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Lie higher derivations on $B(X)$

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Abstract. Let X be a Banach space of $dimX > 2$ and $B(X)$ be the space of bounded linear operators on X. If $L : B(X) \to B(X)$ be a Lie higher derivation on $B(X)$, then there exists an additive higher derivation D and a linear map $\tau : B(X) \to \mathbb{F}I$ vanishing at commutators $[A, B]$ for all $A, B \in B(X)$ such that $L = D + \tau$.

Keywords: Lie higher derivations, higher derivations.

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1. Introduction

Let X be a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real number field $\mathbb{R}$ or the complex field $\mathbb{C}$. Recall that an additive map $L : B(X) \to B(X)$ is called a derivation if

$$L(AB) = L(A)B + AL(B) \qquad \forall A, B \in B(X).$$

More generally, an additive map $L : B(X) \to B(X)$ is called a Lie derivation if $L([A, B]) = [L(A), B]+[A, L(B)]$ for all $A, B \in B(X)$. In recent years, Lie derivations has attracted the attentions of many researchers(see [1, 2, 8], and references therein). On the other hand, higher derivations were introduced and studied mainly in commutative rings and later, also in non-commutative rings and some operator algebras (see, for example, [3, 11] and the references therein). We first recall the concepts of higher derivations and Lie higher derivations.

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Definition 1.1 Let $L = (L_i)_{i\in\mathbb{N}}$ ($\mathbb{N}$ denotes the set of natural numbers including 0) be a sequence of additive maps of a ring $\mathcal{A}$ such that $L_0 = \mathrm{id}_{\mathcal{A}}$. L is said to be a higher derivation if for every $n \in \mathbb{N}$ we have $L_n(AB) = \sum_{i+j=n} L_i(A)L_j(B)$ for all $A, B \in \mathcal{A}$; a Lie higher derivation if for every $n \in \mathbb{N}$ we have $L_n([A,B]) = \sum_{i+j=n}[L_i(A), L_j(B)]$ for all $A, B \in \mathcal{A}$.

It is clear that all higher derivations are Lie higher derivations. However, the converse is not true in general. Assume that $D = (D_n)_{n\in\mathbb{N}}$ is a higher derivation on a ring $\mathcal{A}$. For any $n \in \mathbb{N}$, let $L_n = D_n + h_n$, where h_n is an additive map from $\mathcal{A}$ into its center vanishing on every commutator. It is easily seen that $(L_n)_{n\in\mathbb{N}}$ is a Lie higher derivation, but not a higher derivation if $h_n \neq 0$ for some n. Then a natural question is to ask whether or not every Lie higher derivations have the above form?

In [10], Fei and Chen discussed the properties of Lie higher derivations on nest algebras. Generalized higher derivations, jordan higher derivations and higher derivations were studied by many authors (see [6, 7, 9, 11, 12]). The purpose of this paper is to show that every Lie higher derivations on $B(X)$ is proper.

2. Lie higher derivations on $B(X)$

In this section, we give a characterization of Lie higher derivations of $B(X)$. The following is our main result which generalizes the main result in [4].

Theorem 2.1 Let X be a Banach space of $dimX > 2$ and $L : B(X) \to B(X)$ be a Lie higher derivation on $B(X)$. Then there exists an additive higher derivation D and a linear map $\tau : B(X) \to \mathbb{F}I$ vanishing at commutators $[A,B]$ for all $A, B \in B(X)$ such that $L = D + \tau$.

Proof. In the following, we always assume that $L = (L_n)_{n\in\mathbb{N}}$ is an Lie higher derivation of $B(X)$. We proceed by induction on $n \in \mathbb{N}$.

If $n = 1$, then by the definition of Lie higher derivations, L_1 is a Lie derivation of $B(X)$. So by ([4, Theorem 1.1]), there exists an additive derivation D and a linear map $\tau : B(X) \to \mathbb{F}I$ vanishing at commutators $[A,B]$ for all $A, B \in B(X)$ such that $L = D+\tau$.

