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Bernoulli collocation method with residual correction for solving integral-algebraic equations

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Abstract. The principal aim of this paper is to serve the numerical solution of an integral-algebraic equation (IAE) by using the Bernoulli polynomials and the residual correction method. After implementation of our scheme, the main problem would be transformed into a system of algebraic equations such that its solutions are the unknown Bernoulli coefficients. This method gives an analytic solution when the exact solutions are polynomials. Also, an error analysis based on the use of the Bernoulli polynomials is provided under several mild conditions. Several examples are included to illustrate the efficiency and accuracy of the proposed technique and also the results are compared with the different methods.

Keywords: Integral algebraic equations, Approximate solutions, Bernoulli collocation method, Error analysis.

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1. Introduction

Many physical events, such as biological applications in population dynamics and genetics where impulses arise naturally or are caused by control, can be modeled by the differential equation, integral equation, integro-differential equation or a system of these equations. Systems of Volterra integral equations with identically singular matrices in the principal part are called integral algebraic equations. Such equations and systems frequently arise in many physical and applied problems especially in the fields of dynamic processes in

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chemical reactors [13], identification of memory kernels in heat conduction [27] viscoelastic materials [11, 29], the two-dimensional biharmonic equation in a semi-infinite strip [6, 10], evolution of a chemical reaction within a small cell [12] and Kirchhoff's laws. The theory of IAEs appeared from early attempts by Gear in the 1990 that determined the difficulties of these equations. He introduced the "index reduction procedure" for IAEs system in [7] similar to that in [8] for differential algebraic equations in which if the process is terminated, then the index is determined. This means that under suitable conditions, there is a solution for the resulting regular system of integral equations.
In this article, we study numerical method for solving the system of Volterra integral equations of the first and second kind. More precisely, we consider the integral algebraic equation

$$\mathbf{V}(t)\mathbf{G}(t) = \mathbf{F}(t) + \int_0^t \mathbf{K}(t,s)\mathbf{G}(s)ds, \qquad 0 \leqslant t \leqslant 1, \tag{1}$$

where

$$\begin{aligned} \mathbf{V}(t) &= [v_{ij}(t)], \qquad i,\ j = 1, 2, \ldots, m, \\ \mathbf{G}(t) &= [g_1(t),\ g_2(t),\ \ldots,\ g_m(t)]^T, \\ \mathbf{F}(t) &= [f_1(t),\ f_2(t),\ \ldots,\ f_m(t)]^T, \\ \mathbf{K}(t,s) &= [k_{ij}(t,s)], \qquad i,\ j = 1, 2, \ldots, m, \end{aligned}$$

which $\mathbf{V}(t)$, $\mathbf{F}(\mathbf{t})$ and $\mathbf{K}(t,s)$ are known functions and $\mathbf{G}(t)$ is the solution that should be determined. If $\det \mathbf{V}(t) = 0$, then this system is called as Volterra integral-algebraic equation. Under the condition $\det \mathbf{V}(t) = 0$, the system can have several solutions or no solution at all. Several authors have investigated the existence, uniqueness and numerical analysis of IAEs systems.

Theorem 1.1 ([1]) Assume that the system (1) with $\det \mathbf{V}(t) = 0$ satisfies the following conditions:

(i) $\mathrm{Rank}\mathbf{V}(\mathrm{t}) = \deg(\det[\lambda\mathbf{V}(\mathrm{t}) + \mathbf{K}(\mathrm{t},\mathrm{t})]) = \mathrm{k} =$constant $\forall t \in [0,1]$, where λ is a scalar and $deg(p(\cdot))$ is degree of the polynomial $p(\cdot)$.
(ii) $\mathrm{Rank}\mathbf{V}(0) = \mathrm{Rank}[\mathbf{V}(0)|\mathrm{G}(0)]$.
(iii) $\mathbf{V}(t), \mathbf{G}(t) \in C^1_{[0,1]}$ and $\mathbf{K}(t,s) \in C^1_{\Delta}$, where $\Delta = \{0 \leqslant s \leqslant t \leqslant 1\}$.

Then the system (1) has a unique continuous solution.

Bulatov et al. [5], gave the existence and uniqueness results of solution for IAE systems with convolutions kernels and defined the index in analogy to Gear's approach. Kauthen [14], applied the polynomial spline collocation method for a semi-explicit IAEs with index-1 and established global convergence as well as local superconvergence. Furthermore, Brunner [3] defined the index-1 tractable for a semi-explicit form of IAEs and investigated the existence of a unique solution for these types of systems.
Several researchers have adopted different techniques for solving system of integral algebraic equations. Rabbani et al. have used a modified Taylor series expansion method to reduce the system of integral equations to a linear system of ordinary differential equations, which are solved by constructing appropriate boundary conditions [19]. Tahmasbi and Fard have presented a derivative-free method based on the power series method [24].

Mirzaee have used rationalized Haar functions with their product operational matrix [15] and Fibonacci matrix method [16, 17]. Biazar and Eslami have proposed a modified homotopy perturbation method using a simple modification [2].
In the present paper, we develop Bernoulli collocation method in combination with residual correction method for solving IAEs. The outline of this paper is organized as follows. Section 2 briefly describes the definition and properties of the Bernoulli functions. In Section 3, we apply Bernoulli collocation method for solving volterra integral algebraic equation (1). Section 4 is concerned with the derivation of the residual correction method. A convergence analysis is presented in Section 5. In Section 6, some numerical experiments are reported which confirm the theoretical results of the paper. Section 7 concludes the paper.

