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Subcategories of topological algebras

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Abstract. In addition to exploring constructions and properties of limits and colimits in categories of topological algebras, we study special subcategories of topological algebras and their properties. In particular, under certain conditions, reflective subcategories when paired with topological structures give rise to reflective subcategories and epireflective subcategories give rise to epireflective subcategories.

Keywords: Monotopolocial category, topological category, topological functors, universal algebra, topological algebra, reflective subcategory, coreflective subcategory, epireflective subcategory.

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Essentially a topological algebra is a universal algebra endowed with a topological structure so that algebraic operations are continuous in all variables together. Wyler has generalized the construction of categories of topological algebras (see [15]), by obtaining from what he calls a "top" category (which is equivalent to the concept topological category) $\boldsymbol{C}^s$ and an operational category $\boldsymbol{A}$ over a category $\boldsymbol{C}$ a new category $\boldsymbol{A}^r$ which is "top" over $\boldsymbol{A}$ and operational over $\boldsymbol{C}^s$, with a pullback property. Fay further generalized the categories of topological algebras using a concept called topologically algebraic situation (see [4]). Later Nel ([11]) and Koslowski ([9]) have given descriptions that are adopted in our work. First, let us describe some concepts used in this work.

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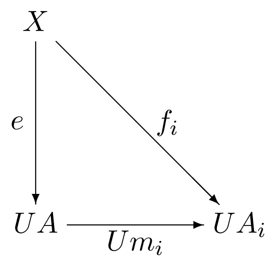
1. Preliminaries

There are several definitions for the term "algebraic functor" in the literature, all of which are equivalent in some special categories, but not in general. We choose to adopt the following popular definition [8, page 243]. A functor $U : \boldsymbol{X} \to \boldsymbol{Y}$ is called **algebraic** iff U has a left adjoint and preserves and reflects regular epimorphisms.

An algebraic functor is faithful ([8], 32.17). If $U : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is an algebraic functor and the category $\boldsymbol{X}$ has coequalizers, then $(\boldsymbol{X}, U)$ is called an **algebraic category over $\boldsymbol{Y}$**. Algebraic category over $\boldsymbol{Set}$ is equivalent to regularly algebraic category over $\boldsymbol{Set}$ in the sense of [1, 20.35, 23.38, 23.39]. The functor from the category $\boldsymbol{Grp}$ of groups into $\boldsymbol{Set}$ is algebraic but not topological. It is known (see [7]) that every algebraic functor into $\boldsymbol{Set}$ is topologically algebraic.

A functor $U : \boldsymbol{A} \to \boldsymbol{X}$ **creates isomorphisms** iff to each $\boldsymbol{X}$-isomorphism $f : X \longrightarrow UA$ corresponds a unique $\boldsymbol{A}$-morphism $g : B \longrightarrow A$ such that $Ug = f$ and g is an $\boldsymbol{A}$-isomorphism. A functor $U : \boldsymbol{A} \to \boldsymbol{X}$ is called **(generating, monosource) - factorizable** if for every source $(X, f_i : X \to UA_i)_{i\in I}$ there exists a generating map $e : X \to UA$ (i.e., e is an $\boldsymbol{X}$-morphism such that for any two $\boldsymbol{X}$-morphisms $r : UA \longrightarrow Y$ and $s : UA \longrightarrow Y$, the equality $r \circ e = s \circ e$ implies $r = s$) and a monosource $(A, m_i : A \to A_i)_{i\in I}$ (i.e., for any two $\boldsymbol{A}$-morphisms $u : B \longrightarrow A$ and $v : B \longrightarrow A$, the equalities $m_i \circ u = m_i \circ v$ for all $i \in I$ implies $u = v$) such that the diagram



commutes. A functor $U : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is said to be **essentially algebraic** [6] provided that it creates isomorphisms and is (generating, monosource) - factorizable. If $U : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is an essentially algebraic functor and faithfull, then $(\boldsymbol{X}, U)$ is called an **essentially algebraic category over $\boldsymbol{Y}$**.

A functor $U : \boldsymbol{A} \longrightarrow \boldsymbol{X}$ is called **uniquely transportable** iff any $\boldsymbol{X}$-isomorphism $f : X \longrightarrow UA$ can be lifted via U to a unique $\boldsymbol{A}$-isomorphism $g : A' \longrightarrow A$. For a later use, we will formulate a result in the following Theorem, whose proof can be found in [1, 23.2].

Theorem 1.1
The following conditions are equivalent for a uniquely transportable (generating, monosource) - factorizable functor $U : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$.
(a) U is essentially algebraic.
(b) U reflects isomorphisms.
(c) U reflects limits.
(d) U reflects equalizers.
(e) U reflects extremal epimorphisms and is faithful.
(f) Every monosource in $\boldsymbol{X}$ is U-initial. ∎

A family $\Omega = (n_j)_{j\in J}$ of natural numbers indexed by some set J is called a **type**. The index set J is called the **order** of Ω. In the following, we let a type $\Omega = (n_j)_{j\in J}$ be fixed. A pair $(|A|, (\omega_j)_{j\in J})$ of a set $|A|$ and a family $\omega_j : |A|^{n_j} \to |A|$ $(j \in J)$ of mappings is called an Ω-**algebra** (see, for example, [3]). For the sake of simplicity, we write A instead of the pair $(|A|, (\omega_j)_{j\in J})$ and $\omega_{j,A}$ for the $\mathbf{n}_j$**-ary operation** ω_j on A. If the Ω-algebra A is clear from the context, we drop the suffix A in denoting its n_j-ary $(j \in J)$ operation. If A and B are Ω-algebras, then a mapping $f : |A| \to |B|$ is said to be an Ω-**morphism** $f : A \to B$ iff for each $j \in J$, $f \circ \omega_{j,A} = \omega_{j,B} \circ f^n$ where $n = n_j$ and $f^n : |A|^n \to |B|^n$ is the mapping with the obvious definition $(a_1, \ldots, a_n) \to (fa_1, \ldots, fa_n)$.

The symbol $\boldsymbol{Alg(\Omega)}$ denotes the category whose objects are Ω-algebras and whose morphisms are Ω-morphisms. $\boldsymbol{Alg(\Omega)}$ is algebraic over $\boldsymbol{Set}$ (see [1, 7.72 (3), 23.6 (1), 23E (a)]).