For the convenience, in the sequel, take $x_0 \in X, f_0 \in X^*$ satisfying $f_0(x_0) = 1$. Let $P = x_0 \otimes f_0$ and $Q = I-P$ be idempotent of $B(X)$, it is obvious that $PQ = QP = 0$. Then $B(X) = B_{11}+B_{12}+B_{21}+B_{22}$, where $B_{11} = PB(X)P$, $B_{12} = PB(X)Q$, $B_{21} = QB(X)P$, $B_{22} = QB(X)Q$.

Thus by ([4, Lemmas 2.2-2.12]) we have;

$$\mathbf{P_1}: \begin{cases} PL_1(P)P + QL_1(P)Q \in \mathbb{F}I; \\ PL_1(Q)P + QL_1(Q)Q \in \mathbb{F}I; \\ \Delta_1(PAQ + QAP) = P\Delta_1(A)Q + Q\Delta_1(A)P \quad \textit{where} \\ \Delta_1(A) = L_1(A) - (AT - TA) \quad \textit{and} \quad T = PL_1(P)Q - QL_1(P)P; \\ \Delta_1(P) \in \mathbb{F}I, \quad \Delta_1(Q) \in \mathbb{F}I; \\ \Delta_1(\mathrm{B_{ij}}) \subseteq \mathrm{B_{ij}} \textit{ for } 1 \leqslant i \neq j \leqslant 2; \\ \Delta_1(X_{ii}) \in B_{ii} + \mathbb{F}I. \end{cases}$$

Assume that $L = (L_n)$ is a Lie higher derivation of $B(X)$. We proceed by induction on $n \in N$. When $n = 1$, the conclusion is true by discussion. We now assume that $L_m(x) = D_m(x) + \tau_m(x)$ holds for all $x \in B(X)$ and for all $m < n \in N$, where τ_m :

$B(X) \to Z(B(X))$ is such that $\tau_m([x,y]) = 0$ for all $x, y \in B(X)$ and $D_m(xy) = \sum_{i+j=m} D_i(x)D_j(y)$ for all $x, y \in B(X)$.
Moreover, we have the following properties;

$$\mathbf{P_m}: \begin{cases} PL_m(P)P + QL_m(P)Q \in \mathbb{F}I; \\ PL_m(Q)P + QL_m(Q)Q \in \mathbb{F}I; \\ \Delta_m(PAQ + QAP) = P\Delta_m(A)Q + Q\Delta_m(A)P \quad where \\ \Delta_m(A) = L_m(A) - (AT - TA) \quad and \quad T = PL_m(P)Q - QL_m(P)P; \\ \Delta_m(P) \in \mathbb{F}I, \quad \Delta_m(Q) \in \mathbb{F}I; \\ .\Delta_m(\mathrm{B_{ij}}) \subseteq \mathrm{B_{ij}} for \ 1 \leqslant i \neq j \leqslant 2; \\ \Delta_m(X_{ii}) \in B_{ii} + \mathbb{F}I. \end{cases}$$

Our aim is to show that L_n also satisfies the similar properties and
that $L_n(x) = D_n(x) + \tau_n(x)$ holds for all $x \in B(X)$ and τ_n is linear map from $B(X)$ into its center satisfying $\tau_n([x,y]) = 0$ for all $x, y \in B(X)$. Therefore, by induction, the theorem is true.
We will prove it by several claims.

Claim 1. $PL_n(P)P + QL_n(P)Q \in \mathbb{F}I$ and $PL_n(Q)P + QL_n(Q)Q \in \mathbb{F}I$.

proof. Let $x \in X$, $f \in X^*$. Then

$$\begin{aligned} L_n(Px \otimes Q^*f) &= L_n([P, Px \otimes Q^*f]) \\ &= \sum_{i+j=n} [L_i(P), L_j(Px \otimes Q^*f)] \\ &= [L_n(P), Px \otimes Q^*f] + [P, L_n(Px \otimes Q^*f)] + \sum_{i+j=n; i \neq 0,n} [L_i(P), L_j(Px \otimes Q^*f)] \\ &= L_n(P)Px \otimes Q^*f - Px \otimes Q^*fL_n(P) + PL_n(Px \otimes Q^*f) \\ &\quad - L_n(Px \otimes Q^*f)P + \sum_{i+j=n; i \neq 0,n} [L_i(P), L_j(Px \otimes Q^*f)] \end{aligned}$$