2. The properties of Bernoulli polynomials

In this section, we recall some properties of the Bernoulli polynomials which will be of fundamental importance in the sequel. Bernoulli polynomials [18] and also Bernoulli functions [20], have received considerable attention in numerical analysis. They appear in the integral representation of the differentiable periodic functions, since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-Maclaurin quadrature rule [20]. The classical Bernoulli polynomials $B_n(x)$ are usually defined by means of the exponential generating functions

$$\frac{\omega e^{t\omega}}{e^{\omega}-1}=\sum_{k=0}^{\infty} B_k(t)\frac{\omega^k}{k!}, \qquad (|\omega|<2\pi).$$

The following familiar expansion

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(t)=(n+1)t^n,$$

is the most primary property of the Bernoulli polynomials. Also, the sequence of Bernoulli polynomials $(B_n)_{n\in N}$ is uniquely defined by the conditions

(1) $B_0(t)=1$.
(2) $\forall\ k\in N, \quad B_k(t+1)-B_k(t)=kt^{k-1}$.
(3) $\forall\ k\in N, \quad \int_0^1 B_k(t)dt=0$.

For instance, it is straightforward to see that

$$\begin{aligned}
B_1(t)&=t-\frac{1}{2},\\
B_2(t)&=t^2-t+\frac{1}{6},\\
B_3(t)&=t^3-\frac{3}{2}t^2+\frac{1}{2},\\
&\vdots
\end{aligned}$$

Moreover, for every nonnegative integer n we have

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} t^k,$$

where $(b_n)_{n\in N}$ is the sequence of Bernoulli numbers that is defined by

$$b_n = B_n(0), \qquad \text{for } n \geqslant 0.$$

Also, for every positive integer n we have

$$b_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k.$$

According to the discussions in [26], the Bernoulli polynomials form a complete basis over the interval $[0, 1]$. We introduce the Bernoulli vector $B(t)$ in the following form

$$B(t) = [B_0(t), B_1(t), \ldots, B_n(t)]^T. \tag{2}$$

Now, suppose that $H = L^2[0, 1]$ and $\{B_0(t), B_1(t), \ldots, B_n(t)\} \subset H$ is the set of the Bernoulli polynomials and

$$Y = span\{B_0(t), B_1(t), \ldots, B_n(t)\},$$

and u is an arbitrary element in H. Since Y is a finite dimensional vector space, u has the unique best approximation belongs to Y such as $\tilde{u}$, that is,

$$\forall y \in Y, \qquad \|u - \tilde{u}\| \leqslant \|u - y\|.$$

Since $\tilde{u} \in Y$, there exist the unique coefficients $u_0,\ u_1, \ldots, u_n$ such that

$$u \simeq \tilde{u} = \sum_{i=0}^{n} u_i B_i(t) = U^T B(t),$$

where $U = [u_0, u_1, \ldots, u_n]^T$.

Moreover, we can write $B(t)$ in the matrix form as

$$B(t) = DX(t), \tag{3}$$

that D is an $(n + 1) \times (n + 1)$ lower triangular matrix with rows

$$D_{i+1} = \left[\binom{i}{i} b_i, \binom{i}{i-1} b_{i-1}, \ldots, \binom{i}{0} b_0, \overbrace{0,\ 0, \ldots,\ 0}^{n-i\ times} \right], \qquad i = 0, 1, \ldots, n,$$

and $X(t) = [1, t, t^2, \ldots, t^n]^T$.

3. Method of the solution

We consider the system of linear Volterra integral algebraic equations with variable coefficients (1). The system also can be written as

$$\sum_{j=1}^{m} v_{ij}(t)g_j(t) = f_i(t) + \sum_{j=1}^{m} \int_0^t k_{ij}(t,s)g_j(s)ds, \qquad 0 \leqslant t \leqslant 1, \tag{4}$$

where $i = 1, 2, \ldots, m$. Bernoulli expansion for the functions g_j, can be written as

$$g_j(t) \simeq G_j^T B(t) = B^T(t)G_j, \tag{5}$$

where $G_j = [g_{j0}, g_{j1}, \ldots, g_{jn}]^T$. Now, the ith equation, from the system (4), can be expanded in terms of Bernoulli expansion as follows

$$\sum_{j=1}^{m} v_{ij}(t)B^T(t)G_j = f_i(t) + \sum_{j=1}^{m} \int_0^t k_{ij}(t,s)B^T(s)G_j ds, \qquad 0 \leqslant t \leqslant 1. \tag{6}$$

By substituting the collocation points $t_l = \frac{2l+1}{2(n+1)}$, $l = 0, 1, \ldots, n$ into (4), we have

$$\sum_{j=1}^{m} v_{ij}(t_l)B^T(t_l)G_j = f_i(t_l) + \sum_{j=1}^{m} \int_0^{t_l} k_{ij}(t_l,s)B^T(s)G_j ds,$$

