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Some results on higher numerical ranges and radii of quaternion matrices

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Abstract. Let n and k be two positive integers, $k \leqslant n$ and A be an $n-$square quaternion matrix. In this paper, some results on the $k-$numerical range of A are investigated. Moreover, the notions of k-numerical radius, right k-spectral radius and k-norm of A are introduced, and some of their algebraic properties are studied.

Keywords: k-Numerical radius; right k-spectral radius; k-norm, quaternion matrices.

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1. Introduction and preliminaries

As usual, let $\mathbb{R}$ and $\mathbb{C}$ denote the field of the real and complex numbers, respectively. Moreover, let $\mathbb{H}$ be the four-dimensional algebra of quaternions over $\mathbb{R}$ with the standard basis $\{1, i, j, k\}$ and multiplication rules:

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik,\ and$$
$$1q = q1 = q\ for\ all\ q \in \{1, i, j, k\}.$$

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If $q \in \mathbb{H}$, then there are $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$q = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k.$$

This representation of q is called the canonical form of q. We define $Re\ q = \alpha_0$, the real part of q; $Co\ q = \alpha_0 + \alpha_1 i$, the complex part of q; $Im\ q = \alpha_1 i + \alpha_2 j + \alpha_3 k$, the imaginary part of q; $\bar{q} = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k$, the conjugate of q; $|q| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2} = (q\bar{q})^{\frac{1}{2}} = (\bar{q}q)^{\frac{1}{2}}$, the norm of q. Moreover, the set of all $q \in \mathbb{H}$ with $Re\ q = 0$ is denoted by $\mathbb{P}$, and $q \in \mathbb{H}$ is called a unit quaternion if $|q| = 1$.

Two quaternions x and y are said to be similar, denoted by $x \sim y$, if there exists a nonzero quaternion $q \in \mathbb{H}$ such that $x = q^{-1}yq$. It is known, e.g., see [4, Theorem 2.2], that $x \in \mathbb{H}$ is similar to $y \in \mathbb{H}$ if and only if $Re\ x = Re\ y$ and $|Im\ x| = |Im\ y|$. Obviously, $\sim$ is an equivalence relation on the quaternions. The equivalence class containing x is denoted by $[x]$.

Let $\mathbb{H}^n$ be the collection of all n-column vectors with entries in $\mathbb{H}$, and $M_{m\times n}(\mathbb{H})$ (for the case $m = n$, $M_n(\mathbb{H})$) be the set of all $m \times n$ quaternion matrices. For any $m \times n$ quaternion matrix $A = (a_{ij}) \in M_{m\times n}(\mathbb{H})$, we define $\bar{A} = (\bar{a}_{ij}) \in M_{m\times n}(\mathbb{H})$, the conjugate of A; $A^T = (a_{ji}) \in M_{n\times m}(\mathbb{H})$, the transpose of A; $A^* = (\bar{A})^T \in M_{n\times m}(\mathbb{H})$, the conjugate transpose of A.

Let $A \in M_n(\mathbb{H})$. The matrix A is said to be normal if $A^*A = AA^*$; Hermitian if $A^* = A$; skew-Hermitian if $A^* = -A$; and unitary if $A^*A = I_n$, where I_n is the $n \times n$ identity matrix. A quaternion λ is called a (right) eigenvalue of A if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$. The set of all right eigenvalues of A is denoted by $\sigma_r(A)$; i.e., the right spectrum of A. Also, the right spectral radius of A is defined as $\rho_r(A) = max\{|z| \ : \ z \in \sigma_r(A)\}$. If λ is an eigenvalue of A, then any element in $[\lambda]$ is also an eigenvalue of A. Moreover, it is known, e.g., see [4, Theorem 5.4], that A has, counting multiplicities, exactly n (right) eigenvalues which are complex numbers with nonnegative imaginary parts. These eigenvalues are called the standard right eigenvalues of A.