Multiplying this equation by P from the left and by Q from the right, we get, for all $x \in X$ and $f \in X^*$, that

$$PL_n(P)P(x \otimes f)Q = P(x \otimes f)QL_n(P)Q - P \sum_{i+j=n; i \neq 0,n} [L_i(P), L_j(Px \otimes Q^*f)]Q.$$

By $\mathbf{P_m}$, $PL_i(P)P + QL_i(P)Q \in \mathbb{F}I$ and $PL_j(P)P + QL_j(P)Q \in \mathbb{FI}$.
So $P \sum_{i+j=n \quad i \neq 0,n} [L_i(P), L_j(Px \otimes Q^*f)]Q = 0$.
Thus $PL_n(P)P = \mu P$ for some $\mu \in \mathbb{F}$. Hence $QL_n(P)Q = \mu Q$, which implies that $PL_n(P)P + QL_n(P)Q = \mu I$. Similarly we can prove $PL_n(Q)P + QL_n(Q)Q \in \mathbb{F}I$.
Now we put $T = PL_n(P)Q - QL_n(P)P$. For $A \in B(X)$, define $\Delta_n(A) = L_n(A) - (AT - TA)$. Also we can easily checked that $\Delta_n[A, B] = [\Delta_n(A), B] + [A, \Delta_n(B)]$ for all $A, B \in B(X)$.

Claim 2.$\Delta_n(P) \in \mathbb{F}I$.

proof. Using Claim 1, we have

$$\Delta_n(P) = L_n(P) - (PT - TP) = PL_n(P)P + QL_n(P)Q \in \mathbb{F}I.$$

Claim 3. $\Delta_n(PAQ + QAP) = P\Delta_n(A)Q + Q\Delta_n(A)P$ for all $A \in B(X)$.

proof. Let $A \in B(X)$, then

$$\begin{aligned}
\Delta_n(PAQ + QAP) &= \Delta_n([P, [P, A]]) \\
&= \sum_{i+j=n} [\Delta_i(P), \Delta_j([P, A]]) \\
&= [\Delta_n(P), [P, A]] + [p, \Delta_n([P, A])] + \sum_{i+j=n \quad i\neq 0,n} [\Delta_i(P), \Delta_j([P, A])] \\
&= [P, [P, \Delta_n(A)]] + \sum_{i+j=n; i\neq 0,n} [\Delta_i(P), \Delta_j([P, A])]
\end{aligned}$$

Since by Claim 2, $\Delta_i(P) \in \mathbb{F}I$, for $i < n$, so $\sum_{i+j=n;i\neq 0,n}[\Delta_i(P), \Delta_j([P, A])] = 0$
So

$$\Delta_n(PAQ + QAP) = [P, [P, \Delta_n(A)]] = P\Delta_n(A)Q + Q\Delta_n(A)P.$$

Claim 4. $\Delta_n(Q) \in \mathbb{F}I$.

proof. Applying Claim 1, we have

$$P\Delta_n(Q)Q + Q\Delta_n(Q)P = \Delta_n(PQQ + QQP) = 0.$$

Thus $\Delta_n(Q) = PL_n(Q)Q + QL_n(Q)Q \in \mathbb{F}$.

Claim 5. $\Delta_n(B_{ij}) \subseteq \mathrm{B_{ij}} \quad$ for $1 \leqslant i \neq j \leqslant 2$.

proof. For any $X \in B_{12}$, we have

$$\begin{aligned}
\Delta_n(X) &= \Delta_n([P, X]) \\
&= \sum_{i+j=n} [\Delta_i(P), \Delta_j(X)] \\
&= [P, \Delta_n(X)] + \sum_{i+j=n; i\neq 0,n} [\Delta_i(P), \Delta_j(X)] \\
&= [P, \Delta_n(X)] + \sum_{j; j\neq 0,n} [P, \Delta_j(X)]
\end{aligned}$$

Since $\Delta_i(P) \in \mathbb{F}I$ is already shown, $\sum_{i+j=n;i\neq 0,n}[\Delta_i(P), \Delta_j(X)] = 0$ immediately follows.

