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Probability of having n^{th}-roots and n-centrality of two classes of groups

M. Hashemi[a*], M. Polkouei[a]

[a] *Faculty of Mathematical Sciences, University of Guilan, P.O.Box 41335-19141, Rasht, Iran.*

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Abstract. In this paper, we consider the finitely 2-generated groups $K(s,l)$ and G_m as follows;

$$K(s,l) = \langle a, b | ab^s = b^l a,\ ba^s = a^l b\rangle,$$
$$G_m = \langle a, b | a^m = b^m = 1,\ [a, b]^a = [a, b],\ [a, b]^b = [a, b]\rangle$$

and find the explicit formulas for the probability of having n^{th}-roots for them. Also we investigate integers n for which, these groups are n-central.

Keywords: Nilpotent groups, n^{th}-roots, n-central groups

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1. Introduction

Let $n > 1$ be an integer. An element a of group G is said to have an n^{th}-root b in G, if $a = b^n$. The probability that a randomly chosen element in G has an n^{th}-root, is given by

$$P_n(G) = \frac{|G^n|}{|G|}$$

*Corresponding author.
E-mail address: m_hashemi@guilan.ac.ir (M. Hashemi).

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where $G^n = \{a \in G | a = b^n, for \quad some \quad b \in G\} = \{x^n | x \in G\}$. In [5], the probability $P_n(G)$ for Dihedral groups D_{2m} and Quaternion groups Q_{2^m} for every integer $m \geqslant 3$ have been computed. Also, in [4] the probability that Hamiltonian groups may have n^{th}-roots have been calculated. For $n > 1$, a group G is said to be n-central if $[x^n, y] = 1$ for all $x, y \in G$. In [6], some aspects of n-central groups have been investigated.

First, we state the following Lemma without proof.

Lemma 1.1 If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:
(i) $[uv, w] = [u, w][v, w]$ and $[u, vw] = [u, v][u, w]$;
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

Now, we state some lemmas which can be found in [1, 2].

Lemma 1.2 The groups $K(s, l) = \langle a, b | ab^s = b^l a, \ ba^s = a^l b\rangle$ where $(s, l) = 1$, have the following properties:
(i) $|K(s, l)| = |l - s|^3$, if $(s, l) = 1$ and is infinite otherwise;
(ii) if $(s, l) = 1$ then $|a| = |b| = (l - s)^2$;
(iii) if $(s, l) = 1$, then $a^{l-s} = b^{s-l}$.

Lemma 1.3 (i) For every $l \geqslant 3$, $K(s, l) \cong K(1, 2 - l)$.
(ii) For every $i \geqslant 2$ and $(s, i) = 1$, $K(s, s + i) \cong K(1, i + 1)$.

Note that if $(s, l) = 1$, then $K(s, l) \cong K(1, l - s + 1)$ which we can write as K_m where $m = l - s + 1$.

Lemma 1.4 Every element of K_m can be uniquely presented by $x = a^{\beta} b^{\gamma} a^{(m-1)\delta}$, where $1 \leqslant \beta, \gamma, \delta \leqslant m - 1$.

Lemma 1.5 In K_m, $[a, b] = b^{m-1} \in Z(K_m)$.

The following lemma can be seen in [3].

Lemma 1.6 Let $G_m = \langle a, b | a^m = b^m = 1, \ [a, b]^a = [a, b], \ [a, b]^b = [a, b]\rangle$ where $m \geqslant 2$, then we have
(i) every element of G_m can be uniquely presented by $a^i b^j [a, b]^t$, where
$1 \leqslant i, j, t \leqslant m$.
(ii) $|G_m| = m^3$.

In this paper, we consider the groups K_m and G_m which are nilpotent groups of nilpotency class two. In section 2, we compute the probability of having n^{th}-root of K_m and G_m. Section 3 is devoted to finding integers n for which, K_m and G_m are n-central.

2. The probability of having n^{th}-roots

In this section we consider groups K_m and G_m and find the probability of having n^{th}-roots. Here for $m \in \mathbb{Z}$, by m^* we mean the arithmetic inverse of m.