A subcategory of an algebraic category, in general, may not be algebraic. However, this is guaranteed by several equivalent conditions for a special type of subcategory. To state this result we need the definition of an isomorphism closed subcategory. A subcategory $\boldsymbol{A}$ of $\boldsymbol{B}$ is called **isomorphism closed** iff every isomorphism in $\boldsymbol{B}$ whose domain or codomain belongs to $\boldsymbol{A}$ is a morphism in $\boldsymbol{A}$. We state two results in this direction whose proofs can be found in the stated references.

Theorem 1.2 If $(\boldsymbol{B}, U)$ is an algebraic category and $\boldsymbol{A}$ is a full isomorphism closed subcategory of $\boldsymbol{B}$ with embedding $E : \boldsymbol{A} \hookrightarrow \boldsymbol{B}$ such that $\boldsymbol{A}$ is closed under the formation of subobjects in $\boldsymbol{B}$, then the following are equivalent [8, 38.2]:
(a) $(\boldsymbol{A}, U \circ E)$ is algebraic.
(b) $\boldsymbol{A}$ is reflective in $\boldsymbol{B}$.
(c) $\boldsymbol{A}$ is a complete subcategory of $\boldsymbol{B}$.
(d) $\boldsymbol{A}$ is closed under the formation of products in $\boldsymbol{B}$. ▎

Theorem 1.3 An isomorphism closed full subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Omega)}$ is epireflective in $\boldsymbol{Alg(\Omega)}$ iff $\boldsymbol{A}$ is closed under the formation of products and subalgebras ([7], [1, 20.18, 23.6(1), 23.12(1), 16.9]). ▎

Now we can conclude, as a consequence of these two results, that a full isomorphism closed epireflective subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Omega)}$ is algebraic over $\boldsymbol{Set}$ and hence admits free Ω-algebras and has regular factorizations because algebraic category over $\boldsymbol{Set}$ means regularly algebraic category over $\boldsymbol{Set}$ in the sense of [1, 23.35,23.38, 23.39]. A full isomorphism closed epireflective subcategory of $\boldsymbol{Alg(\Omega)}$ is usually referred to as an **SP-class** or as a **quasiprimitive category** of algebras. A full subcategory $\boldsymbol{A}$ of the category $\boldsymbol{Alg(\Omega)}$ is a variety (in the sense of [3]) iff $\boldsymbol{A}$ is closed under subalgebras, homomorphic images and direct products. A variety is also called an **HSP-class** or a **primitive category** of algebras.

A variety is an epireflective subcategory of $\boldsymbol{Alg(\Omega)}$ (by Theorem 3) and is algebraic over $\boldsymbol{Set}$ (by Theorem 2). Thus every nontrivial variety has free algebras. Since every algebraic construct is topologically algebraic, both SP-classes and HSP-classes are topologically algebraic over $\boldsymbol{Set}$.

The following theorem, whose proof can be found in [1, 23.8, 23.13], sheds some light on essentially algebraic subcategories of $\boldsymbol{Alg(\Omega)}$.

Theorem 1.4
A concrete category $(\boldsymbol{A}, U)$ is essentially algebraic iff U creates isomorphisms, U is adjoint, and $\boldsymbol{A}$ is (epi, monosource) - factorizable.
Every essentially algebraic construct is complete, cocomplete, and wellpowered. ▎

Thus products, equalizers, coequalizers, intersections and free objects exist in any essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$. Moreover, in such categories any monosource is point separating (because essentially algebraic functors preserve monosources, see [1, 23A]) and products are concrete (since any essentially algebraic functor preserves products). Since any category that has (epi, monosource) - factorizations is an (extremal epi, monosource) - category (see [1, 15.10]), an essentially algebraic construct is an (extremal epi, monosource) - category.

2. Paired Categories

Let $\boldsymbol{X}$ be a construct with finite concrete powers and $\boldsymbol{A}$ be a subcategory of $\boldsymbol{Alg(\Omega)}$. By a **paired object** (from $\boldsymbol{X}$ and $\boldsymbol{A}$) is meant an ordered pair (X, A) where X and A are objects in $\boldsymbol{X}$ and $\boldsymbol{A}$ respectively with the same underlying set such that, for each $j \in J$, the $n(= n_j)$-ary operation $\omega_{j,A} : |A|^n \to |A|$ on A is an $\boldsymbol{X}$-morphism $\omega_{j,A} : X^n \to X$. In this case, we write $\omega_{j,X}$ for the $\boldsymbol{X}$-morphism from X^n to X whose underlying function is $\omega_{j,A}$. If (X, A) and (X', A') are two paired objects (from $\boldsymbol{X}$ and $\boldsymbol{A}$), then an $\boldsymbol{X}$-morphism $f : X \to X'$ that is also an $\boldsymbol{A}$-morphism $f : A \to A'$ is called a **paired morphism** (from $\boldsymbol{X}$ and $\boldsymbol{A}$) and is denoted by $f : (X, A) \to (X', A')$. The category of all paired objects (from $\boldsymbol{X}$ and $\boldsymbol{A}$) together with paired morphisms (from $\boldsymbol{X}$ and $\boldsymbol{A}$) is called the **paired category** (from $\boldsymbol{X}$ and $\boldsymbol{A}$). We denote this category by $\boldsymbol{X}\diamond\boldsymbol{A}$.

In this work, we assume that all subcategories are full isomorphism closed. The fact that the most of the natural subcategories fall into this class justifies our assumption. Unless otherwise stated, $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote arbitrary constructs with finite concrete powers, and $\boldsymbol{A}$ represents any subcategory of $\boldsymbol{Alg(\Omega)}$. For the sake of simplicity, we will denote an object (X, A) in the paired category $\boldsymbol{X}\diamond\boldsymbol{A}$ (from $\boldsymbol{X}$ and $\boldsymbol{A}$) either by X or by A. We will use a similar identification for morphisms in the paired category.

To see some examples of paired categories, notice that the category of topological groups with continuous homomorphisms is the paired category $\boldsymbol{Top}\diamond\boldsymbol{Grp}$ from $\boldsymbol{Top}$ and $\boldsymbol{Grp}$.

***Example* 2.1** The category $\boldsymbol{Ab}$ of abelian groups and group homomorphisms can be viewed as the paired category from $\boldsymbol{Grp}$ and $\boldsymbol{Grp}$.