So

$$\Delta_n(X) = P\Delta_n(x)Q - Q\Delta_n(X)P$$

Multiplying above equation by P from the right, we get

$$P\Delta_n(X)P = Q\Delta_n(X)P = Q\Delta_n(X)Q = 0$$

Thus

$$\Delta_n(X) = P\Delta_n(X)P \in \mathrm{B}_{12}$$

Similarly we can prove that $\Delta_n(y) \subseteq \mathrm{B}_{21}$ for any $y \in \mathrm{B}_{21}$.

Claim 6. There is a functional $f_{ni} : \mathrm{B_{ii}} \to \mathbb{F}$ such that $\Delta_n(X_{ii}) - f_{ni}(X_{ii})I \in \mathrm{B_{ii}}$ for all $X_{ii} \in \mathrm{B_{ii}}$, $1 \leqslant i \leqslant 2$.

proof. Let $X_{ii} \in \mathrm{B_{ii}}$, Applying Claim 3, we obtain

$$0 = \Delta_n(0) = \Delta_n(PX_{ii}Q + QX_{ii}P) = P\Delta_n(X_{ii})Q + Q\Delta_n(X_{ii})P$$

Then we can assume that $\Delta_n(X_{11}) = a_{11}+a_{22}$ and $\Delta_n(X_{22}) = b_{11}+b_{22}$, where $a_{11}, b_{11} \in \mathrm{B}_{11}$, $a_{22}, b_{22} \in \mathrm{B}_{22}$. Also

$$\begin{aligned} 0 &= \Delta_n([X_{11}, X_{22}]) \\ &= \sum_{i+j=n} [\Delta_i(X_{11}), \Delta_j(X_{22})] \\ &= [a_{22}, X_{22}] + [X_{11}, a_{11}] + \sum_{i+j=n; i\neq 0,n} [\Delta_i(X_{11}), \Delta_j(X_{22})]. \end{aligned}$$

By the property $\mathbf{P_m}$, $\Delta_i(X_{11}) \in B_{11} + \mathbb{F}I$ and $\Delta_j(X_{22}) \in \mathrm{B}_{22} + \mathbb{F}I$.
one gets $\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{11}), \Delta_j(X_{22})] = 0$ which implies that $[a_{22}, X_{22}] = 0$ for all $X_{22} \in \mathrm{B}_{22}$ and $[X_{11}, a_{11}] = 0$ for all $X_{11} \in \mathrm{B}_{11}$.
Therefore, there exist scalars $f_{n1}(X_{11})$ and $f_{n2}(X_{22})$ such that $a_{22} = f_{n1}(X_{11})Q$ and $b_{11} = f_{n2}(X_{22})P$. So $\Delta_n(X_{11}) - f_{n1}(X_{11})I \in B_{11}$ and $\Delta_n(X_{22}) - f_{n2}(X_{22})I \in \mathrm{B}_{22}$.

Claim 7.Δ_n is additive on B_{12} and B_{21}.

proof. Let $X_{22} \in \mathrm{B}_{22}$,$X_{12} \in \mathrm{B}_{12}$ and $Y_{21} \in \mathrm{B}_{21}$, by applying Claim 5, 6 we have

$$\begin{aligned}\Delta_n[X_{22}+X_{21},Y_{21}] &= \sum_{i+j=n}[\Delta_i(X_{22}+X_{21}),\Delta_j(Y_{21})]\\ &= [\Delta_n(X_{22}+X_{21}),Y_{21}]+[X_{22}+X_{21},\Delta_n(Y_{21})]\\ &\quad+\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22}+X_{21}),\Delta_j(Y_{21})]\\ &= [\Delta_n(X_{22}+X_{21}),Y_{21}]+[X_{22},\Delta_n(Y_{21})]\\ &\quad+\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22}+X_{21}),\Delta_j(Y_{21})]\\ \Delta_n[X_{22}+X_{21},Y_{21}] &= \Delta_n[X_{22},Y_{21}]\\ &= \sum_{i+j=n}[\Delta_i(X_{22}),\Delta_j(Y_{21})]\\ &= [\Delta_n(X_{22})+\Delta_n(X_{21}),Y_{21}]+[X_{22},\Delta_n(Y_{21})]\\ &\quad+\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22})+\Delta_i(X_{21}),\Delta_j(Y_{21})]\end{aligned}$$