Proposition 2.1 For integers $m, n \geqslant 2$;

(1) If $G = K_m$ and $x \in G$, then we have

$$x^n = a^{n\beta} b^{n\gamma} a^{(m-1)(n\delta + \frac{n(n-1)}{2}\beta\gamma)};$$

(2) If $G = G_m$ and $x \in G$, then we have

$$x^n = a^{ni} b^{nj} [a, b]^{nt - \frac{n(n-1)}{2} ij}.$$

Proof. We use an induction method on n. By Lemma 1.4, the assertion holds for $n = 1$. Now, let

$$x^n = a^{n\beta} b^{n\gamma} a^{(m-1)(n\delta + \frac{n(n-1)}{2}\beta\gamma)}.$$

Then

$$x^{n+1} = a^{\beta} b^{\gamma} a^{(m-1)\delta} a^{n\beta} b^{n\gamma} a^{(m-1)(n\delta + \frac{n(n-1)}{2}\beta\gamma)}$$

By Lemma 1.2, $a^{(m-1)\delta} = b^{(1-m)\delta}$. So

$$\begin{aligned} x^{n+1} &= a^{\beta} b^{\gamma} a^{n\beta} b^{n\gamma} a^{(m-1)((n+1)\delta + \frac{n(n-1)}{2}\beta\gamma)} \\ &= a^{(n+1)\beta} [b, a]^{n\beta\gamma} b^{(n+1)\gamma} a^{(m-1)((n+1)\delta + \frac{n(n-1)}{2}\beta\gamma)}. \end{aligned}$$

Since K_m is a group of nilpotency class two, $G' \subseteq Z(G)$. Hence by Lemma 1.1 we have

$$x^{n+1} = a^{(n+1)\beta} b^{(n+1)\gamma} a^{(m-1)((n+1)\delta + \frac{n(n+1)}{2}\beta\gamma)}.$$

The second part can be proved similarly. ■

Theorem 2.2 Let $G = K_m$, where $m \geqslant 2$. Then

$$P_n(G) = \begin{cases} \frac{2}{d^3} \; if \; n \; be \; even, \; (\frac{n}{2}, m-1) = \frac{d}{2} \; and \; \frac{m-1}{d} \; be \; odd; \\ \frac{1}{d^3} \; otherwise, \end{cases}$$

where $(n, m-1) = d$.

Proof. Let $a^{\beta} b^{\gamma} a^{(m-1)\delta}$ be an element of G^n where $1 \leqslant \beta, \gamma, \delta \leqslant m-1$. If $x = (x_1)^n$ when $a^{\beta_1} b^{\gamma_1} a^{(m-1)\delta_1} \in G$, $1 \leqslant \beta_1, \gamma_1, \delta_1 \leqslant m-1$, then by Proposition 2.1 we have

$$\begin{aligned} a^{\beta} b^{\gamma} a^{(m-1)\delta} &= (a^{\beta_1} b^{\gamma_1} a^{(m-1)\delta_1})^n \\ &= a^{n\beta_1} b^{n\gamma_1} a^{(m-1)(n\delta_1 + \frac{n(n-1)}{2}\beta_1\gamma_1)}. \end{aligned}$$

By uniqueness of presentation of G, we obtain

$$\begin{cases} n\beta_1 \equiv \beta \quad (mod \; m-1) \\ n\gamma_1 \equiv \gamma \quad (mod \; m-1) \\ n\delta_1 + \frac{n(n-1)}{2}\beta_1\gamma_1 \equiv \delta \quad (mod \; m-1). \end{cases} \qquad (1)$$

Now let $(n, m-1) = d$. The first congruence of the system (1) has the solution

$$\beta_1 \equiv (\frac{n}{d})^* (\frac{\beta}{d}) \quad (mod \; \frac{m-1}{d})$$

if and only if $d \mid \beta$. Then

$$\beta \in \{d, 2d, \ . \ . \ . \ , \frac{m-1}{d} \times d\}.$$

This means that β has $\frac{m-1}{d}$ choices. Similarly, by second equation of System (1) we get

$$\gamma \in \{d, 2d, \ . \ . \ . \ , \frac{m-1}{d} \times d\}.$$

So γ admits $\frac{m-1}{d}$ values.
Now for finding the number of values of δ, we consider two cases, where n is odd or even.

First let n be an odd integers. Then

$$n(\delta_1 + \frac{n(n-1)}{2}\beta_1\gamma_1) \equiv \delta \quad (mod\ m-1).$$

Since $(n, m-1) = d$, we get

$$\delta_1 \equiv (\frac{n}{d})^* \frac{\delta}{d} - \frac{n(n-1)}{2}\beta_1\gamma_1 \quad (mod\ \frac{m-1}{d})$$

provided that $d \mid \delta$. So

$$\delta \in \{d, 2d, \ . \ . \ . \ , \frac{m-1}{d} \times d\}.$$

Therefore in this case we have $\frac{m-1}{d}$ choices for δ. By the above facts, we have

$$\begin{aligned} \mid G^n \mid &= \mid \{a^{\beta}b^{\gamma}a^{(m-1)\delta} \mid \beta \in \{d, . . ., \frac{m-1}{d}d\}, \gamma \in \{d, . . ., \frac{m-1}{d}d\}, \delta \in \{d, . . ., \frac{m-1}{d}d\}\} \mid \\ &= \mid \{(\beta, \gamma, \delta) \mid \{\beta \in \{d, . . ., \frac{m-1}{d}d\}, \gamma \in \{d, . . ., \frac{m-1}{d}d\}, \delta \in \{d, . . ., \frac{m-1}{d}d\}\} \mid \\ &= \frac{m-1}{d} \times \frac{m-1}{d} \times \frac{m-1}{d} = (\frac{m-1}{d})^3. \end{aligned}$$