Indeed, Suppose G is a set, and $(G, \cdot, ^{-1})$ and $(G, \odot, *)$ are groups, the first operation is group multiplication and the second operation is group inversion, such that

$$\odot : (G, \cdot, ^{-1}) \times (G, \cdot, ^{-1}) \to (G, \cdot, ^{-1})$$

and

$$* : (G, \cdot, ^{-1}) \to (G, \cdot, ^{-1})$$

are group homomorphisms. If e and E are the identity elements in $(G, \cdot, ^{-1})$ and $(G, \odot, *)$ respectively, then $e \odot e = e$ and $e* = e$, because (e, e) is the identity element in $(G, \cdot, ^{-1}) \times (G, \cdot, ^{-1})$ and any group homomorphism maps the identity element in the domain group to the identity element of the codomain group. Combining these two equalities, we have

$$E = e \odot e* = e \odot e = e.$$

For any x and x' in G, since

$$\odot((x, e) \cdot (e, x')) = (x \odot e) \cdot (e \odot x'),$$

we have

$$x \odot x' = x \cdot x'.$$

This shows that $\odot = \cdot$ and $* = ^{-1}$. Consequently, the group inversion being a group homomorphism, group $(G, \cdot, ^{-1})$ has to be abelian.

Example* 2.2** Similarly the paired category ***Grp◇Rng is nothing but ***Ab***. (Note that, in a ring, if $(a+b)\cdot(c+d) = a\cdot c + b\cdot d$, then, taking $b = c = 0$, $a \cdot d = 0$ for all a and d.)

For the sake of simplicity, we assume that all the subcategories are isomorphism closed. The fact that most of the natural subcategories fall into this class justifies our assumption. We also use the convention that using the same symbol for morphisms in different constructs indicates that their underlying functions are the same [thus underlying sets for the domain objects (respectively, for the codomain objects) are the same]. However, for the sake of clarity, in some instances we may use different symbols for morphisms in different categories with the same underlying functions.

Unless otherwise stated, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are assumed to be constructs admitting concrete finite powers while $\boldsymbol{A}$ and $\boldsymbol{B}$ are subcategories of $\boldsymbol{Alg(\Omega)}$.

3. Subcategories

Let us first discuss the construction of subcategories of $\boldsymbol{X}\diamond\boldsymbol{A}$ from subcategories of $\boldsymbol{X}$ and of $\boldsymbol{A}$. It is clear that if $\boldsymbol{B}$ is a subcategory of $\boldsymbol{A}$, then $\boldsymbol{X}\diamond\boldsymbol{B}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$. On the other hand, if $\boldsymbol{Y}$ is a subcategory of $\boldsymbol{X}$ then the category $\boldsymbol{Y}\diamond\boldsymbol{A}$ need not be a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$ because concrete powers in $\boldsymbol{Y}$ need not agree with the concrete powers in $\boldsymbol{X}$. Here is an example:

***Example* 3.1** The additive group $I\!R$ of real numbers with its usual topology, i.e., with the nearness structure

$$\xi := \{\mathcal{A} \in \boldsymbol{P}^2(I\!R) : \cap\{\bar{A} : A \in \mathcal{A}\} \neq \phi\}$$

constitutes a counterexample since $I\!R$ is a topological group but not a nearness group: The addition $+ : I\!R \times I\!R \longrightarrow I\!R$ is not uniformly continuous with respect to the ***Near*** product structure on $I\!R \times I\!R$ (for a detailed proof, see [2]). ❙

However, we have the following result.

Theorem 3.2 If $\boldsymbol{Y}$ is a subcategory of $\boldsymbol{X}$ such that concrete powers in $\boldsymbol{Y}$ agree with concrete powers in $\boldsymbol{X}$ then $\boldsymbol{Y}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

In particular, if $\boldsymbol{Y}$ is an epireflective subcategory of $\boldsymbol{X}$, then $\boldsymbol{Y}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

Proof. Let (Y, A) be any object in $\boldsymbol{Y}\diamond\boldsymbol{A}$. For each $j \in J$, the n_j-th product Y^{n_j} of Y in the category $\boldsymbol{Y}$ is the same as the n_j-th product of Y in the category $\boldsymbol{X}$ and the n_j-ary operation $\omega_{j,Y} : Y^{n_j} \rightarrow Y$, being a morphism in $\boldsymbol{Y}$, must be a morphism in $\boldsymbol{X}$. Thus (Y, A) is also an $\boldsymbol{X}\diamond\boldsymbol{A}$-object. Obviously $\boldsymbol{Y}\diamond\boldsymbol{A}$-morphisms are also morphisms in $\boldsymbol{X}\diamond\boldsymbol{A}$.

If $\boldsymbol{Y}$ is an epireflective subcategory of $\boldsymbol{X}$, then the products in $\boldsymbol{Y}$ do agree with those in $\boldsymbol{X}$ so that $\boldsymbol{Y}\diamond\boldsymbol{A}$ is a subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$ by what was proved above. ■

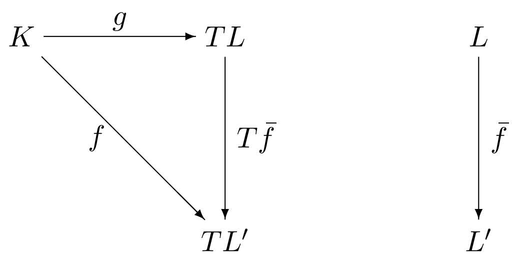
If $\boldsymbol{Y}$ is a coreflective subcategory of $\boldsymbol{X}$, then any object in $\boldsymbol{X} \diamond \boldsymbol{A}$ is also in $\boldsymbol{Y} \diamond \boldsymbol{A}$ if its first part is already in $\boldsymbol{Y}$. In other words:

Theorem 3.3 If $\boldsymbol{Y}$ is a coreflective subcategory of $\boldsymbol{X}$ and the pair (X, A) is an object in $\boldsymbol{X} \diamond \boldsymbol{A}$ such that X is an object in $\boldsymbol{Y}$ then (X, A) is also an object in $\boldsymbol{Y} \diamond \boldsymbol{A}$.