So, the associated matrix-vector form of (1) has the following form

$$\mathcal{V}\mathcal{G} = \mathcal{F} + \mathcal{K}\mathcal{G}, \tag{7}$$

where

$$\mathcal{V} = \begin{pmatrix} v_{11}(t_0)B^T(t_0) & v_{12}(t_0)B^T(t_0) & \cdots & v_{1m}(t_0)B^T(t_0) \\ v_{11}(t_1)B^T(t_1) & v_{12}(t_1)B^T(t_1) & \cdots & v_{1m}(t_1)B^T(t_1) \\ \vdots & \vdots & & \vdots \\ v_{11}(t_n)B^T(t_n) & v_{12}(t_n)B^T(t_n) & \cdots & v_{1m}(t_n)B^T(t_n) \\ \vdots & \vdots & & \vdots \\ v_{m1}(t_0)B^T(t_0) & v_{m2}(t_0)B^T(t_0) & \cdots & v_{mm}(t_0)B^T(t_0) \\ \vdots & \vdots & & \vdots \\ v_{m1}(t_n)B^T(t_n) & v_{m2}(t_n)B^T(t_n) & \cdots & v_{mm}(t_n)B^T(t_n) \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_m \end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix} \int_0^{t_0} k_{11}(t_0,s)B^T(s)ds & \int_0^{t_0} k_{12}(t_0,s)B^T(s)ds & \cdots & \int_0^{t_0} k_{1m}(t_0,s)B^T(s)ds \\ \int_0^{t_0} k_{11}(t_1,s)B^T(s)ds & \int_0^{t_1} k_{12}(t_1,s)B^T(s)ds & \cdots & \int_0^{t_1} k_{1m}(t_1,s)B^T(s)ds \\ \vdots & \vdots & & \vdots \\ \int_0^{t_n} k_{11}(t_0,s)B^T(s)ds & \int_0^{t_n} k_{12}(t_n,s)B^T(s)ds & \cdots & \int_0^{t_n} k_{1m}(t_n,s)B^T(s)ds \\ \vdots & \vdots & & \vdots \\ \int_0^{t_0} k_{m1}(t_0,s)B^T(s)ds & \int_0^{t_0} k_{m2}(t_0,s)B^T(s)ds & \cdots & \int_0^{t_0} k_{mm}(t_0,s)B^T(s)ds \\ \vdots & \vdots & & \vdots \\ \int_0^{t_n} k_{m1}(t_0,s)B^T(s)ds & \int_0^{t_n} k_{m2}(t_n,s)B^T(s)ds & \cdots & \int_0^{t_n} k_{mm}(t_n,s)B^T(s)ds \end{pmatrix},$$

and $\mathcal{F} = [f_1(t_0),\ f_1(t_1),\ \cdots,\ f_1(t_n),\ \cdots,\ f_m(t_0),\ \cdots,\ f_m(t_n)]^T$. Hence, the fundamental matrix equation corresponding to Eq. (1) can be written in the form

$$(\mathcal{V} - \mathcal{K})\mathcal{G} = \mathcal{F},$$

or

$$\mathcal{A}\mathcal{G} = \mathcal{F}. \tag{8}$$

After solving this algebraic system, we obtain the approximated solutions of (1).

4. Residual correction and error estimation

In this section, we will give an error estimation for the Bernoulli collocation method and the residual correction of the Bernoulli approximate solution. For our purpose, we can define the vector of residual functions of the Bernoulli collocation method as

$$R_n(t) = L[\mathbf{G}_n(t)] - \mathbf{F}(t), \tag{9}$$

where $\mathbf{G}_n(t)$, which is the Bernoulli polynomial solution defined by (5), is the approximate solution of the problem (1). Hence $\mathbf{G}_n(t)$ satisfies the problem

$$L[\mathbf{G}_n(t)] = \mathbf{V}(t)\mathbf{G}_n(t) - \int_0^t \mathbf{K}(t,s)\mathbf{G}_n(s)ds = \mathbf{F}(t) + R_n(t), \tag{10}$$

Also, the error function $e_n(t)$ can be defined as

$$e_n(t) = \mathbf{G}(t) - \mathbf{G}_n(t), \tag{11}$$

where $\mathbf{G}(t)$ is the exact solution of the problem (1). Substituting (11) into (1) and using (9) and (10), we have the error integral algebraic equation

$$L[e_n(t)] = L[\mathbf{G}(t)] - L[\mathbf{G}_n(t)] = -R_n(t),$$

or

$$\mathbf{V}(t)e_n(t) - \int_0^t \mathbf{K}(t,s)e_n(s)ds = -R_n(t).$$

Solving above problem in the same way as Section 3, we get the approximation $e_{nm}(t)$ to $e_n(t)$, $(m \geqslant n)$ which is the vector of error functions based on the residual function $R_n(t)$.
Consequently, by means of $\mathbf{G}_n(t)$ and $e_{nm}(t)$, $(m \geqslant n)$, we obtain the corrected Bernoulli polynomial solution $\mathbf{G}_{nm}(t) = \mathbf{G}_n(t) + e_{nm}(t)$, we construct the Bernoulli error function $e_n(t) = \mathbf{G}(t) - \mathbf{G}_n(t)$, the corrected Bernoulli error function $E_{nm}(t) = e_n(t) - e_{nm}(t) = \mathbf{G}(t) - \mathbf{G}_{nm}(t)$ and the estimated error function $e_{nm}(t)$.

5. Convergence analysis

In this section, we will try to provide an error analysis which theoretically justifies the convergence of the proposed method.

Corollary 5.1 ([25]) Assume that $u \in L^2[0,1]$ is an arbitrary function and also is approximated by the truncated Bernoulli series $\sum_{i=0}^{\infty} u_i B_i(t)$, then the coefficients u_i for all $i = 0, 1, \ldots$ can be calculated from the following relation

$$u_i = \frac{1}{i!}\int_0^1 u^{(i)}(s)ds.$$

In practice one can use finite terms of the above series.

Corollary 5.2 ([25]) Assume that one approximates the function u on the interval $[0,1]$ by the Bernoulli polynomials as discussed in Corollary (5.1). Then, the coefficients u_i decay as follows

$$u_i \leqslant \frac{U_i}{i!},$$

where U_i denotes the maximum of $u^{(i)}$ in the interval $[0,1]$.