Throughout the paper, we assume that k and n are positive integers, and $k \leqslant n$. A matrix $X \in M_{n\times k}(\mathbb{H})$ is called an isometry if $X^*X = I_k$, and the set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n\times k}$. For the case $k = n$, $\mathcal{X}_{n\times n}$ is denoted by $\mathcal{U}_n$ which is the set of all $n \times n$ quaternionic unitary matrices. For $A \in M_n(\mathbb{H})$, the notion of k-numerical range of A which was first introduced in [1], is defined and denoted by

$$W^k(A) = \{\frac{1}{k} tr(X^*AX) : X \in \mathcal{X}_{n\times k}\}. \tag{1}$$

The sets $W^k(A)$, where $k \in \{1, 2, \ldots, n\}$, are generally called the higher numerical ranges of A. Let A have the standard right eigenvalues $\lambda_1, \ldots, \lambda_n$, counting multiplicities. The right $k-$spectrum of A is defined and denoted by

$$\sigma_r^k(A) = \{\frac{1}{k}\sum_{j=1}^{k} \alpha_{i_j} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n,\ \alpha_{i_j} \in [\lambda_{i_j}]\}.$$

Obviously, if $\alpha \in \sigma_r^k(A)$, then $[\alpha] \subseteq \sigma_r^k(A)$. Moreover, $\sigma_r^k(A) \subseteq W^k(A)$, $\sigma_r^1(A) = \sigma_r(A)$, and

$$W^1(A) = W(A) := \{x^*Ax : x \in \mathbb{H}^n, x^*x = 1\}$$

is the standard quaternionic numerical range of A, which was first studied in 1951 by Kippenhahn [2]. The numerical radius of A is also defined as $r(A) = max\{|z| \ : \ z \in W(A)\}$. Now, in the following theorem, we list some other properties of the $k-$numerical range of quaternion matrices which can be found in [1].

Theorem 1.1 Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:
(a) $W^k(\alpha I + \beta A) = \alpha + \beta W^k(A)$, where $\alpha, \beta \in \mathbb{R}$;
(b) $W^k(A + B) \subseteq W^k(A) + W^k(B)$, where $B \in M_n(\mathbb{H})$;
(c) $W^k(U^*AU) = W^k(A)$, where $U \in \mathcal{U}_n$;
(d) $\bar{\alpha}W^k(A)\alpha = W^k(A)$, where $\alpha \in \mathbb{H}$ is such that $|\alpha| = 1$;
(e) $W^k(A^*) = W^k(A)$;
(f) $W^{k+1}(A) \subseteq conv(W^k(A))$;
(g) $W^k(A) \subseteq \mathbb{R}$ if and only if A is Hermitian;
(h) $W^n(A) = \{\frac{1}{n}trA\}$ if and only if A is Hermitian.

In this paper, we are going to study some properties of the $k-$numerical ranges and radii of quaternionic matrices. To this end, in the next section, we state some other properties of the $k-$numerical range of quaternion matrices. We also introduce and study, as in the complex case, the notions of right $k-$spectral, $k-$numerical radius and the $k-$norm of quaternion matrices. Moreover, we establish some relations among them.

2. Main results

We begin this section by a result about quaternion numbers which is important to study some properties of the $k-$numerical range of quaternion matrices.

Theorem 2.1 Let $S \subseteq \mathbb{H}$ be such that $\lambda \in S$ implies that $[\lambda] \subseteq S$. Then

$$conv(\mathbb{C} \bigcap S) = \mathbb{C} \bigcap conv(S).$$

Proof. It is clear that $conv(\mathbb{C} \bigcap S) \subseteq \mathbb{C} \bigcap conv(S)$.
Conversely, let $\lambda = \sum_{l=1}^{m} \theta_l(a_l + b_l i + c_l j + d_l k) \in \mathbb{C} \bigcap conv(S)$, where $\theta_l \geqslant 0$, $\sum_{l=1}^{m} \theta_l = 1$, and $a_l + b_l i + c_l j + d_l k \in S$ for all $l = 1, \ldots, m$. Thus, we have

$$\lambda = \sum_{l=1}^{m} \theta_l(a_l + b_l i), \ and \ \sum_{l=1}^{m} \theta_l(c_l j + d_l k) = 0.$$