Proof. We need to prove that each algebraic operation on A is a $\boldsymbol{Y}$-morphism. Let $j \in J$ and $n = n_j$. To avoid ambiguity, let us use the symbol Y to indicate X regarded as a $\boldsymbol{Y}$-object and write Y^n and X^n for the products of X to itself n times in the categories $\boldsymbol{Y}$ and $\boldsymbol{X}$ respectively. Since $\boldsymbol{Y}$ is coreflective in $\boldsymbol{X}$, the $\boldsymbol{Y}$-product Y^n is the $\boldsymbol{Y}$-coreflection of the $\boldsymbol{X}$-product X^n. Therefore, any $\boldsymbol{X}$-morphism $X^n \longrightarrow X$ is also a $\boldsymbol{Y}$-morphism Y^n $\longrightarrow$Y. In particular, the n_j-ary operation on A being an $\boldsymbol{X}$-morphism $\omega_{j,X} : X^n \longrightarrow X$ is indeed a $\boldsymbol{Y}$-morphism $\omega_{j,Y} : Y^n \longrightarrow$Y. ■

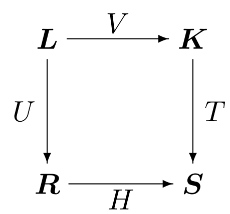
Fay [4] proved, among other things, an (E, M)-topological version of Wyler's taut lift theorem (see [15]). Tholen (see [14]) generalized and discussed some applications of Wyler's theorem. In this section we will show another application, namely, an essentially algebraic functor between two subcategories of $\boldsymbol{Alg}(\Omega')$ and $\boldsymbol{Alg(\Omega)}$ can be extended to the associated paired categories. We use (E, M)-topological version of Wyler's taut lift theorem due to Fay ([4, Theorem (5.3)]) restated here as a lemma in a slightly different form using the hypothesis that is necessary so that the proof given by him works for the lemma. The lemma as stated here is also a consequence of the proof of Theorem (4.1) in [4]. Although it appears closer to Tholen's Theorem (4.1) in [4], we chose to use Fay's result because of its notational advantage. First we will explain a concept.

If $T : \boldsymbol{L} \longrightarrow \boldsymbol{K}$ is a functor and M is a class of sources in $\boldsymbol{K}$, then M_T denotes the class of all T-initial sources $(L, f_i : L \longrightarrow L_i)_{i \in I}$ in $\boldsymbol{L}$ with $(TL, Tf_i : TL \longrightarrow TL_i)_{i \in I}$ a source in M. A pair (g, L) of an $\boldsymbol{L}$-object L and $\boldsymbol{K}$-morphism $g : K \longrightarrow TL$ is said to be **T-universal map for $\boldsymbol{K}$** ([8, 26.1]) iff for each $\boldsymbol{L}$-object L' and each $\boldsymbol{K}$-morphism $f : K \longrightarrow TL'$, there exists a unique $\boldsymbol{L}$-morphism $\bar{f} : L \longrightarrow L'$ such that the triangle



commutes.

Lemma 3.4 Consider the following commutative square of categories and functors



where $\boldsymbol{R}$ has (E, M) - factorizations, $\boldsymbol{L}$ has U-initial lifts for sources in M, M' is the class of all sources in $\boldsymbol{S}$ which have T-initial lifts, H has a left adjoint, H sends M-sources to M'-sources, and V sends M_U- sources to $(M')_T$- sources.

Then V has a left adjoint.

Moreover, if for each $\boldsymbol{S}$-object S there exists an $\boldsymbol{H}$-universal map (e, R) for S where $e : S \longrightarrow HR$ is an epimorphism in $\boldsymbol{S}$, then for each $\boldsymbol{K}$-object K there corresponds a V-universal map (f, L) for K such that $f : K \longrightarrow VL$ is an epimorphism in $\boldsymbol{K}$. ∎

Theorem 3.5 Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are subcategories of $\boldsymbol{Alg(\Omega)}$.

(a) If $\boldsymbol{B}$ is an essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$ that is reflective in a subcategory $\boldsymbol{A}$, then $\boldsymbol{X}\diamond\boldsymbol{B}$ is a reflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

(b) If $\boldsymbol{B}$ is further epireflective in $\boldsymbol{A}$, then $\boldsymbol{X}\diamond\boldsymbol{B}$ is an epireflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

Proof. Let $H : \boldsymbol{B} \longrightarrow \boldsymbol{A}$ be the inclusion map and define the concrete functor $\tilde{H} : \boldsymbol{X}\diamond\boldsymbol{B} \longrightarrow \boldsymbol{X}\diamond\boldsymbol{A}$ by $\tilde{H}(X, B) := (X, HB)$, for any $\boldsymbol{X}\diamond\boldsymbol{B}$-object (X, B). We have the commutative square diagram

$$\begin{array}{ccc} \boldsymbol{X}\diamond\boldsymbol{B} & \xrightarrow{\tilde{H}} & \boldsymbol{X}\diamond\boldsymbol{A} \\ T \big\downarrow & & \big\downarrow T' \\ \boldsymbol{B} & \xrightarrow[H]{} & \boldsymbol{A} \end{array},$$

where T and T' are forgetful functors.

If $\boldsymbol{B}$ is an essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$ that is reflective in a subcategory $\boldsymbol{A}$, then H is essentially algebraic and hence $\tilde{H}$ is essentially algebraic (see [5, Theorem 3.2 (c)]), which proves $\boldsymbol{X}\diamond\boldsymbol{B}$ is a reflective subcategory of $\boldsymbol{X}\diamond\boldsymbol{A}$.

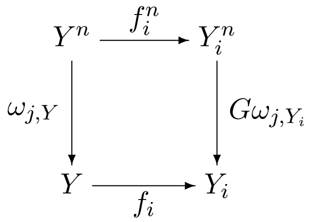
The second part is a direct application of the Lemma 1. ■

4. Limits and Colimits

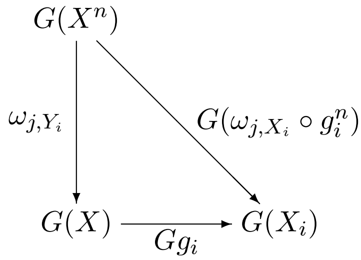
In [15]), Wyler shows, among other things, that if $\boldsymbol{X}$ is a topological category then all categorical limits and colimits can be lifted from a category $\boldsymbol{A}$ of algebras to the category $\boldsymbol{X}\diamond\boldsymbol{A}$. In particular, if $\boldsymbol{X}$ is a topological category and $\boldsymbol{A}$ is complete and cocomplete then $\boldsymbol{X}\diamond\boldsymbol{A}$ is complete and cocomplete. Since each monotopological category is an epireflective subcategory of a topological category, similar results are true if $\boldsymbol{X}$ is a monotopological category. In this section we intend to describe some limits and colimits in the paired category $\boldsymbol{X}\diamond\boldsymbol{A}$ under the assumption that $\boldsymbol{X}$ is monotopological. We begin with a Theorem that is very useful in our work.