The above corollary implies that the Bernoulli coefficients are decayed rapidly as the increasing of i.

Theorem 5.3 ([26]) Suppose that $u(t)$ is an enough smooth function in the interval $[0,1]$ and is approximated by the Bernoulli polynomials as done in Corollary 1. With more details, assume that $P_n[u](t)$ is the approximate polynomial of $u(t)$ in terms of the

Bernoulli polynomials and $r_n[u](t)$ is the remainder term. Then, the associated formulas are stated as follows

$$u(t) = P_n[u](t) + r_n[u](t), \qquad t \in [0,1],$$
$$P_n[u](t) = \int_0^1 u(t)dt + \sum_{j=1}^{n} \frac{B_j(t)}{j!}\left(u^{(j-1)}(1) - u^{(j-1)}(0)\right),$$
$$r_n[u](t) = -\frac{1}{n!}\int_0^1 B_n^*(t-s)u^{(n)}(s)ds,$$

where $B_n^*(t) = (t - [t])$ and $[t]$ denotes the largest integer not greater than t.

Trivially, the algebraic degree of exactness of the operator $P_n[\,\cdot\,]$ is n.

Theorem 5.4 ([26]) Suppose $g(t) \in C^\infty[0,1]$ (with bounded derivatives) and $g_n(t)$ is the approximated polynomial using Bernoulli polynomials. Then the error bound would be obtained as follows

$$\|error(g(t))\|_\infty \leqslant \rho\mathcal{M}(2\pi)^{-n}, \qquad t \in [0,1],$$

where $\mathcal{M}$ denotes a bound for all the derivatives of function $g(t)$ (i.e., $\|g^{(i)}(t)\|_\infty$, for $i = 0, 1, \ldots$) and ρ is a positive constant.

Theorem 5.5 ([23]) Let $\|\cdot\| : C^{n\times n} \to R$ be a consistent matrix norm. For any matrix, A, of order n, if $\|A\| \leqslant 1$, then, $I - A$ is nonsingular. Moreover

$$\|x(I-A)^{-1}\| \leqslant \frac{\|x\|}{1-\|A\|}, \qquad \|I-(I-A)^{-1}\| \leqslant \frac{\|A\|}{1-\|A\|}.$$

Theorem 5.6 ([9]) If A is a nonsingular matrix and $\|\delta A\| \leqslant \frac{1}{\|A^{-1}\|}$ then $A + \delta A$ is nonsingular. Moreover, let $b \neq 0$ and let x and $\tilde{x} = x + \delta x$ be solutions of $Ax = b$ and $(A + \delta A)\tilde{x} = b$, respectively. Then

$$\|\delta x\| \leqslant \|A^{-1}\|\|\delta A\|\|\tilde{x}\|.$$

In what follows, we formulate the assumptions under which Eq. (1) will be investigated. Namely, we assume the following hypotheses.

- (H_1) Let $m = 1$ and $v(t) = v_{ij}(t)$, for clarity of presentation.
- (H_2) $y = g(t)$ and $p_n(t)$ are the exact and approximate Bernoulli series solutions with the mentioned assumptions.
- (H_3) $\bar{\mathcal{A}} = \mathcal{A} + \delta\mathcal{A}$ is the coefficient matrix of Eq. (8), where $\delta\mathcal{A}$ represents the computational error.
- (H_4) There exists positive constant α, such that $\|\delta\mathcal{A}\| \leqslant \alpha$.

Theorem 5.7 Under the tacit assumptions, $(H_1) - (H_4)$, above, if $\|\bar{\mathcal{A}}^{-1}\|_\infty\|\delta\mathcal{A}\|_\infty < 1$, then, the absolute error, $\|p_n - g\|_\infty$, of Eq. (1) has the following upper bound

$$\|p_n - g\|_\infty \leqslant \frac{\alpha\|\mathcal{G}\|_1\|\bar{\mathcal{A}}^{-1}\|_\infty}{1-\alpha\|\bar{\mathcal{A}}^{-1}\|_\infty}\left(\sum_{i=0}^{n}\binom{n}{i}|b_i|\right) + \rho\mathcal{M}(2\pi)^{-n},$$

where $\mathcal{G}$ is the solution of Eq. (8), $\mathcal{M}$ denotes a bound for all the derivatives of function $g(t)$ and ρ is a positive constant.

Proof. According to the assumptions, the basic Eq. (1) will be changed into the following equation

$$v(t)g(t) = f(t) + \int_0^t k(t,s)g(s)ds, \qquad 0 \leqslant t \leqslant 1.$$

Since the exact solution $y = g(x)$ is continuous on $[0,1]$, by using Theorem (5.4), we get

$$\begin{aligned} \|p_n - y\|_\infty &= \|p_n - p_n g + p_n g - y\|_\infty \\ &\leqslant \|p_n - p_n g\|_\infty + \|p_n g - y\|_\infty \\ &\leqslant \|p_n - p_n g\|_\infty + \rho\mathcal{M}(2\pi)^{-n}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|p_n - p_n g\|_\infty &= \|\sum_{i=0}^n \tilde{g}_i B_i - \sum_{i=0}^n g_i B_i\|_\infty \\ &\leqslant \|\sum_{i=0}^n (\tilde{g}_i - g_i) B_i\|_\infty \\ &\leqslant \|[\tilde{g}_0 - g_0,\ \tilde{g}_1 - g_1,\ \ldots,\ \tilde{g}_n - g_n]B\|_\infty \\ &\leqslant \|[\tilde{g}_0 - g_0,\ \tilde{g}_1 - g_1,\ \ldots, \tilde{g}_n - g_n]\|_\infty \|B\|_\infty \\ &\leqslant \|\mathcal{A}^{-1}\|_\infty \|\delta\mathcal{A}\|_\infty \|\mathcal{G}\|_1 \|D\|_\infty \|X(1)\|_\infty \\ &= \|(\bar{\mathcal{A}} - \delta\mathcal{A})^{-1}\|_\infty \|\delta\mathcal{A}\|_1 \|\mathcal{G}\|_\infty \|D\|_\infty. \end{aligned}$$