Since $a_l \pm i\sqrt{b_l^2 + c_l^2 + d_l^2} \in [a_l + b_l i + c_l j + d_l k]$, by our assumption, we have $a_l \pm i\sqrt{b_l^2 + c_l^2 + d_l^2} \in \mathbb{C} \bigcap S$ for all $l = 1, \ldots, m$. So, for every $l \in \{1, \ldots, m\}$, we have $a_l + b_l i = t(a_l + i\sqrt{b_l^2 + c_l^2 + d_l^2}) + (1 - t)(a_l - i\sqrt{b_l^2 + c_l^2 + d_l^2}) \in conv(\mathbb{C} \bigcap S)$, where $t = \frac{b_l + \sqrt{b_l^2 + c_l^2 + d_l^2}}{2\sqrt{b_l^2 + c_l^2 + d_l^2}}$ for the case $\sqrt{b_l^2 + c_l^2 + d_l^2} \neq 0$, and for the case $b_l = c_l = d_l = 0$, $t \in [0, 1]$ is arbitrary. Therefore, $\lambda \in conv(\mathbb{C} \bigcap S)$. Hence, $\mathbb{C} \bigcap conv(S) \subseteq conv(\mathbb{C} \bigcap S)$. This completes the proof. ■

By Theorem 2.1, we have the following results.

Corollary 2.2 Let $A \in M_n(\mathbb{H})$.Then

$$conv(\mathbb{C} \bigcap W^k(A)) = \mathbb{C} \bigcap conv(W^k(A)).$$

Corollary 2.3 (see also [1, Theorem 2.4(b)]); Let $A \in M_n(\mathbb{H})$.Then

$$conv(\mathbb{C} \bigcap \sigma_r^k(A)) = \mathbb{C} \bigcap conv(\sigma_r^k(A)).$$

Now, we introduce the notions of right $k-$spectral, $k-$numerical radius and the $k-$norm of quaternion matrices. To access more information about the similar results in the complex case, see [3].

Definition 2.4 Let $A \in M_n(\mathbb{H})$. The right $k-$spectral radius, the $k-$numerical radius, and the $k-$norm of A are defined and denoted, respectively, by

$$\rho_r^{(k)}(A) = max\{|z| \ : \ z \in \sigma_r^k(A)\},$$

$$r^{(k)}(A) = max\{|z| \ : \ z \in W^k(A)\}, \ and$$

$$\|A\|_{(k)} = \frac{1}{k} max\{|tr(X^*AY)| \ : \ X, \ Y \in \mathcal{X}_{n\times k}\}.$$

It is clear that $\rho_r^{(1)}(A) = \rho_r(A)$ and $r^{(1)}(A) = r(A)$. So, the notions of right k-spectral radius and k-numerical radius are generalizations of the calssical spectral radius and numerical radius, respectively. In the following theorem, we state some basic properties of $r^{(k)}(.)$.

Theorem 2.5 Let $A, B \in M_n(\mathbb{H})$ and $c \in \mathbb{R}$. Then the following assertions are true:
(a) $r^{(k)}(A) \geqslant 0$;
(b) $r^{(k)}(cA) = |c| r^{(k)}(A)$;
(c) $r^{(k)}(U^*AU) = r^{(k)}(A)$, where $U \in \mathcal{U}_n$;
(d) $r^{(k)}(A) = r^{(k)}(A^*)$;
(e) Let $k < n$. Then $r^{(k)}(A) = 0$ if and only if $A = 0$. For the case $k = n$, $r^{(n)}(A) = 0$ if and only if A is Hermitian and $trA = 0$;
(f) $r^{(k)}(A + B) \leqslant r^{(k)}(A) + r^{(k)}(B)$;
(g) $r^{(n)}(A) \leqslant r^{(n-1)}(A) \leqslant \cdots \leqslant r^{(1)}(A) = r(A)$.

Proof. The part (a) follows from Definition 2.4. The parts (b), (c), (d) and (f) follow easily from Theorem 1.1.
To prove (e), at first, we assume that $r^{(k)}(A) = 0$ and $k < n$. We will show that $A = 0$. Since $r^{(k)}(A) = 0$, for any $z \in W^k(A)$, $|z| = 0$. Therefore, $W^k(A) = \{0\}$, and hence, by Theorem 1.1(g), A is Hermitian. Now, since $k < n$, by a simple calculation we see that $A = 0$. The converse is trivial. For the case $k = n$, let A have the standard right eigenvalues $\lambda_1, \ldots, \lambda_n$, counting multiplicities, and $r^{(n)}(A) = 0$. Then $W^n(A) = \{0\}$. Since $\frac{1}{n} trA \in \sigma^n(A)$, by [1, Theorem 2.5(e)], $W^n(A) = \{0\} = \{\frac{1}{n} trA\}$. So, $trA = 0$ and also by Theorem 1.1(h), A is Hermitian. The converse is trivial.
To prove (g), let $1 < k \leqslant n$ be given. Then by Theorem 1.1(f), we have $W^k(A) \subseteq conv(W^{k-1}(A))$. Now, let $r^{(k)}(A) = |\mu|$ for some $\mu \in W^k(A)$. Hence, $\mu \in conv(W^{k-1}(A))$.