which implies that

$$\begin{aligned}&[\Delta_n(X_{22}+X_{21})-\Delta_n(X_{22})-\Delta_n(X_{21}),Y_{21}]+\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22}+X_{21}),\Delta_j(Y_{21})]\\ &-\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22}),\Delta_j(Y_{21})]=0\end{aligned}$$

But by $\mathbf{P_m}$, we have

$$\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22}+X_{21}),\Delta_j(Y_{21})]=\sum_{i+j=n;i\neq 0,n}[\Delta_i(X_{22})+\Delta_i(X_{22}),\Delta_j(Y_{21})].$$

Thus we conclude that $[\Delta_n(X_{22}+X_{21})-\Delta_n(X_{22})-\Delta_n(X_{21}),Y_{21}]=0$. By a similar way to we can prove that Δ_n is additive on B_{21} and similarly additivity of Δ_n on B_{12} can be deduced easily.

Now we define $D_n(A)=\Delta_n(PAQ)+\Delta_n(QAP)+\Delta_n(PAP)+\Delta_n(QAQ)-(f_1(PAP)-f_2(QAQ))I$. Then by Claim 5, 6 we have

Claim 8. For $B_{ij}\in\mathrm{B_{ij}}$, $1\leqslant i,j\leqslant 2$ we have

1. (1) $D_n(B_{ij})\in\mathrm{B_{ij}}$, $1\leqslant i,j\leqslant 2$;
2. (2) $D_n(B_{12})=\Delta_n(\mathrm{B}_{12})$ and $D_n(B_{21})=\Delta_n(\mathrm{B}_{21})$;
3. (3) $D_n(B_{11}+B_{12}+B_{21}+B_{22})=D_n(B_{11})+D_n(B_{12})+D_n(B_{21})+D_n(B_{22})$.

The following claims immediately follows from Claim 7 and Claim 8.

Claim 9. D_n is additive on B_{12} and B_{21}.

Claim 10. D_n is additive on B_{11} and B_{22}.
It follows similarly to proof of Lemma 2.12 in [4].

Claim 11. D_n is additive.
It follows similarly to proof of Lemma 2.13 in [4].

Claim 12. D_n has the following properties:

(1) $D_n(A_{ii}B_{ij}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij}) \quad 1 \leqslant i \neq j \leqslant 2$
(2) $D_n(B_{ij}A_{jj}) = \sum_{t+k=n} D_t(B_{ij})D_k(A_{jj}) \quad 1 \leqslant i \neq j \leqslant 2$
(3) $D_n(A_{ii}B_{ii}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ii})$

proof. Let $A_{ii} \in \mathrm{B_{ii}}$ and $B_{ij} \in \mathrm{B_{ij}}$, then

$$\begin{aligned} D_n(A_{ii}B_{ij}) &= \Delta_n(A_{ii}B_{ij}) \\ &= \sum_{t+k=n} [D_t(A_{ii}), D_k(B_{ij})] \\ &= \sum_{t+k=n} [D_t(A_{ii}) + \tau_t(A_{ii}), D_k(B_{ij}) + \tau_t(B_{ij})] \\ &= \sum_{t+k=n} [D_t(A_{ii}), D_k(B_{ij})] \end{aligned}$$

Since $D_t(A_{ii}) \in \mathrm{B_{ii}} + \mathbb{F}I$ and $D_k(B_{ij}) \in \mathrm{B_{ij}}$, then

$$D_n(A_{ii}B_{ij}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij}).$$

Also

$$\begin{aligned} D_n(B_{ij}A_{jj}) &= \Delta_n(B_{ij}A_{jj}) \\ &= \sum_{t+k=n} [D_t(B_{ij}), D_k(A_{jj})] \\ &= \sum_{t+k=n} [D_t(B_{ij}) + \tau_t(A_{jj}), D_k(B_{ij}) + \tau_t(A_{jj})] \\ &= \sum_{t+k=n} [D_t(B_{ij}), D_k(A_{jj})] \end{aligned}$$