So

$$P_n(G) = \frac{\mid G^n \mid}{\mid G \mid} = \frac{(m-1/d)^3}{(m-1)^3} = \frac{1}{d^3}.$$

Now suppose n be an even integer. Then $(\frac{n}{2}, m-1) = d$ or $(\frac{n}{2}, m-1) = \frac{d}{2}$.
Case 1. Let $(\frac{n}{2}, m-1) = d$. Then

$$\frac{n}{2}(2\delta_1 + (n-1)\beta_1\gamma_1) \equiv \delta \quad (mod\ m-1).$$

So

$$2\delta_1 \equiv (\frac{n}{2d})^* \frac{\delta}{d} - (n-1)\beta_1\gamma_1 \quad (mod\ \frac{m-1}{d}).$$

Since $(\frac{n}{2}, m-1) = d$, $(\frac{m-1}{d}, 2) = 1$. Hence, the above congruence holds if and only if $d \mid \delta$. Therefore

$$\delta \in \{d, 2d, \ \ldots, \frac{m-1}{d} \times d\}.$$

So

$$\mid G^n \mid = \mid \{(\beta, \gamma, \delta) \mid \{\beta \in \{d, \ldots, \frac{m-1}{d} d\}, \gamma \in \{d, \ldots, \frac{m-1}{d} d\}, \delta \in \{d, \ldots, \frac{m-1}{d} d\}\} \mid$$
$$= (\frac{m-1}{d})^3$$

and consequently

$$P_n(G) = \frac{1}{d^3}.$$

Case 2. Let $(\frac{n}{2}, m-1) = \frac{d}{2}$. Then

$$\frac{n}{d}(2\delta_1 + (n-1)\beta_1\gamma_1) \equiv \frac{2\delta}{d} \quad (mod \ \frac{2(m-1)}{d}).$$

Hence

$$2\delta_1 \equiv (\frac{n}{d})^* \frac{2\delta}{d} - (n-1)\beta_1\gamma_1 \quad (mod \ \frac{2(m-1)}{d}). \quad (2)$$

So, we must have $2 \mid \beta_1\gamma_1$. Suppose $2 \mid \gamma_1$. Now by congruence

$$\gamma_1 \equiv (\frac{n}{d})^* \frac{\gamma}{d} \quad (mod \ \frac{m-1}{d}) \quad (3)$$

we consider two subcases:

Subcase 2.a. Let $\frac{(m-1)}{d}$ be an even integer. Now since

$$\frac{n}{d}(\frac{n}{d})^* \equiv 1 \quad (mod \ \frac{m-1}{d}),$$

both $\frac{n}{d}$ and $(\frac{n}{d})^*$ are odd. Since $2 \mid \gamma_1$, By congruence (3) we get $2 \mid \frac{\gamma}{d}$. It means that

$$\gamma \in \{2d, 4d, \ldots, \frac{m-1}{2d} \times 2d\}.$$

Hence the number of values of γ is $\frac{m-1}{2d}$. On the other hand according to congruence (2), $\frac{d}{2} \mid \delta$. Therefore

$$\delta \in \{\frac{d}{2}, d, \ldots, \frac{2(m-1)}{d} \times \frac{d}{2}\}.$$

So δ admits $\frac{2(m-1)}{d}$ values. Consequently

$$| G^n |= \frac{m-1}{d} \times \frac{m-1}{2d} \times \frac{2(m-1)}{d} = (\frac{m-1}{d})^3$$

and

$$P_n(G) = \frac{1}{d^3}.$$

Case 2.b. Let $\frac{(m-1)}{d}$ be an odd integer and $\gamma \in \{d, 2d, \ldots, \frac{m-1}{d}d\}$. If

$$\gamma_1 \equiv \frac{n}{d}(\frac{n}{d})^* \ (mod\ \frac{m-1}{d})$$

and γ_1 be an even integer, then we get the desired result. Otherwise, instead of γ_1, we put $\gamma_1 + \frac{m-1}{d}$. So for each

$$\gamma \in \{d, 2d, \ldots, \frac{m-1}{d} \times d\},$$

the congruence holds. It means that the number of choices for γ is equal to $\frac{m-1}{d}$. Finally, we get