Theorem 4.1 Suppose $G : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is a concrete functor which preserves concrete finite powers, X is an $\boldsymbol{X}$-object, $((X_i, A_i))_{i\in I}$ is a family of $\boldsymbol{X}\diamond\boldsymbol{A}$-objects, $((Y, A), f_i : (Y, A) \longrightarrow (GX_i, A_i))_{i\in I}$ is a source in $\boldsymbol{Y}\diamond\boldsymbol{A}$, and $(X, g_i : X \longrightarrow X_i)_{i\in I}$ is a G-initial source in $\boldsymbol{X}$. If $Y = GX$ and $Gg_i = f_i$ as $\boldsymbol{Y}$-morphisms for each $i \in I$, then (X, A) is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object.

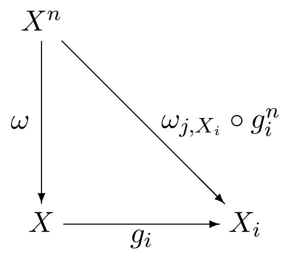
Proof. Since $GX = Y$, we have that $|X| = |Y|(= |A|)$. Because X and A are objects in the categories $\boldsymbol{X}$ and $\boldsymbol{A}$ respectively, we only have to show that for each $j \in J$, the n_j-ary operation $\omega_{j,A}$ on A is a morphism in $\boldsymbol{X}$. Fix $j \in J$ and write n for n_j. For any $i \in I$, since $f_i : A \longrightarrow A_i$ is an $\boldsymbol{A}$-homomorphism we have the commutative diagram of $\boldsymbol{Y}$-morphisms:



where $Y_i = GX_i$. Since G preserves concrete finite powers and $GX = Y$, we have $Y^n = (GX)^n = G(X^n)$ and $G\omega_{j,X_i} \circ f_i = G\omega_{j,X_i} \circ (Gg_i)^n = G\omega_{j,X_i} \circ G(g_i) = G(\omega_{j,X_i} \circ g_i)$, hence the above diagram can be viewed as the following:



Since $(X, g_i : X \longrightarrow X_i)_{i \in I}$ is G-initial there exists a unique $\boldsymbol{X}$-morphism $\omega : X^n \longrightarrow X$ such that $G\omega = \omega_{j,Y}$ and the diagram



commutes. Since G is concrete, ω and $\omega_{j,Y}$ have the same underlying functions. Thus $\omega_{j,A}$ is an $\boldsymbol{X}$-morphism. This shows that (X, A) is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object. ∎

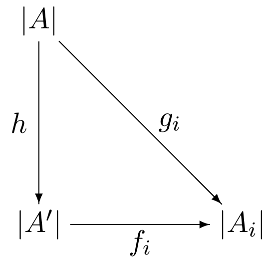
As a consequence of this result we describe a construction of $\boldsymbol{X}\diamond\boldsymbol{A}$-objects from sources in the category $\boldsymbol{A}$.

Corollary 4.2 Let $((X_i, A_i))_{i \in I}$ be a family of objects in the category $\boldsymbol{X}\diamond\boldsymbol{A}$ and $(A, g_i : A \longrightarrow A_i)_{i \in I}$ be a source in $\boldsymbol{A}$. If X is an object in $\boldsymbol{X}$ having the same underlying set as A and is initial with respect to the source $(X, g_i : X \longrightarrow X_i)_{i \in I}$, then the pair (X, A) lies in $\boldsymbol{X}\diamond\boldsymbol{A}$.

In addition to the above hypothesis, if $\boldsymbol{A}$ is essentially algebraic and $(|A|, g_i : |A| \longrightarrow |A_i|)_{i \in I}$ is a monosource in $\boldsymbol{Set}$, then $((X, A), g_i : (X, A) \longrightarrow (X_i, A_i))_{i \in I}$ is initial in $\boldsymbol{X}\diamond\boldsymbol{A}$.

Proof. Since $G : \boldsymbol{X} \longrightarrow \boldsymbol{Set}$ is a concrete functor preserving concrete finite powers, the first part follows from the above Theorem.

In order to prove the second part, let us assume that $((X', A'), f_i : (X', A') \longrightarrow (X_i, A_i))_{i \in I}$ is an $\boldsymbol{X}\diamond\boldsymbol{A}$-source and $h : |A'| \longrightarrow |A|$ is a function such that the diagram



commutes for each $i \in I$. Since any source in $\boldsymbol{A}$ that is a monosource in $\boldsymbol{Set}$ is initial in $\boldsymbol{A}$ (see [1, 23.2(6)]), the source $(A, g_i : A \longrightarrow A_i)_{i\in I}$ is initial in $\boldsymbol{A}$ so that h is an $\boldsymbol{A}$-morphism. It is also an $\boldsymbol{X}$-morphism since $(X, g_i : X \longrightarrow X_i)_{i\in I}$ is G-initial. Hence h is an $\boldsymbol{X}\diamond\boldsymbol{A}$-morphism. ■

The preceding corollary ensures, as shown in the following Theorem, the existence of products, subobjects, equalizers and intersections in the category $\boldsymbol{X}\diamond\boldsymbol{A}$ which proves that $\boldsymbol{X}\diamond\boldsymbol{A}$ is complete and is an (extremal epi, mono) - category, provided $\boldsymbol{X}$ is monotopological and $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$.

Theorem 4.3 Suppose $\boldsymbol{X}$ is a monotopological category and $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg(\Omega)}$. Then the following hold.

(a) For any family $((X_i, A_i))_{i\in I}$ of objects in $\boldsymbol{X}\diamond\boldsymbol{A}$ such that the concrete product A of $(A_i)_{i\in I}$ exists in $\boldsymbol{A}$, then (X, A), where X is given the initial structure in $\boldsymbol{X}$ with respect to the natural projections $\pi_i : A \longrightarrow A_i$, is the product in $\boldsymbol{X}\diamond\boldsymbol{A}$.

(b) If (X, A) is an object in $\boldsymbol{X}\diamond\boldsymbol{A}$, A' is an $\boldsymbol{A}$-object which is an Ω-subalgebra of A and X' is an $\boldsymbol{X}$-object having the same underlying set as A' with the initial structure with respect to the inclusion $u : X' \hookrightarrow X$, then (X', A') is an object in $\boldsymbol{X}\diamond\boldsymbol{A}$ which is a subobject of (X, A).