Since $D_k(A_{jj}) \in \mathrm{B_{jj}} + \mathbb{F}I$ and $D_t(B_{ij}) \in \mathrm{B_{ij}}$, then

$$D_n(B_{ij}A_{jj}) = \sum_{t+k=n} D_t(B_{ij})D_k(A_{jj}).$$

Furthermore

$$\begin{aligned} D_n(A_{ii}B_{ii}C_{ij}) &= \sum_{t+k=n} D_t(A_{ii}B_{ii})D_k(C_{ij}) \\ &= \sum_{t+k=n,k\neq 0} D_t(A_{ii}B_{ii})D_k(C_{ij}) + D_n(A_{ii}B_{ii})C_{ij} \end{aligned}$$

But

$$\begin{aligned} D_n(A_{ii}B_{ii}C_{ij}) &= \sum_{t+k=n} D_t(A_{ii})D_k(B_{ii}C_{ij}) \\ &= \sum_{t+k+l=n} D_t(A_{ii})D_k(B_{ij})D_l(C_{ij}) \\ &= \sum_{t+k+l=n,L\neq 0} D_t(A_{ii})D_k(B_{ij})D_l(C_{ij}) + \sum_{t+k=n,l\neq 0} D_t(A_{ii})D_k(B_{ij})C_{ij} \end{aligned}$$

Thus

$$D_n(A_{ii}B_{ii})C_{ij} = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij})C_{ij}$$

Which implies that

$$D_n(A_{ii}B_{ii}) = \sum_{t+k=n} D_t(A_{ii})D_k(B_{ij}).$$

Claim 14. $\Delta_n(B_{11}+C_{22}) - \Delta_n(B_{11}) - \Delta_n(C_{22}) \in \mathbb{F}I$.

Claim 15. $\Delta_n(A_{ij}B_{ji}) = \sum_{i+j=n}[\Delta_i(A_{ij}, \Delta_j(B_{ji})] \quad 1 \leqslant i \neq j \leqslant 2$

proof. Note that $[A_{12}, B_{21}] = [[A_{12},, P_2], B_{21}]$ holds true for all $A_{12} \in \mathrm{B}_{12}$ and $B_{21} \in \mathrm{B}_{21}$. So we have

$$\begin{aligned} \Delta_n([B_{21}, C_{12}]) - D_n([B_{21}, C_{12}]) &= \sum_{i+j=n} [\Delta_i(B_{21}, \Delta_j(C_{12})] - D_n(B_{21})C_{12} - D_n(C_{12}B_{21}) \\ &= [\Delta_n(B_{21}), C_{12}] + [B_{21}, \Delta_n(C_{12})] \\ &\quad + \sum_{i+j=n;i\neq 0,n} [\Delta_i(B_{21}), \Delta_j(D_{12})] - D_n(B_{21})C_{12} - D_n(C_{12}B_{21}) \\ &= [D_n(B_{21}), C_{12}] + [B_{21}, D_n(C_{12})] \\ &\quad + D_n(B_{21}C_{12}) - D_n(B_{21}C_{12}) \\ &= D_n(A_{12})B_{21} - P_2D_n(A_{12})B_{21} - B_{21}D_n(A_{12})P_2 + B_{21}D_n(A_{12}) \\ &\quad + A_{12}D_n(B_{21}) - D_n(B_{21})A_{12} - D_n(A_{12}B_{21}) + D_n(B_{21}A_{12}) \end{aligned}$$

Which implies that

$$(D_n(A_{12})B_{21}+A_{12}D_n(B_{21})-D_n(A_{12}B_{21}))+(D_n(B_{21}A_{12})-D_n(B_{21})A_{12}-B_{21}D_n(A_{12})) = K \in \mathbb{F}I.$$