Using Theorems (5.5) and (5.6), we conclude that

$$\begin{aligned} &\leqslant \|\bar{\mathcal{A}}^{-1}\|_\infty \|(I - \bar{\mathcal{A}}^{-1}\delta\mathcal{A})^{-1}\|_\infty \|\delta\mathcal{A}\|_\infty \|\mathcal{G}\|_1 \|D\|_\infty \\ &\leqslant \frac{\|\bar{\mathcal{A}}^{-1}\|_\infty \|\delta\mathcal{A}\|_\infty \|\mathcal{G}\|_1}{1 - \|\bar{\mathcal{A}}^{-1}\delta\mathcal{A}\|_\infty} \|D\|_\infty \\ &\leqslant \frac{\|\bar{\mathcal{A}}^{-1}\|_\infty \|\delta\mathcal{A}\|_\infty \|\mathcal{G}\|_1}{1 - \|\bar{\mathcal{A}}^{-1}\|_\infty \|\delta\mathcal{A}\|_\infty} \|D\|_\infty \\ &\leqslant \frac{\alpha\|\bar{\mathcal{A}}^{-1}\|_\infty \|\mathcal{G}\|_1}{1 - \alpha\|\bar{\mathcal{A}}^{-1}\|_\infty} \|D\|_\infty \\ &\leqslant \frac{\alpha\|\bar{\mathcal{A}}^{-1}\|_\infty \|\mathcal{G}\|_1}{1 - \alpha\|\bar{\mathcal{A}}^{-1}\|_\infty} \sum_{i=0}^n \binom{n}{i} |b_i|. \end{aligned}$$

■

Theorem 5.8 Let $\mathbf{G}(t)$ be the exact solution and $\mathbf{G}_n(t) = \mathcal{G}^T B(t)$ be the approximated solution of (1) where the unknown Bernoulli coefficient vector $\mathcal{G}$ is determined by solving the algebraic system of equations (8). Moreover, assume that there is positive number β where $\|\mathbf{K}(t,s)\| \leqslant \beta$ and $-\beta < \|I - \mathbf{V}(t)\| < 1 - \beta$. Then $\mathbf{G}_n(t)$ converges to $\mathbf{G}(t)$.

Proof. Assume that the terms of $\mathbf{G}(t)$ are approximated by Bernoulli polynomials as described by (5). Then the obtained solution is an approximated polynomial; $\mathbf{G}_n(t)$ and we have

$$\mathbf{V}(t)\mathbf{G}(t) - \mathbf{V}(t)\mathbf{G}_n(t) = \int_0^t \mathbf{K}(t,s)\mathbf{G}(s)ds - \int_0^t \mathbf{K}(t,s)\mathbf{G}_n(s)ds.$$

So

$$\mathbf{G}(t) - \mathbf{G}_n(t) = (I - \mathbf{V}(t))[\mathbf{G}(t) - \mathbf{G}_n(t)] + \int_0^t \mathbf{K}(t,s)\left[\mathbf{G}(s) - \mathbf{G}_n(s)\right] ds.$$

Therefore

$$\|\mathbf{G}(t) - \mathbf{G}_n(t)\| \leqslant \|I - \mathbf{V}(t)\|\|\mathbf{G}(t) - \mathbf{G}_n(t)\| + \|\mathbf{K}(t,s)\|\|\mathbf{G}(s) - \mathbf{G}_n(s)\|.$$

Hence

$$\|\mathbf{G}(t) - \mathbf{G}_n(t)\|\left(1 - \|I - \mathbf{V}(t)\| - \|\mathbf{K}(t,s)\|\right) \leqslant 0.$$

From above equation and assumption $\|\mathbf{K}(t,s)\| \leqslant \beta$ we get $-\beta < \|I - \mathbf{V}(t)\| < 1 - \beta$. So by increasing n, $G_n(t)$ converges to $\mathbf{G}(t)$ as $n \to \infty$. ■

6. Illustrative examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness of the method. In this regard, we report in tables and figures, the values of the Bernoulli error (Bern. err.), corrected Bernoulli error (Corr. Bern. err.) and the estimated error (Est. err.) at the selected points of the given interval. All of the numerical computations have been performed on computer using a program written in MATLAB.

***Example* 6.1** ([28]) Consider the following Volterra integral-algebraic equation

$$\begin{pmatrix} 2t & t \\ t & -2t \end{pmatrix}\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} 2t \\ t - \frac{5}{3}t^3 + \frac{7}{6}t^4 \end{pmatrix} + \int_0^t \begin{pmatrix} 3s & 2t+1 \\ 2(t+s) & 2s(t+s) \end{pmatrix}\begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds, \quad (12)$$

with the exact solution $g_1(t) = 1 + t,\ g_2(t) = -t$.

