \\ &\geqslant \frac{p-2}{2p(\lambda K_f)^{\frac{2}{p-2}}}\left(\frac{(\|(u,v)\|^2_W)^p}{\big(\int_\Omega(|u|^p + |v|^p)dx\big)^2}\right)^{\frac{1}{p-2}} \\ &\geqslant \frac{p-2}{2p(\lambda K_f)^{\frac{2}{p-2}}}\left(\frac{1}{4S_p^{2p}}\right)^{\frac{1}{p-2}} \geqslant \frac{\delta}{\lambda^{\frac{2}{p-2}}}, \end{aligned} \tag{17}$$

where δ is independent of (u,v). Now, we are going to prove that there exists $\Lambda_1 \subset (\mathbb{R}^+)^2$ such that for all non-zero $(u,v) \in W$, $\phi_{u,v}(t_{max}) > 0$, provided that $(\lambda,\mu) \in \Lambda_1$. To do this, first note that from (16), (17) and (3) for $1 \leqslant \alpha < 2^*$

$$(t_{max})^\alpha \int_\Omega (|u|^\alpha + |v|^\alpha)dx \leqslant 2S_\alpha^\alpha \left(\frac{\|(u,v)\|_W^2}{\lambda F(u,v)}\right)^{\frac{\alpha}{p-2}} (\|(u,v)\|_W^2)^{\frac{\alpha}{2}} \\ = 2S_\alpha^\alpha \left(\frac{(\|(u,v)\|_W^2)^p}{(\lambda F(u,v))^2}\right)^{\frac{\alpha}{2(p-2)}} \\ = 2S_\alpha^\alpha (\frac{2p}{p-2})^{\frac{\alpha}{2}} (R_\lambda(t_{max}))^{\frac{\alpha}{2}} = c_1 (R_\lambda(t_{max}))^{\frac{\alpha}{2}}, \tag{18}$$

similarly, $(t_{max})^\alpha \int_{\partial\Omega} (|u|^\alpha + |v|^\alpha)dx = \bar{c}_1 (R_\lambda(t_{max}))^{\frac{\alpha}{2}}$. By computing (2), (5), (14) and (18) we find

$$S_\mu(t_{max}) = \frac{1}{q}\mu(t_{max})^q H(u,v) + t_{max}G(u,v) + \frac{1}{r}(t_{max})^r J(u,v) \\ \leqslant \frac{1}{q}\mu K_h (t_{max})^q \int_{\partial\Omega} (|u|^q + |v|^q)d\sigma \\ + K_g t_{max} \int_\Omega (|u| + |v|)dx + \frac{1}{r}K_j (t_{max})^r \int_{\partial\Omega} (|u|^r + |v|^r)d\sigma \\ \leqslant \mu C_1 \big(R_\lambda(t_{max})\big)^{\frac{q}{2}} + C_2 \big(R_\lambda(t_{max})\big)^{\frac{1}{2}} + C_3 \big(R_\lambda(t_{max})\big)^{\frac{r}{2}}, \tag{19}$$

where C_1, C_2 and C_3 are positive constants and independent of (u,v). Hence using (7), (17) and (19) we observe that

$$\phi_{u,v}(t_{max}) = R_\lambda(t_{max}) - S_\mu(t_{max}) \\ \geqslant \mathfrak{R}\left(1 - \mu C_1 \mathfrak{R}^{\frac{q-2}{2}} - C_2 \mathfrak{R}^{\frac{-1}{2}} - C_3 \mathfrak{R}^{\frac{r-2}{2}}\right) \\ \geqslant \frac{\delta}{\lambda^{\frac{2}{p-2}}}\left(1 - \mu C_1 \delta^{\frac{q-2}{2}} \lambda^{\frac{2-q}{p-2}} - C_2 \delta^{\frac{-1}{2}} \lambda^{\frac{1}{p-2}} - C_3 \delta^{\frac{r-2}{2}} \lambda^{\frac{2-r}{p-2}}\right),$$

where $\mathfrak{R} = R_\lambda(t_{max})$. Since $1 < q, r < 2 < p$, so there exist $\Lambda_1 \subset (\mathbb{R}^+)^2$ and $\epsilon > 0$ such that if $(\lambda, \mu) \in \Lambda_1$, then $\phi_{u,v}(t_{max}) > \epsilon > 0$ for all nonzero (u,v) and this completes the proof.

Corollary 2.4 If $(\lambda, \mu) \in \Lambda_1$, then $\ell_{\lambda,\mu}(u,v) > \epsilon > 0$ for all $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$.

