

Primal and dual robust counterparts of uncertain linear programs: an application to portfolio selection

P. Hanafizadeh

Assistant Professor, Department of Industrial Management, Allame Tabataba'ee University, Tehran, Iran

A. Seifi

Associate Professor, Department of Industrial Engineering, Amirkabir University of Technology, Tehran, Iran

K. Ponnambalam

Professor, Department of Systems Design Engineering, University of Waterloo, Waterloo, Canada

Abstract

This paper proposes a family of robust counterpart for uncertain linear programs (LP) which is obtained for a general definition of the uncertainty region. The relationship between uncertainty sets using norm bodies and their corresponding robust counterparts defined by dual norms is presented. Those properties lead us to characterize primal and dual robust counterparts. The researchers show that when the uncertainty region is small the corresponding robust counterpart is less conservative than the one for a larger region. Therefore, the model can be adjusted by choosing an appropriate norm body and the radius of the uncertainty region. We show how to apply a robust modeling approach to single and multi-period portfolio selection problems and illustrate the model properties with numerical examples.

Keywords: Robust optimization; Linear programming; Data uncertainty; Portfolio selection

1. Introduction

Robust optimization methodology (ROM) has been developed to deal with uncertainty in convex programming problems to design solutions that are immune to data uncertainty. In 1970, a robust formulation for uncertain linear programs (LP) was proposed by Sengupta [23] based on the statistical approach. He analyzed the implications of a non-normal but independent distribution of the random parameters of an LP in the framework of probabilistic linear programming. Afterwards, this was addressed by Soyster [25] who considered uncertainty in the columns of constraint coefficients belonging to a convex set. In 1991, Sengupta defined robust solutions based on non-parametric methods which are applicable in situations of incomplete knowledge and partial uncertainty. He had shown that this class of methods provides a measure of robustness through the adoption of a cautious policy [24].

Recently, ROM has become more popular (El Ghaoui, et al. [12], Ben-Tal and Nemirovski [3-4]). A unified robust modeling approach has been also suggested for Design Centering in engineering application by Seifi et al. [22]. The reasons for the popularity of ROM are: (i) The size of the robust counterpart essentially remains the same as the original problem, (ii) No need to generate scenarios, and (iii) Usually results in convex programming problems that can be efficiently solved using interior point methods [6]. Bertsimas and Melvyn [8] have a relaxed robust counterpart using general conic optimization to reduce the computational complexity especially for robust semi-definite programming problems (SDP). Generalization of robust LP with any l_p -norms has also been done in Hanafizadeh and Seifi constraint-wise [15] and also by Bertsimas, et al. column-wise [7]. An extension of the robust optimization for uncertain linear programs is called the adjustable robust

* Corresponding author. E-mail: hanafizadeh@gmail.com

solution as proposed by Ben-Tal et al. [1]. They define adjustable variables to represent "wait and see" decisions, that is, those that can be made when a part of the uncertain data becomes known.

In ROM, the uncertain parameters are assumed to be bounded in an interval or an ellipsoidal region. Considering the worst-case behavior of the parameters, the robust counterpart of an uncertain LP becomes another LP or a second-order cone programming (SOCP) model. Ben-Tal and Nemirovski [5] claim that the ellipsoidal model of uncertainty is significantly less conservative than the interval region, and thus, leads to more practical solutions. In this paper, the researchers define a family of robust counterpart (FRC) for uncertain LP using different norms and present its properties in terms of the uncertainty sets, feasibility regions and the value of objective functions. The authors show that using higher norms in the robust counterpart leads to less conservative model than using lower norms. Based on the properties, the authors define the primal and dual robust counterpart of an uncertain LP. In order to make the advantages of these properties clear, the authors bring examples from portfolio selection problems in single and multiple periods.