(c) Suppose $f : (X, A) \longrightarrow (X', A')$ and $g : (X, A) \longrightarrow (X', A')$ are two morphisms such that the pair $f : A \longrightarrow A'$ and $g : A \longrightarrow A'$ admits an equalizer (E, e) in $\boldsymbol{A}$. Then, the $\boldsymbol{X}$-object Z, having the same underlying set as E, initial with respect to $e : E \longrightarrow X$ leads to an object (Z, E) in $\boldsymbol{X}\diamond\boldsymbol{A}$ and $((Z, E), e)$ is an equalizer of the pair of morphisms f and g.

(d) If $((X_i, A_i))_{i\in I}$ is a family of $\boldsymbol{X}\diamond\boldsymbol{A}$-subobjects of (X, A) such that the intersection $\cap A_i$ of $(A_i)_{i\in I}$ exists in $\boldsymbol{A}$, then the $\boldsymbol{X}\diamond\boldsymbol{A}$-object $(\cap X_i, \cap A_i)$, where $\cap X_i$ has the initial structure with respect to inclusions into $X_i's$, is the intersection of $((X_i, A_i))_{i\in I}$ in $\boldsymbol{X}\diamond\boldsymbol{A}$.

In particular,

(e) If $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$, then the category $\boldsymbol{X}\diamond\boldsymbol{A}$ is complete and is an (extremal epi, mono) - category.

Proof. In order to prove (a), first note that the source $(|A|, \pi_i : |A| \longrightarrow |A_i|)_{i\in I}$ is point separating and hence is a monosource in $\boldsymbol{Set}$. Since $\boldsymbol{X}$ is monotopological, we can find an $\boldsymbol{X}$-object X, having the same underlying set as A, initial with respect to the source $(X, \pi_i : X \longrightarrow X_i)_{i\in I}$. By Corollary 9, (X, A) is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object and is the product of $((X_i, A_i))_{i\in I}$.

Analogously, statements $(b) - (d)$ follow from the observation that any inclusion is a monosource in $\boldsymbol{Set}$.

To establish statement (e), let us assume that $\boldsymbol{A}$ is an essentially algebraic subcategory of $\boldsymbol{Alg(\Omega)}$. Then, by Theorem 4 and the discussion following it, $\boldsymbol{A}$ is complete and wellpowered, and $\boldsymbol{A}$ has concrete products. It is straight forward to see that $\boldsymbol{X}\diamond\boldsymbol{A}$

is wellpowered. Statements $(a), (c)$, and (d) imply that $\boldsymbol{X}\diamond\boldsymbol{A}$ has arbitrary products, equalizers, and intersections. Because any category having products and equalizers is complete (see [8, 23.8]) and because every wellpowered finitely complete category having intersections is an (extremal epi, mono) - category (see [8, 34.1]), $\boldsymbol{X}\diamond\boldsymbol{A}$ is complete and is an (extremal epi, mono) - category. ■

If (X, A) is an object in $\boldsymbol{X}\diamond\boldsymbol{A}$, X' is a quotient of X with respect to an $\boldsymbol{X}$-epimorphism $f : X \longrightarrow X'$ and A' is a quotient of A with respect to an $\boldsymbol{A}$-epimorphism $f : A \longrightarrow A'$ then, in general, (X', A') may not be a quotient of (X, A). In fact, it need not be an object in the category $\boldsymbol{X}\diamond\boldsymbol{A}$ as is seen in the following example (where $\boldsymbol{X}$ is the category ***Haus*** of Hausdorff topological spaces and $\boldsymbol{A}$ is the category ***Sgrp*** of semigroups) due to Lawson and Madison [10]. After presenting the example, in Theorem 11 we establish that an additional condition on the category $\boldsymbol{X}$ will eliminate the pathology.

***Example* 4.4** Let $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ be the two dimensional Euclidean space with the product topology inherited from the usual topology on the real line $\mathbb{R}$, $\mathbb{Q}$ be the set of all rational numbers, X be the subspace of $\mathbb{R}^2$ defined by

$$X := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ and } x_2 \geq 0\right\}$$

and

$$I := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ and } x_2 = 0\right\}.$$

Define a multiplication $\cdot$ on X by the formula

$$(x_1, x_2) \cdot (x_1', x_2') = (x_1 + x_1', \min(x_2, x_2')),$$

where $+$ is the usual addition of real numbers. Clearly X is a semigroup with multiplication $\cdot$ and I is a closed ideal of X (i.e., $X \cdot I \subseteq I$). Hence the so-called Rees congruence θ on X relative to the ideal I, given by $x\theta x'$ iff either $x = x'$ or I contains both x and x', leads to the semigroup X/θ under the multiplication defined by $(\pi(x), \pi(x')) \longrightarrow \pi(x \cdot x')$, where $\pi(z)$ is the θ-class containing z, i.e., $\pi(z) := I$ whenever $z \in I$ and $\pi(z) := \{z\}$ for any $z \notin I$. Thus $\pi : X \longrightarrow X/\theta$ is a quotient map in ***Sgrp*** (see [13, page 9]).

We give X/θ the quotient topology induced by the natural map $\pi : X \longrightarrow X/\theta$. It can be shown that that X/θ is not a topological semigroup. In fact, the multiplication on X/θ is not continuous at $(\pi(0, 0), \pi(0, 0))$ (see [10]).

Lemma 4.5 Suppose $\boldsymbol{X}$ has finitely productive quotients, (X, A) is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object, $f : A \longrightarrow A'$ is an $\boldsymbol{A}$-morphism, and $f : X \longrightarrow X'$ is a quotient map in $\boldsymbol{X}$. Then (X', A') is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object.