Proof. Since $(u,v) \in N^-_{\lambda,\mu}$ thus, $\phi_{u,v}$ has a positive global maximum at $t = 1$, i.e.

$$\ell_{\lambda,\mu}(u,v) = \phi_{u,v}(1) \geqslant \phi_{u,v}(t_{max}) > \epsilon > 0.$$

Corollary 2.5 For $(\lambda, \mu) \in \Lambda_1$, $\phi'_{u,v}(t) = R'_\lambda(t) - S'_\mu(t)$ takes on positive values for all non-zero $(u,v) \in W$.

Proof. Let $(u,v) \in W$, using (7) $\phi_{u,v}(0) = 0$, and by Lemma 2.3, $\phi_{u,v}(t_{max}) > 0$. So there exists $0 < \tau < t_{max}$, such that $\phi'_{u,v}(\tau) > 0$.

To state our main results, we now present some important properties of $N^-_{\lambda,\mu}$ and $N^+_{\lambda,\mu}$.

Corollary 2.6 If $G(u,v)>0$, then for $(u,v)\in W\setminus\{(0,0)\}$ and $(\lambda,\mu)\in\Lambda_1$, we have
(i) there exists $t_1>0$ such that $(t_1u,t_1v)\in N^+_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<0$.
(ii) if $F(u,v)>0$, then there exists $0<t_1<t_2$ such that $(t_1u,t_1v)\in N^+_{\lambda,\mu}$, $(t_2u,t_2v)\in N^-_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<0$.

Proof. (i) From the definition of $\phi'_{u,v}(t)$, we know $\phi'_{u,v}(0)<0$ and by Corollary 2.5, we obtain that $\phi'_{u,v}(\tau)>0$ for suitable $\tau>0$, so exists $0<t_1<\tau$ such that $\phi'_{u,v}(t_1)=0$ and $\phi''_{u,v}(t_1)>0$. Therefore, we conclude that $(t_1u,t_1v)\in N^+_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<\phi_{u,v}(0)=0$.
Proof. (ii) As in the proof of (i), we have that $\phi'_{u,v}(0)<0$ and $\phi'_{u,v}(\tau)>0$. Moreover $\lim_{t\to\infty}\phi'_{u,v}(t)=-\infty$, so there exist t_1,t_2 such that $0<t_1<\tau<t_2$ and $\phi'_{u,v}(t_1)=\phi'_{u,v}(t_2)=0$. Furthermore $(t_1u,t_1v)\in N^+_{\lambda,\mu}$, $(t_2u,t_2v)\in N^-_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<\phi_{u,v}(0)=0$.

3. Existence of solutions

In order to prove of Theorem 1.3, we need to show the existence of local minimum for $\ell_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$, for this, we need the following remark:

Remark 1 *By using relation (2) we have $f(x,u,v)\leqslant K_f(|u|^p+|v|^p)$, $g(x,u,v)\leqslant K_g(|u|+|v|)$, $h(x,u,v)\leqslant K_h(|u|^q+|v|^q)$ and $j(x,u,v)\leqslant K_j(|u|^r+|v|^r)$ for $1<q,\ r<2<p<2^*$. Hence from compactness of the embeddings $W^{1,2}(\Omega)\hookrightarrow \mathrm{L}^\alpha(\Omega)$ and $W^{1,2}(\Omega)\hookrightarrow \mathrm{L}^\alpha(\partial\Omega)$ for $1\leqslant\alpha<2^*$ (Rellich-Kondrachov Theorem [5]) and the fact that the functions $f(x,u,v)$, $g(x,u,v)$, $h(x,u,v)$ and $j(x,u,v)$ are continuous, we conclude that the functionals $I_1(u,v)=\int_\Omega f(x,u,v)dx, I_2(u,v)=\int_\Omega g(x,u,v)dx, I_3(u,v)=\int_{\partial\Omega} h(x,u,v)d\sigma$ and $I_4(u,v)=\int_{\partial\Omega} j(x,u,v)d\sigma$ are weakly continuous, i.e. if $(u_n,v_n)\rightharpoonup(u,v)$, then $I_i(u_n,v_n)\to I_i(u,v)$, i=1,2,3,4.*

Definition 3.1 A sequence $y_n=(u_n,v_n)\subset W$ is called a Palais-Smale sequence ((PS)-sequence) if $\{\ell_{\lambda,\mu}(y_n)\}$ is bounded and $\ell'_{\lambda,\mu}(y_n)\to 0$ as $n\to\infty$. It is said that the functional $\ell_{\lambda,\mu}$ satisfies the Palais-Smale condition if each Palais-Smale sequence has a convergent subsequence.