Then there are nonnegative real numbers $t_1, \ldots, t_n \in \mathbb{R}$ summing to 1, and $\alpha_1, \ldots, \alpha_n \in W^{k-1}(A)$ such that $\mu = \sum_{i=1}^{n} t_i \alpha_i$. Therefore,

$$r^{(k)}(A) = |\mu| \leqslant \sum_{i=1}^{n} t_i |\alpha_i| \leqslant \sum_{i=1}^{n} t_i r^{(k-1)}(A) = r^{(k-1)}(A).$$

This completes the proof. ■

Using Definition 2.4 and this fact that $\sigma_r^k(A) \subseteq W^k(A)$, we have the following result which states the relation between $\rho_r^{(k)}(.)$, $r^{(k)}(.)$ and $\|.\|_{(k)}$.

Proposition 2.6 Let $A \in M_n(\mathbb{H})$. Then

$$\rho_r^{(k)}(A) \leqslant r^{(k)}(A) \leqslant \|A\|_{(k)}.$$

The following example shows that in Proposition 2.6, the equality $\rho_r^{(k)}(A) = r^{(k)}(A)$ does not hold in general.

***Example* 2.7** Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{H})$. Then A is a matrix with eigenvalues $1, 0, 0$. Therefore, $\rho_r^{(2)}(A) = \frac{1}{2}$. By a simple calculation, we have $r^{(2)}(A) = 1$. So, $\rho_r^{(2)}(A) = \frac{1}{2} < 1 = r^{(2)}(A)$.

In the following proposition, we show that the left inequality in Proposition 2.6 is sharp. It follows easily from [1, Theorem 2.13].

Proposition 2.8 Let $A \in M_n(\mathbb{H})$ be a Hermitian matrix. Then

$$\rho_r^{(k)}(A) = r^{(k)}(A).$$

In the following theorem, we state some basic properties of $\|A\|_{(k)}$.

Theorem 2.9 Let $A, B \in M_n(\mathbb{H})$ and $c \in \mathbb{R}$. Then the following assertions are true:
(a) $\|A\|_{(k)} \geqslant 0$;
(b) $\|cA\|_{(k)} = |c|\|A\|_{(k)}$;
(c) Let $k < n$. Then $\|A\|_{(k)} = 0$ if and only if $A = 0$;
(d) $\|A + B\|_{(k)} \leqslant \|A\|_{(k)} + \|B\|_{(k)}$;
(e) $\|A\|_{(n)} \leqslant \|A\|_{(n-1)} \leqslant \ldots \leqslant \|A\|_{(1)}$.

Proof. The assertions in (a), (b), and (d) follow easily from Definition 2.4.
To prove (c), at first, we assume that $\|A\|_{(k)} = 0$ and $k < n$. Then by Theorem 2.5(e) and Proposition 2.6, we have $A = 0$. The converse is trivial.
For (e), let $1 < k \leqslant n$. Moreover, let $X = [x_1, \ldots, x_n], Y = [y_1, \ldots, y_n] \in \mathcal{X}_{n\times k}$ be given.

Therefore, we have

$$\begin{aligned}
\frac{1}{k}|\sum_{j=1}^{k} x_j^* A y_j| &= \frac{1}{k}|\sum_{j=1}^{k} \frac{1}{k-1} \sum_{\substack{i=1\\ i\neq j}}^{k} x_i^* A y_i| \\
&\leqslant \frac{1}{k}\sum_{j=1}^{k} \frac{1}{k-1}|\sum_{\substack{i=1\\ i\neq j}}^{k} x_i^* A y_i| \\
&\leqslant \frac{1}{k}\sum_{j=1}^{k} \|A\|_{(k-1)} \\
&= \|A\|_{(k-1)}.
\end{aligned}$$

So, $\|A\|_{(k)} \leqslant \|A|_{(k-1)}$. This completes the proof. ■

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