Proof. Let $j \in J$, $n = n_j$ and $\omega = \omega_{j,A}$, $\omega' = \omega_{j,A'}$ be n-ary operations on A and A' respectively. Since $\boldsymbol{X}$ has finitely productive quotients, X'^n is a quotient of X^n with respect to $f^n : X^n \longrightarrow X'^n$. Thus ω' is an $\boldsymbol{X}$-morphism iff $\omega' \circ f^n$ is an $\boldsymbol{X}$-morphism. However, because f is an Ω-homomorphism, we have the commutative diagram,

$$\begin{array}{ccc} |X^n| & \xrightarrow{\omega} & |X| \\ {\scriptstyle f^n}\downarrow & & \downarrow{\scriptstyle f} \\ |X'^n| & \xrightarrow[\omega']{} & |X'|, \end{array}$$

which shows that $\omega' \circ f^n$, being equal to $f \circ \omega$, is an $\boldsymbol{X}$-morphism. This being true for each $j \in J$, (X', A') is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object. ■

Theorem 4.6 If $\boldsymbol{X}$ has finitely productive quotients, (X, A) is an object in $\boldsymbol{X}\diamond\boldsymbol{A}$, $f : X \longrightarrow X'$ is a quotient map in $\boldsymbol{X}$ and $f : A \longrightarrow A'$ is a quotient map in $\boldsymbol{A}$, then (X', A') is an object in $\boldsymbol{X}\diamond\boldsymbol{A}$ and $f : (X, A) \longrightarrow (X', A')$ is a quotient map in $\boldsymbol{X}\diamond\boldsymbol{A}$.

Proof. By the virtue of Lemma 2, it remains to show that $f : (X, A) \longrightarrow (X', A')$ is a quotient map in $\boldsymbol{X}\diamond\boldsymbol{A}$. Let (X'', A'') be any object in $\boldsymbol{X}\diamond\boldsymbol{A}$ and $g : |X'| \longrightarrow |X''|$ be a function between the two sets such that $g \circ f$ is an $\boldsymbol{X}\diamond\boldsymbol{A}$-morphism. Then g is an $\boldsymbol{X}$-morphism between X' and X'' because $g \circ f$ is one such and $f : X \longrightarrow X'$ is a quotient map in $\boldsymbol{X}$. Similarly g is also an Ω-homomorphism. Thus $g : (X', A') \longrightarrow (X'', A'')$ is an $\boldsymbol{X}\diamond\boldsymbol{A}$-morphism. ■

Thus we have proved that if $\boldsymbol{X}$ has finitely productive quotients then quotients of objects in $\boldsymbol{X}$ and $\boldsymbol{A}$ with the same underlying set resulting from the same underlying function can be paired to form quotients in the category $\boldsymbol{X}\diamond\boldsymbol{A}$.

A similar construction is possible for coproducts under an additional condition that final epi sinks are finitely productive in $\boldsymbol{X}$ (in particular, under the assumption that $\boldsymbol{X}$ is well-fibred monotopological and cartesian closed, see [12]). As we will see later in Theorem 12, one can give other sufficient conditions that coproducts exist in $\boldsymbol{X}\diamond\boldsymbol{A}$.

First we need the following definitions. We say that **final epi sinks are finitely productive** in $\boldsymbol{X}$ iff the product $(f_i \times g_k : X_i \times Y_k \longrightarrow X \times Y, X \times Y)_{i\in I, k\in K}$ of any two final epi sinks $(f_i : X_i \longrightarrow X, X)_{i\in I}$ and $(g_k : Y_k \longrightarrow Y, Y)_{k\in K}$ in $\boldsymbol{X}$ is final in $\boldsymbol{X}$. A class $\mathcal{F}$ of functions is said to be **Ω-admissible** to an Ω-algebra A iff each function in $\mathcal{F}$ has the codomain $|A|$, and for each $j \in J$, $n = n_j$, $f_1, \ldots, f_n \in \mathcal{F}$,

$$\omega_{j,A} \circ (f_1 \times \ldots \times f_n) \in \mathcal{F}.$$

Lemma 4.7 Suppose final epi sinks are finitely productive in $\boldsymbol{X}$, A is an $\boldsymbol{A}$-object, $(X_i)_{i\in I}$ is a family of $\boldsymbol{X}$-objects, and $(f_i : |X_i| \longrightarrow |A|)_{i\in I}$ is a class of functions Ω-admissible to A. If X is an $\boldsymbol{X}$-object with the same underlying set as A such that $(f_i : X_i \longrightarrow X, X)_{i\in I}$ is a final epi sink in $\boldsymbol{X}$, then (X, A) is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object.

Proof. Since X has the final structure with respect to $f_i : X_i \longrightarrow X$, for any positive integer n, X^n has the final structure with respect to $f_{i_1} \times \ldots \times f_{i_n}$, $i_1 \in I$, .., $i_n \in I$, by hypothesis. Let $j \in J$ and $n = n_j$. We have to show that $\omega_{j,A}$ is an $\boldsymbol{X}$-morphism. However, $\omega_{j,A} \circ (f_{i_1} \times \ldots \times f_{i_n})$, $i_1 \in I$, $\ldots$, $i_n \in I$, being one of the $f_i's$ as (f_i) is Ω-admissible to A, is an $\boldsymbol{X}$-morphism. Consequently, the n_j-ary operation $\omega_{j,A}$ on A is an $\boldsymbol{X}$-morphism $\omega_{j,x} : X^n \longrightarrow X$. This being true for each $j \in J$, (X, A) is an $\boldsymbol{X}\diamond\boldsymbol{A}$-object. ■

Lemma 4.8 Suppose A is an Ω-algebra and $(C_i)_{i\in I}$ is a family of sets and $(f_i : C_i$

$\longrightarrow |A|)_{i\in I}$ is a class of functions. Then there exists the smallest class $\mathcal{F}$ of functions containing $(f_i)_{i\in I}$ that is Ω-admissible to A, i.e., there exists a class $\mathcal{F}$ of functions containing $(f_i)_{i\in I}$ such that $\mathcal{F}$ is Ω-admissible to A and any class of functions containing $(f_i)_{i\in I}$that is Ω-admissible to A contains $\mathcal{F}$.

Moreover, any member f of $\mathcal{F}$ has a domain of the form $C_{i_1}\times\ldots\times C_{i_n}$ $(i_1,\ldots,i_n\in I)$ and the codomain $|A|$.