Now we prove the boundedness of Palais-Smale sequence.

Lemma 3.2 If $\{(u_n,v_n)\}$ is a $(PS)-$ sequence for $\ell_{\lambda,\mu}$, then $\{(u_n,v_n)\}$ is bounded in W.

Proof. By using Young inequality and from (2), (3), (5) and (7) we get

$$\begin{aligned}
&\ell_{\lambda,\mu}(u_n,v_n)-\frac{1}{p}\langle\ell'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle\\
&=\frac{p-2}{2p}M(u_n,v_n)-\mu\frac{p-q}{pq}H(u_n,v_n)\\
&-\frac{p-1}{p}G(u_n,v_n)-\frac{p-r}{pr}J(u_n,v_n)\\
&\geqslant\frac{p-2}{2p}M(u_n,v_n)-\frac{p-q}{pq}2\mu\bar{S}_q^qK_hM(u_n,v_n)^{\frac{q}{2}}\\
&-\frac{p-1}{p}2S_1K_gM(u_n,v_n)^{\frac{1}{2}}-\frac{p-r}{pr}2\bar{S}_r^rK_jM(u_n,v_n)^{\frac{r}{2}}\\
&\geqslant\frac{p-2}{2p}M(u_n,v_n)-\frac{(p-2)}{4pq}\left(\frac{2-q}{2}\big(\frac{4(p-q)}{(p-2)}2\mu\bar{S}_q^qK_h\big)^{\frac{2}{2-q}}+\frac{q}{2}M(u,v)\right)\\
&-\frac{(p-2)}{4p}\left(\frac{1}{2}\big(\frac{4(p-1)}{(p-2)}2S_1K_g\big)^2+\frac{1}{2}M(u_n,v_n)\right)\\
&-\frac{(p-2)}{4pr}\left(\frac{2-r}{2}\big(\frac{4(p-r)}{(p-2)}2\bar{S}_r^rK_j\big)^{\frac{2}{2-r}}+\frac{r}{2}M(u,v)\right)\geqslant\frac{p-2}{8p}\|(u_n,v_n)\|_W^2-L,
\end{aligned}$$

where

$$\begin{aligned}
L=&\frac{(p-2)(2-q)}{8pq}\big(\frac{4(p-q)}{(p-2)}2\mu\bar{S}_q^qK_h\big)^{\frac{2}{2-q}}\\
&+\frac{(p-2)}{8p}\big(\frac{4(p-1)}{(p-2)}2S_1K_g\big)^2+\frac{(p-2)(2-r)}{8pr}\big(\frac{4(p-r)}{(p-2)}2\bar{S}_r^rK_j\big)^{\frac{2}{2-r}},
\end{aligned}$$

so $\{(u_n,v_n)\}$ is bounded in W.

Now, we establish the existence of local minimum for $\ell_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$. For simplicity, let $\Lambda^*=\{\Lambda_0\cap\Lambda_1\}$ and $\Lambda^{**}=\Lambda_1$ where Λ_0 and Λ_1 are given in the previous section.

First, we establish the existence of local minimum for $\ell_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$.

Proposition 3.3 If $G(u,v)>0$, then for $(\lambda,\mu)\in\Lambda^*$ we have
(i) there exists a minimizer of $\ell_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$,
(ii) if $q\leqslant r$ and $J(u,v)\leqslant 0$, then there exists a minimizer of $\ell_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}(\Omega)$.

Proof. (i) By arguing as in Lemma 1.1, $\ell_{\lambda,\mu}$ is bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$ and so on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$. Let $\{(u_n,v_n)\}$ be a minimizing sequence for $\ell_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$, i.e.

$$\lim_{n\to\infty}\ell_{\lambda,\mu}(u_n,v_n)=\inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}}\ell_{\lambda,\mu}(u,v)=c.$$

From Ekeland's variational principle [10] we have

$$\langle\ell'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle\to 0,$$

combining the compact embedding Theorem [5] and Lemma 3.2, we obtain that there exists a subsequence $\{(u_n, v_n)\}$ and (u_1, v_1) in W such that

$$\begin{cases} u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1 \text{ weakly in } \quad W^{1,2}(\Omega), \\ u_n \to u_1, \quad v_n \to v_1 \quad \text{strongly in } L^\alpha(\Omega) \, \text{and} \, L^\alpha(\partial\Omega) \, \text{for} 1 \leqslant \alpha < 2^*, \end{cases} \tag{20}$$

and $(u_n(x), v_n(x)) \to (u_1(x), v_1(x))$, a.e.