$$| G^n |= \frac{m-1}{d} \times \frac{m-1}{d} \times \frac{2(m-1)}{d} = 2(\frac{m-1}{d})^3$$

and

$$P_n(G) = \frac{2}{d^3}.$$

■

Theorem 2.3 Let $G = G_m$, where $m \geqslant 2$. Then

$$P_n(G) = \begin{cases} \frac{2}{d^3} \ if\ n\ be\ even,\ (\frac{n}{2}, m) = \frac{d}{2}\ and\ \frac{m}{d}\ be\ odd; \\ \frac{1}{d^3}\ otherwise, \end{cases}$$

where $(n, m) = d$.

Proof. Let $a^i b^j [a, b]^t$ be an element of G^n where $1 \leqslant i, j, t \leqslant m$. If $x = (x_1)^n$ when $a^{i_1} b^j [a, b]^{t_1} \in G$, $1 \leqslant i_1, j_1, t_1 \leqslant m$, then by Proposition 2.1 we have

$$\begin{aligned} a^i b^j [a, b]^t &= (a^{i_1} b^{j_1} [a, b]^{t_1})^n \\ &= a^{ni_1} b^{nj_1} [a, b]^{nt_1 - \frac{n(n-1)}{2} i_1 j_1}. \end{aligned}$$

By uniqueness of presentation of G, we obtain

$$\begin{cases} ni_1 \equiv i \quad (mod\ m) \\ nj_1 \equiv j \quad (mod\ m) \\ nt_1 - \frac{n(n-1)}{2} i_1 j_1 \equiv t \quad (mod\ m). \end{cases}$$

The obtained congruence system is exactly similar to System (1). So it can be solve, similarly. ■

3. n-centrality

In this section, we again consider groups K_m, G_m and investigate n-centrality for them.

Theorem 3.1 Let $G = K_m$, where $m \geqslant 2$. Then for $n > 1$, the group G is n-central if and only if $m-1 \mid n$.

Proof. By Proposition 2.1 and Lemma 1.1, we get

$$x^n y = a^{n\beta_1+\beta_2} b^{n\gamma_1+\gamma_2} a^{(m-1)(n\delta_1+\delta_2+\frac{n(n-1)}{2}\beta_1\gamma_1+n\beta_2\gamma_1)}.$$

Also we obtain

$$yx^n = a^{n\beta_1+\beta_2} b^{n\gamma_1+\gamma_2} a^{(m-1)(n\delta_1+\delta_2+\frac{n(n-1)}{2}\beta_1\gamma_1+n\beta_1\gamma_2)}.$$

We know that G is n-central if and only if $x^n y = yx^n$, for all $x, y \in G$. Furthermore by uniqueness of presentation of $x^n y$ and yx^n, we see that $x^n y = yx^n$ if and only if

$$n\delta_1 + \delta_2 + \frac{n(n-1)}{2}\beta_1\gamma_1 + n\beta_2\gamma_1 \equiv n\delta_1 + \delta_2 + \frac{n(n-1)}{2}\beta_1\gamma_1 + n\beta_1\gamma_2 \pmod{m-1}.$$

This is equivalent to

$$n(\beta_1\gamma_2 - \beta_2\gamma_1) \equiv 0 \pmod{m-1}.$$

Now since this holds for all $x, y \in G$, $m-1 \mid n$. ■

Theorem 3.2 Let $G = G_m$, where $m \geqslant 2$. Then for $n > 1$, the group G is n-central if and only if $m \mid n$.

Proof. By Proposition 2.1 and Lemma 1.1, we get

$$x^n y = a^{ni_1+i_2} b^{nj_1+j_2} [a, b]^{nt_1+t_2-\frac{n(n-1)}{2}i_1j_1-ni_2j_1}.$$

Also we obtain

$$yx^n = a^{ni_1+i_2} b^{nj_1+j_2} [a, b]^{nt_1+t_2-\frac{n(n-1)}{2}i_1j_1-ni_1j_2}.$$

We know that G is n-central if and only if $x^n y = yx^n$, for all $x, y \in G$. Furthermore by uniqueness of presentation of $x^n y$ and yx^n, we see that $x^n y = yx^n$ if and only if

$$nt_1 + t_2 - \frac{n(n-1)}{2}i_1j_1 - ni_2j_1 \equiv nt_1 + t_2 - \frac{n(n-1)}{2}i_1j_1 - ni_1j_2 \pmod{m}.$$

This is equivalent to

$$n(i_1j_2 - i_2j_1) \equiv 0 \pmod{m}.$$

Now since this holds for all $x, y \in G$, $m \mid n$. ■

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