Using Claim 12 and multiplying B_{22} from the left in the above equality, We obtain

$$B_{22}D_n(B_{21}A_{12}) - B_{22}D_n(B_{21})A_{12} - B_{22}B_{21}D_n(A_{12}) = B_{22}K.$$

So

$$K = D_n(A_{12}B_{21}+A_{12}D_n(B_{21})-D_n(A_{12}B_{21}))+(D_n(B_{21}A_{12})-D_n(B_{21})A_{12}-B_{21}D_n(A_{12}) = 0.$$

Therefore

$$\begin{aligned} D_n(A_{12})B_{21} + A_{12}D_n(B_{21}) - D_n(A_{12}B_{21}) &= -(D_n(B_{21}A_{12}) - D_n(B_{21})A_{12} - B_{21}D_n(A_{12})) \\ &\in \mathrm{B}_{11} \bigcap \mathrm{B}_{22} \end{aligned}$$

So we have
$D_n(A_{12})B_{21} + A_{12}D_n(B_{21}) - D_n(A_{12}B_{21}) = 0$ and $D_n(B_{21}A_{12}) - D_n(B_{21})A_{12} - B_{21}D_n(A_{12}) = 0$
So D_n is a derivation.

Claim 16. For all $A \in B(X)$, $\Delta_n(A) - \Delta_n(PAQ) - \Delta_n(QAP) \in \mathbb{F}I$.
For any $A \in B(X)$ and Claim it is easy to checked that $\Delta_n(A)-\Delta_n(PAQ)-\Delta_n(QAP) \in \mathbb{F}I$.

Claim 17. $\tau_n([X,Y]) = 0$ for all $X, Y \in B(X)$.

proof. For any $X, Y \in B(X)$, we have

$$\begin{aligned} \tau_n([X,Y]) &= \Delta_n([X,Y]) - D_n([X,Y]) \\ &= \sum_{i+j=n} [\Delta_i(X), \Delta_j(Y)] - D_n(XY) - D_n(YX) \\ &= \sum_{i+j=n} [D_i(X) + \tau_i(X), D_j(Y) + \tau_j(Y)] - \sum_{i+j=n} D_i(X)D_j(Y) - D_j(Y)D_j(X) \\ &= \sum_{i+j=n} [D_i(X), D_j(Y) - \sum_{i+j=n} D_i(X)D_j(Y) - D_j(Y)D_j(X) \\ &= 0 \end{aligned}$$

The proof of theorem is completed. ■

References

[1] W. Cheung, *Lie derivations on triangular matrices*, Linear Multilinear Algebra **55** (2007), 619.626.
[2] Y. Du, Y. Wang, *Lie derivations of generalized matrix algebras*, Linear Algebra Appl. **433** (2012), 2719-2726.
[3] M. Ferrero, C. Haetinger, *Higher derivations of semiprime rings*, Comm. Algebra. **30** (2002), 2321-2333.
[4] F. Lu ., B. Liu, *Lie derivable maps on $B(X)$* , J. Math. Anals. Appl. **372**(2010), 369-376. 2321-2333

[5] Y. B. Li, Z. K. Xiao, *Additivity of maps on generalized matrix algebras*, Electron. J. Linear Algebra **22**, (2011) 743757.

[6] A. Nakajima, it On generalized higher derivations, Turk. J. Math. **24**(2000), 295-311.

[7] X. F. Qi,*chacterization of Lie higher derivation on triangular algebras*, Acta Mathematica Sinica. **26** (2013), 1007-1018.

[8] G. A. Swain, P.S. Blau,*Lie derivations in prime rings with involution* , Canad. Math. Bull. **42**, (1999) 401-411.

[9] Z.-K Xiao, F. Wei,*Jordan higher derivations on triangular algebras*, Linear Algebra and its Applications. **432** (2010), 26152622.

[10] Q. X.- Fei, H. J. Chen, *Lie higher derivations on nest algebras*, Comm. Math. Research.(2010) 131-143.

[11] Z.-K Xiao, F. Wei , *Higher derivations of triangular algebras and its generalizations*, Linear Algebra Appl. **435** (2011),1034-1054.

[12] W. Y. Yu, J. H. Zhang, *Chacterization of Lie higher and Lie triple derivation on triangular algebras*, J. Korean Math. Soc. **49** (2012), 419-433.