By Corollary 2.6(i) for $(u_1, v_1) \in W \setminus \{(0,0)\}$, there exists t_1 such that $(t_1u_1, t_1v_1) \in N^+_{\lambda,\mu}$ and so $\phi'_{u_1,v_1}(t_1) = 0$. Now we show that $(u_n, v_n) \to (u_1, v_1)$ in W. Suppose this is false, then

$$M(u_1, v_1) < \liminf_{n\to\infty} M(u_n, v_n), \tag{21}$$

also we have

$$\begin{aligned} \phi^{'}_{u_n,v_n}(t) &= tM(u_n, v_n) - \lambda t^{p-1} F(u_n, v_n) \\ &\quad - \mu t^{q-1} H(u_n, v_n) - G(u_n, v_n) - t^{r-1} J(u_n, v_n), \end{aligned} \tag{22}$$

and

$$\begin{aligned} \phi^{'}_{u_1,v_1}(t) &= tM(u_1, v_1) - \lambda t^{p-1} F(u_1, v_1) \\ &\quad - \mu t^{q-1} H(u_1, v_1) - G(u_1, v_1) - t^{r-1} J(u_1, v_1), \end{aligned} \tag{23}$$

so from (20)–(23) and Remark *1*, $\phi^{'}_{u_n,v_n}(t_1) > \phi'_{u_1,v_1}(t_1) = 0$ for n sufficiently large. Since $\{(u_n, v_n)\} \subseteq N^+_{\lambda,\mu}(\Omega)$, by considering the possible fibering maps it is easy to see that, $\phi'_{u_n,v_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n,v_n}(1) = 0$ for all n. Hence, we must have $t_1 > 1$, but $(t_1u_1, t_1v_1) \in N^+_{\lambda,\mu}$ and so

$$\begin{aligned} \ell_{\lambda,\mu}(t_1u_1, t_1v_1) &= \phi_{u_1,v_1}(t_1) < \phi_{u_1,v_1}(1) \\ &< \lim_{n\to\infty} \phi_{u_n,v_n}(1) = \lim_{n\to\infty} \ell_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} \ell_{\lambda,\mu}(u, v), \end{aligned}$$

which is a contradiction. Therefore, $(u_nv_n) \to (u_1, v_1)$ in W and so

$$\ell_{\lambda,\mu}(u_1, v_1) = \lim_{n\to\infty} \ell_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} \ell_{\lambda,\mu}(u, v).$$

Thus, (u_1, v_1) is a minimizer for $\ell_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$.
Proof. (ii) By Corollary 2.4, we have $\ell_{\lambda,\mu}(u, v) > \epsilon > 0$ for all $(u, v) \in \mathcal{N}^-_{\lambda,\mu}$, i.e.

$$\inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} \ell_{\lambda,\mu}(u, v) > 0,$$

hence, there exists a minimizing sequence $\{(u_n, v_n)\} \subseteq \mathcal{N}^-_{\lambda,\mu}(\Omega)$ such that

$$\lim_{n\to\infty} \ell_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} \ell_{\lambda,\mu}(u, v) > 0. \tag{24}$$

Similar to the argument in the proof of (i), we find that, $\{(u_n, v_n)\}$ is bounded in W and also the results obtained in (20) are satisfied for $\{(u_n, v_n)\}$ and $\{(u_2, v_2)\}$. Since $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}(\Omega)$, so by (8) $\phi'_{u_n,v_n}(1) = 0, \phi''_{u_n,v_n}(1) < 0$ and by Lemma 2.2, $F(u_n, v_n) > 0$. Letting $n \to \infty$, we see that $\phi'_{u_2,v_2}(1) = 0, \phi''_{u_2,v_2}(1) \leqslant 0$ and $F(u_2, v_2) \geqslant 0$. If $F(u_2, v_2) = 0$, then by (7) and (8) we have

$$(2-q)M(u,v) + (q-1)G(u,v) + (q-r)J(u,v) \leqslant 0,$$

which is a contradiction with our assumptions. So $F(u_2, v_2) > 0$ and by Corollary 2.6(ii) there exists $t_2 > 0$ such that $(t_2u_2, t_2v_2) \in \mathcal{N}^-_{\lambda,\mu}(\Omega)$. We claim that $(u_n, v_n) \to (u_2, v_2)$ in W, suppose that this is false, so