By approximating the solution $\mathbf{G}(t)$ by the truncated Bernoulli series with $n = 2$ and after some operations, we obtain the following matrices

$$\mathcal{V}=\begin{pmatrix} 0.3333 & -0.1111 & 0.0093 & 0.1667 & -0.0556 & 0.0046\\ 1.0000 & 0 & -0.0833 & 0.5000 & 0 & -0.0417\\ 1.6667 & 0.5556 & 0.0463 & 0.8333 & 0.2778 & 0.0231\\ 0.1667 & -0.0556 & 0.0046 & -0.3333 & 0.1111 & -0.0093\\ 0.5000 & 0 & -0.0417 & -1.0000 & 0 & 0.0833\\ 0.8333 & 0.2778 & 0.0231 & -1.6667 & -0.5556 & -0.0463 \end{pmatrix},$$

$$\mathcal{K}=\begin{pmatrix} \frac{1}{24} & -\frac{7}{432} & \frac{5}{1728} & \frac{2}{9} & -\frac{5}{54} & \frac{5}{243}\\ \frac{3}{8} & -\frac{1}{16} & -\frac{1}{64} & 1 & -\frac{1}{4} & 0\\ \frac{25}{24} & \frac{25}{432} & -\frac{25}{576} & \frac{20}{9} & -\frac{5}{27} & -\frac{10}{243}\\ \frac{1}{12} & -\frac{11}{324} & \frac{55}{7776} & \frac{5}{648} & -\frac{23}{7776} & \frac{13}{25920}\\ \frac{3}{4} & -\frac{1}{6} & -\frac{1}{96} & \frac{5}{24} & -\frac{1}{32} & -\frac{29}{2880}\\ \frac{5}{12} & -\frac{25}{324} & -\frac{425}{7776} & \frac{625}{648} & \frac{625}{7776} & -\frac{625}{15552} \end{pmatrix},$$

and

$$\mathcal{F}=[0.3333,\quad 1.0000,\quad 1.6667,\quad 0.1599,\quad 0.3646,\quad 0.4315]^T.$$

Solving this system and rounding the numbers to 15 digits, the unknown Bernoulli coefficient matrix is obtained as

$$G_1=[1.5,\quad 1,\quad 0]^T,\qquad G_2=[-0.5,\quad -1,\quad 0]^T.$$

Therefore, by substituting the Bernoulli coefficient matrix into equation (5), we obtain the approximate solution $g_1(t)=1+t,\ g_2(t)=-t$, which is the exact solution.

***Example* 6.2** ([1]) Consider the following Volterra integral-algebraic equation

$$\begin{pmatrix} 0 & 0\\ t & -2t \end{pmatrix}\begin{pmatrix} g_1(t)\\ g_2(t) \end{pmatrix}=\begin{pmatrix} -t^2\\ t-\frac{5}{3}t^3+\frac{7}{6}t^4 \end{pmatrix}+\int_0^t\begin{pmatrix} 3s & 2t+1\\ 2(t+s) & 2s(t+s) \end{pmatrix}\begin{pmatrix} g_1(s)\\ g_2(s) \end{pmatrix}ds, \qquad (13)$$

with the exact solution $g_1(t)=1+t,\ g_2(t)=-t$.

In Table 1, we compare the maximum absolute error for the present method with $n=5$ and $m=10$ and block pulse functions method [1]. Fig. 1 contains numerical comparison of errors of our solutions using the present method with $n=5$ and $m=10$.

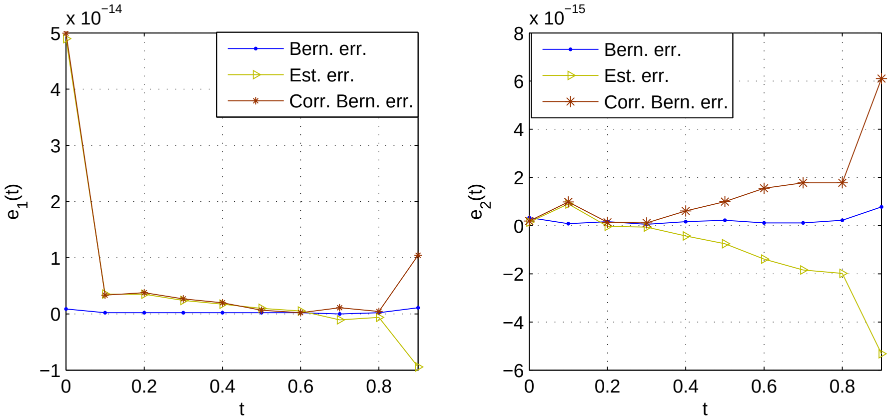

Figure 1. Comparison of the error functions for Example (2) with $n = 5$ and $m = 10$.

***Example* 6.3** ([1, 22]) Consider the following Volterra integral-algebraic equation

$$\begin{pmatrix} 1 & t \\ -2t & 1 \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} \sin t + t\cos t \\ \cos t - 2t\sin t \end{pmatrix} + \int_0^t \begin{pmatrix} t^2\cos s & -t^2\sin s \\ \sin t\cos s & -\sin t\sin s \end{pmatrix} \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds, \quad (14)$$

with the exact solution $g_1(t) = \sin t,\ g_2(t) = \cos t$.

In Table 2, we compare the maximum absolute error for the present method with $n = 5$ and $m = 10$, block pulse functions method [1] and Bessel polynomials method [22]. Fig. 2 contains numerical comparison of errors of our solutions using the present method with $n = 5$ and $m = 10$.