Proof. Let $\mathcal{F}$ be the intersection of all classes of functions that are Ω-admissible to A and contain $(f_i)_{i\in I}$. Clearly $\mathcal{F}$ has the property described in the lemma. If $\mathcal{G}$ is the family of all functions of the form explained in the last statement of the lemma, $\mathcal{G}$ is certainly Ω-admissible to A and contains $(f_i)_{i\in I}$ so that $\mathcal{F}\subseteq\mathcal{G}$. ■

For example, if A is an $\boldsymbol{Alg(1)}$ -object with the unary operation u and $(f_i : C_i \longrightarrow |A|)_{i\in I}$ is a class of functions, then $(u^n\circ f_i : C_i \longrightarrow |A|)_{i\in I, n\geq 0}$ is the smallest Ω-admissible class containing $(f_i : C_i \longrightarrow |A|)_{i\in I}$. If A is a group with the multiplication $\cdot$ and the inversion β_1 and $f : C \longrightarrow |A|$ is a function then write

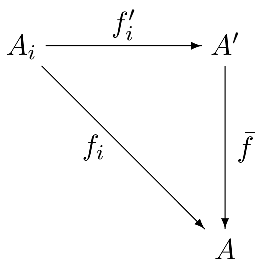
$$f_{n,\mu,\sigma} := \left[(\beta_{\mu(1)}\circ f)\cdot(\beta_{\mu(2)}\circ f)\cdot\ldots\cdot(\beta_{\mu(n)}\circ f)\right]\circ r_\sigma$$

where n is a positive integer, μ is a function from $\mathbf{N}_n := \{1,2,\ldots,n\}$ with values 0 and 1, σ is a bijection from $\mathbf{N}_n$ onto $\mathbf{N}_n$, r_σ is the function $r_\sigma : C^n \longrightarrow C^n$ given by $r_\sigma(c_1,\ldots,c_n) = (c_{\sigma(1)},\ldots c_{\sigma(n)})$, and β_0 :=id_A. The family $(f_{n,\mu,\sigma} : C^n \longrightarrow |A|)$ is the smallest Ω-admissible class containing $f : C \longrightarrow |A|$.

Theorem 4.9 Suppose final epi sinks are finitely productive in $\boldsymbol{X}$, $((X_i, A_i))_{i\in I}$ is a family of $\boldsymbol{X\diamond A}$-objects, and $(f_i : A_i \longrightarrow A, A)_{i\in I}$ is a coproduct in $\boldsymbol{A}$.

Let $(g_k : |X'_k| \longrightarrow |A|)_{k\in K}$ be the smallest class of functions containing $(f_i)_{i\in I}$ that is Ω-admissible to A, where X'_k is of the form $X_{i_1}\times\ldots\times X_{i_n}$ $(i_1,\ldots,i_n\in I)$ for each $k\in K$, the product being formed in the category $\boldsymbol{X}$. Let X be an $\boldsymbol{X}$-object with the same underlying set as A such that $(g_k : X'_k \longrightarrow X, X)_{k\in K}$ is a final epi sink in $\boldsymbol{X}$. Then the pair (X, A) is an $\boldsymbol{X\diamond A}$-object and the sink $(f_i : (X_i, A_i) \longrightarrow (X, A), (X, A))_{i\in I}$ is a coproduct in $\boldsymbol{X\diamond A}$.

Proof. (X, A) is an $\boldsymbol{X\diamond A}$-object by Lemma 3. Since $(g_k)_{k\in K}$ contains $(f_i)_{i\in I}$, f_i : $(X_i, A_i) \longrightarrow (X, A)$ is an $\boldsymbol{X\diamond A}$-morphism for each $i\in I$. We now show that $(f_i)_{i\in I}$ is a coproduct in $\boldsymbol{X\diamond A}$. Let (X', A') be any $\boldsymbol{X\diamond A}$-object and $(f'_i : (X_i, A_i) \longrightarrow (X', A'))_{i\in I}$ be a family of $\boldsymbol{X\diamond A}$-morphisms. Since $(f_i : A_i \longrightarrow A, A)_{i\in I}$ is a coproduct in $\boldsymbol{A}$, there exists a unique $\boldsymbol{A}$-morphism $\bar f : A \longrightarrow A'$ such that the diagram of $\boldsymbol{A}$-morphisms



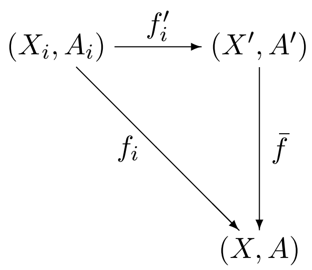
commutes for each $i\in I$. We show that $\bar f : X \longrightarrow X'$ is an $\boldsymbol{X}$-morphism. Let $\mathcal{H}$ be the family of all functions $h : |X_{i_1}\times\ldots\times X_{i_n}| \longrightarrow |A|$ $(i_1\in I,\ldots,i_n\in I)$ such that $\bar f\circ h$ is an $\boldsymbol{X}$-morphism. $\mathcal{H}$ contains $(f_i)_{i\in I}$ because $\bar f\circ f_i = f'_i$ is an $\boldsymbol{X}$-morphism for each $i\in$I. Assume $j\in J, n = n_j$, and $h_1\in\mathcal{H},\ldots,h_n\in\mathcal{H}$. Then

$$
\begin{aligned}
\bar{f} \circ [\omega_{j,A} \circ (h_1 \times \ldots \times h_n)] \\
&= (\bar{f} \circ \omega_{j,A}) \circ (h_1 \times \ldots \times h_n) \\
&= (\omega_{j,A'} \circ \bar{f}^n) \circ (h_1 \times \ldots \times h_n), \text{ since } \bar{f} \text{ is an } \boldsymbol{A}\text{-morphism}, \\
&= \omega_{j,A'} \circ [\bar{f}^n \circ (h_1 \times \ldots \times h_n)] \\
&= \omega_{j,X'} \circ (\bar{f} \circ h_1 \times \ldots \times \bar{f} \circ h_n).
\end{aligned}
$$

Since $\bar{f} \circ h_1, \ldots, \bar{f} \circ h_n$, and $\omega_{j,X'}$ are $\boldsymbol{X}$-morphisms, $\bar{f} \circ [\omega_{j,A} \circ (h_1 \times \ldots \times h_n)]$ is an $\boldsymbol{X}$-morphism. Thus

$$\omega_{j,A} \circ (h_1 \times \ldots \times h_n) \in \mathcal{H}.$$

This shows that $\mathcal{H}$ is Ω-admissible to A containing $(f_i)_{i \in I}$. Since $\mathcal{F}$ is the smallest such class, we conclude that $g_k \in \mathcal{H}$, i.e., $\bar{f} \circ g_k$ is an $\boldsymbol{X}$-morphism for each $k \in K$. Since $(g_k)_{k \in K}$ is final in $\boldsymbol{X}$, $\bar{f}$ is an $\boldsymbol{X}$-morphism. Thus we have the commutative diagram of $\boldsymbol{X}\diamond\boldsymbol{A}$-objects and $\boldsymbol{X}\diamond\boldsymbol{A}$-morphisms:



Of course, $\bar{f}$ is unique. ■

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