$$M(u_2, v_2) < \liminf_{n\to\infty} M(u_n, v_n), \tag{25}$$

but $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$ and so $\ell_{\lambda,\mu}(u_n, v_n) \geqslant \ell_{\lambda,\mu}(tu_n, tv_n)$ for all $t \geqslant 0$. Therefore, by considering (7), (24) and (25) and Remark *1*, we can write

$$\begin{aligned}\ell_{\lambda,\mu}(t_2u_2, t_2v_2) &= \phi_{u_2,v_2}(t_2) < \lim_{n\to\infty} \phi_{u_2,v_2}(t_2)\\ &= \lim_{n\to\infty} \ell_{\lambda,\mu}(t_2u_n, t_2v_n) \leqslant \lim_{n\to\infty} \ell_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} \ell_{\lambda,\mu}(u,v),\end{aligned}$$

which is a contradiction. Therefore, $(u_n, v_n) \to (u_2, v_2)$ in W and so the proof is complete.

Lemma 3.4 If $G(u,v) > 0$, then for $(\lambda,\mu) \in \Lambda^*$, the functional $\ell_{\lambda,\mu}(u,v)$ satisfies (PS) condition on W.

Proof. If $\ell_{\lambda,\mu}(u_n, v_n)$ is bounded and $\ell_{\lambda,\mu}(u_n, v_n) \to 0$, then using Lemma 3.2, (u_n, v_n) is bounded in W. Also, similar to the argument in the proof of Proposition 3.3(i) we find that, the sequence (u_n, v_n), has a convergent subsequence and this completes the proof.

Proof of Theorem 1.3. By Proposition 3.3 there exist $(u_1, v_1) \in N^+_{\lambda,\mu}(\Omega)$ and $(u_2, v_2) \in N^-_{\lambda,\mu}(\Omega)$ such that $\ell_{\lambda,\mu}(u_1, v_1) = \inf_{(u,v)\in N^+_{\lambda,\mu}} \ell_{\lambda,\mu}(u,v)$ and $\ell_{\lambda,\mu}(u_2, v_2) = \inf_{(u,v)\in N^-_{\lambda,mu}} \ell_{\lambda,\mu}(u,v)$ and by Lemmas 1.2 and 2.1, (u_1, v_1) and (u_2, v_2) are critical points of $\ell_{\lambda,\mu}$ on W and hence are weak solutions of problem (1). On the other hand, $\ell_{\lambda,\mu}(u,v) = \ell_{\lambda,\mu}(|u|,|v|)$, so we may assume that (u_1, v_1) and (u_2, v_2) are positive solutions. Also, since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, this implies that (u_1, v_1) and (u_2, v_2) are distinct and the proof is complete.

For the proof of Theorem 1.4. we need the Mountain Pass Lemma.

Mountain Pass Lemma. (see [16]) Let X be a real Banach space with the norm $\|.\|$ and $J \in C^1(X,\mathbb{R})$, $J(0) = 0$. Assume
(i) the function $J(u)$ on X satisfies the (PS) condition;
(ii) there are β, $\rho > 0$ such that $J(u) \geqslant \beta$, $\|u\| = \rho$;
(iii) there is $e \in X$, $\|e\| \geqslant \rho$ such that $J(e) \leqslant 0$.
then $c_0 = \inf_{\phi\in\Psi} \max_{t\in[0,1]} J(\phi(t))$ is a critical value of $J(u)$ with $0 < \beta \leqslant c_0 < \infty$, where $\Psi = \{\phi \in (C[0,1], X), \phi(0) = 0, \phi(1) = e\}$.

Proof of Theorem 1.4. From (7), it is clear that $\ell_{\lambda,\mu}(u,v) \in C^1(W,R)$ and

$\ell_{\lambda,\mu}(0,0) = 0$. Since $F(u,v) > 0$, then $\lim_{t\to\infty} \ell(tu,tv) = -\infty$, this means that there exists $t_0 > 0$ such that $\ell_{\lambda,\mu}(t_0 u, t_0 v) < 0$. Also by using corollary 2.4. we know that $\ell_{\lambda,\mu}(u,v) > \epsilon > 0$ and by Lemma 3.4. $\ell_{\lambda,\mu}(u,v)$ satisfies the (PS) condition on W. Now application of the Mountain Pass Lemma gives Theorem 1.4.

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