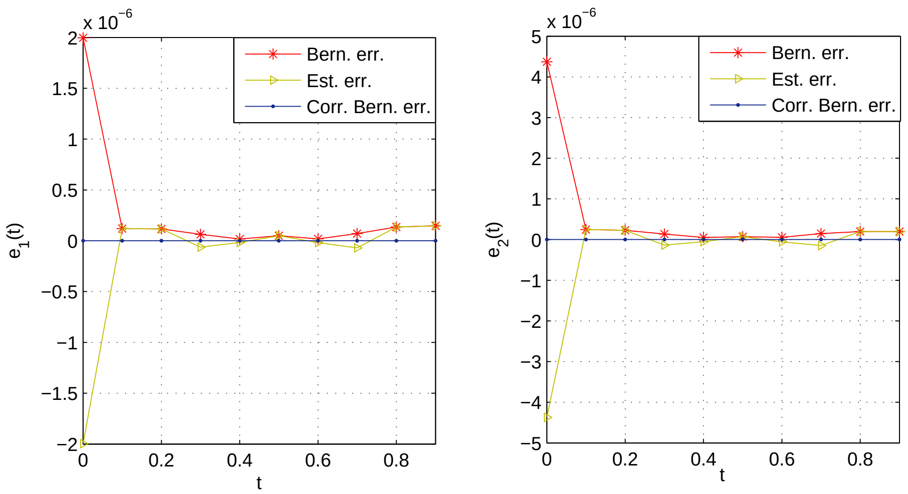

Figure 2. Comparison of the error functions for Example (3) with $n = 5$ and $m = 10$

***Example* 6.4** ([1, 22]) Consider the following Volterra integral-algebraic equation

$$\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} - t\cos 2t - t^2 - \frac{1}{2}t\sin 2t \\ te^{2t} + \frac{5}{4}\cos 2t - \frac{1}{4} - \frac{3}{4}e^t - \frac{3}{2}te^{3t} + \frac{3}{4}e^{3t} \end{pmatrix} + \int_0^t \begin{pmatrix} te^{-2s} & t \\ 3se^t & t-s \end{pmatrix} \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds, \tag{15}$$

with the exact solution $g_1(t) = e^{2t},\ g_2(t) = \cos 2t$.

In Table 3, we compare the maximum absolute error for the present method with $n = 5$ and $m = 10$, block pulse functions method [1] and Bessel polynomials method [22]. Fig. 3 contains numerical comparison of errors of our solutions using the present method with $n = 5$ and $m = 10$.

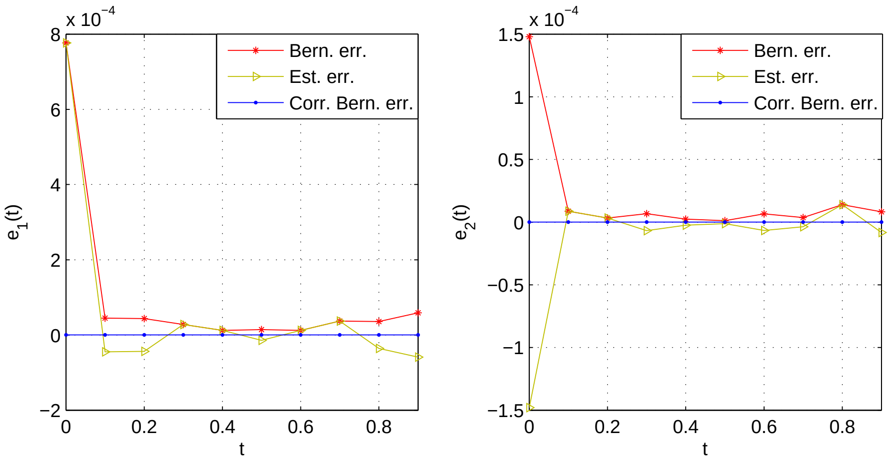

Figure 3. Comparison of the error functions for Example (4) with $n = 5$ and $m = 10$

***Example* 6.5** ([1, 4]) Consider the following Volterra integral-algebraic equation

$$\begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} te^{-t} + e^t(t+1) \\ 2te^t + 1 + e^{-t}(t^2-1) \end{pmatrix} + \int_0^t \begin{pmatrix} -e^{t-s} & 0 \\ -e^{-2s} & -e^{t+s} \end{pmatrix} \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds, \tag{16}$$

with the exact solution $g_1(t) = e^t,\ g_2(t) = e^{-t}$.

In Table 4, we compare the maximum absolute error for the present method with $n = 5$ and $m = 10$, block pulse functions method [1] and multistep Method [4]. Fig. 4 contains numerical comparison of errors of our solutions using the present method with $n = 5$ and $m = 10$.

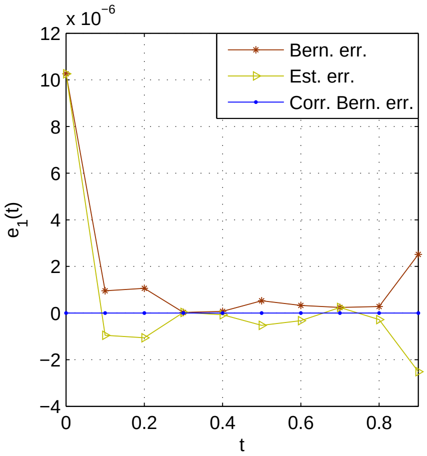

Figure 4. Comparison of the error functions for Example (5) with $n = 5$ and $m = 10$

***Example* 6.6** ([1, 21]) Consider the following Volterra integral-algebraic equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + \int_0^t \begin{pmatrix} t^3 + s + 1 & \cos 3s + 1 \\ t + s + 2 & \sin 3s + 2 \end{pmatrix} \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds, \qquad (17)$$

with

$$f_1(t) = 1 - (1 + t + t^3)\sin t - \frac{1}{3}(3 + \cos 3t)(\sin^2 \frac{3t}{2})$$

$$f_2(t) = 1 - \cos t - 2(1 + t)\sin t + \frac{1}{12}(-8 - 6t + 8\cos 3t + \sin 6t)$$

and the exact solution $g_1(t) = \cos t,\ g_2(t) = \sin 3t$.

In Table 5, we compare the maximum absolute error for the present method with $n = 5$ and $m = 10$, block pulse functions method [1] and Legendre collocation method [21]. Fig. 5 contains numerical comparison of errors of our solutions using the present method with $n = 5$ and $m = 10$.

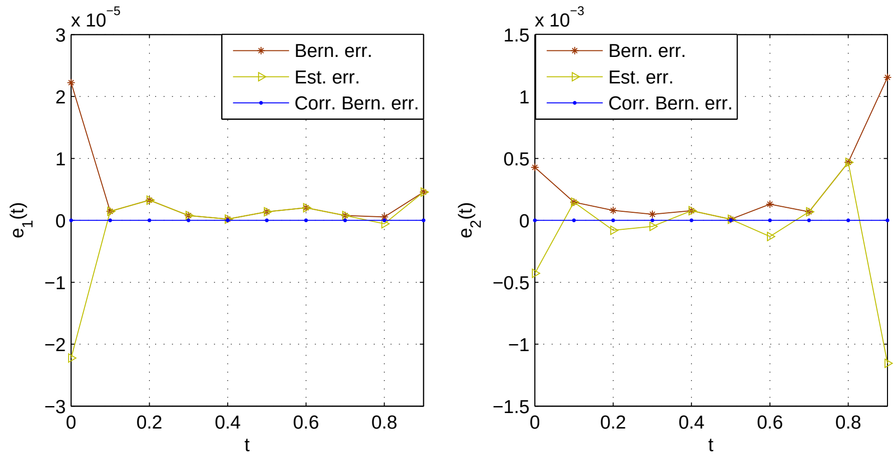

Figure 5. Comparison of the error functions for Example (6) with $n = 5$ and $m = 10$

7. Conclusion

In this study, an approximate method based on Bernoulli polynomials and collocation points has been presented to obtain the solution of integral-algebraic equations. Moreover, the convergence and error analysis of the proposed method were established. Numerical examples are included to demonstrate the validity and the applicability of the technique. The results confirm the theoretical prediction. The obtained numerical results show that this method can solve the problem effectively. One of the considerable advantages of the method is that the approximate solutions are found very easily by using the computer code written in MATLAB. Another interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial of degree n or less than n. Shorter computation time and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy are some of the advantages of our method.

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Table 1. Comparison of the maximum absolute error for Example (2)

	Present method			Method of [1]	
Error	Bern. err.	$\lvert Est.err. \rvert$	Corr. Bern. err.	n=32	n=64
$e_1(t)$	1.11e-015	4.90e-014	4.99e-014	2.50e - 002	9.32e - 003
$e_2(t)$	7.77e-016	5.32e-015	6.11e-015	2.57e - 002	1.29e - 002

Table 2. Comparison of the maximum absolute error for Example (3)

	Present method			Method of [1]		Method of [22]	
Error	Bern. err.	$\lvert Est.err. \rvert$	Corr. Bern. err.	n=32	n=64	n=2	n=5
$e_1(t)$	1.99e-006	1.99e-006	5.76e-013	1.56e - 002	7.81e - 003	2.21e-002	6.29e-003
$e_2(t)$	4.37e-006	4.37e-005	0.24e-013	7.74e - 003	3.80e - 003	1.12e-002	5.25e-005

Table 3. Comparison of the maximum absolute error for Example (4)

	Present method			Method of [1]		Method of [22]
Error	Bern. err.	$\lvert Est.err. \rvert$	Corr. Bern. err.	n=32	n=64	n=10
$e_1(t)$	7.77e-004	7.77e-004	3.13e-009	1.05e - 001	4.78e - 002	1.53e-005
$e_2(t)$	1.48e-003	1.48e-004	1.04e-009	2.55e - 002	1.29e - 002	7.92e-006

Table 4. Comparison of the maximum absolute error for Example (5)

	Present method			Method of [1]		Method of [4]	
Error	Bern. err.	$\lvert Est.err. \rvert$	Corr. Bern. err.	n=32	n=64	n=5	n=10
$e_1(t)$	1.03e-005	1.03e-005	1.68e-012	2.83e - 002	1.31e - 002	2.30e-003	1.10e-004
$e_2(t)$	6.05e-006	6.05e-006	1.48e-012	2.34e - 002	1.04e - 002	2.30e-003	1.10e-004

Table 5. Comparison of the maximum absolute error for Example (6)

	Present method			Method of [1]		Method of [21]	
Error	Bern. err.	$\lvert Est.err. \rvert$	Corr. Bern. err.	n=32	n=64	n=4	n=6
$e_1(t)$	2.22e-005	2.22e-005	3.68e-011	7.91e - 003	3.85e - 003	0.25e-004	2.25e-006
$e_2(t)$	1.20e-003	1.20e-003	2.54e-009	4.02e - 002	1.77e - 002	9.60e-004	1.53e-006

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