Portfolio selection is an area where researchers have been interested in applied ROM. Ben-Tal, Margalit and Nemirovski [2] formulated a multi-period portfolio selection problem. Their model, the uncertain outcomes of earlier stages have an effect on the decisions of the later stages and the decision variables must be chosen to satisfy certain balance constraints. El Ghoui, Oks and Oustry [11] modeled the portfolio selection problem to maximize Value at Risk (VaR) ratio. This problem was reformulated as a SDP. Hall-dorson and Tutuncu [14] considered mean vector and covariance matrices in interval regions and made a saddle point nonlinear program. Goldfarb and Iyengar [13] applied robust optimization to portfolio selection with mean-variance, maximum Sharpe ratio and VaR measures and showed that their robust counterparts are SOCPs. They proposed statistical procedures for estimating the uncertainty regions. Tutuncu and Koenig [26] presented a new formulation for identifying robust portfolios with the largest Sharpe ratio and also addressed the issue of generating uncertainty sets. Pinar and Tutuncu [21] proposed the concept of a robust profit opportunity as an alternative to arbitrage opportunities and formulated the problem of finding the "most robust" profit opportunity. They showed that it can be solved as a convex quadratic programming problem.

This paper also shows how to apply a robust modeling approach to single and multi period portfolio selection problems. It is also shown how the robust model of an uncertain portfolio selection can be adjusted according to the chosen l_p -norms.

The rest of the paper is outlined as follows: Section 2 presents FRC, its properties and primal and dual robust counterparts. In Section 3, the authors apply the proposed model to a single-period portfolio selection problem and describe the numerical example. In Section 4, robust multi-period portfolio selection problems and the corresponding numerical results are presented. Section 5 concludes the whole paper.

2. Family of robust counterparts

An uncertain linear optimization problem is defined as:

$$\begin{aligned} & \text{Max } c^T x \\ & \text{Subject to :} \\ & (a_i ; b_i)^T x \leq 0, (a_i ; b_i) \in U_i, i = 1, \dots, m \\ & x \geq 0. \end{aligned} \quad (1)$$

Let $(a_i ; b_i)$ denote the column vector obtained from the column vector a_i by appending the scalar b_i . Similarly, $x = (\hat{x} ; -1)^T$. It is assumed that $(a_i ; b_i)$ are not known exactly; but must lie in a given uncertainty set U_i for $i = 1, \dots, m$. The uncertainty in c may also be absorbed in the constraints, as shown later in Equation (8). An l_q -norm body is defined as:

$$B_q(r) = \left\{ u \mid \|u\|_q \leq r \right\}. \quad (2)$$

The authors are interested in the ellipsoidal-norm body, which is an ellipsoidal region and is defined by the quadratic-norm as follows:

$$\|u\|_C = (u^T C^{-1} u)^{1/2} \text{ (quadratic-norm)}, \quad (3)$$

where C is a symmetric and positive definite $n \times n$ matrix .

Definition 1. The Partial uncertainty set which is associated with the i -th constraint is defined as:

$$U_i(q, r) = \left\{ (a_i; b_i) : (a_i; b_i) = (a_i^0; b_i^0) + W^i u \mid u \in B_q(r) \right\}, \quad (4)$$

where, W^i is an $n \times n$ symmetric and positive definite matrix, $(a_i^0; b_i^0)$ is the nominal value for $(a_i; b_i)$.

Definition 2. The complete uncertainty set U for all uncertain parameters of an LP in (1) is the Cartesian product of the partial uncertainty sets U_i for $i = 0, 1, \dots, m$:

$$U = U_0 \times U_1 \times \dots \times U_m.$$

U_0 is the partial uncertainty set related to random parameters in the objective function. Constructing the complete uncertainty set is constraint-wise. Each U_i is the set of all possible realizations of i -th row in the constraint matrix and is obtained by the projection of U on to the space of data of the i -th constraint. U does not consider the dependencies (if any) among the uncertain parameters in different constraints.

Definition 3. Any generation of partial uncertainty set is related to using different degrees of norm in its norm-body and is denoted by $U_i(q, r)$, $q = 1, 2, \dots, \infty$. Any generation of complete uncertainty set is related to any distinct combination of different generation of partial uncertainty sets and is denoted by $U(q_0, q_1, \dots, q_m, r)$, $q_i = 1, 2, \dots, \infty$ for $i = 0, 1, \dots, m$. $U(q, r)$ is used when the degree of norms for all partial uncertainty sets, characterizing the complete uncertainty set, is the same.

Proposition 1. Using different generation of partial uncertainty sets brings the following relation for the generation of the complete uncertainty sets:

$$U(1, r) \subseteq U(q_0, q_1, \dots, q_m, r) \subseteq U(\infty, r),$$

$$\forall q_i, 1 < q_i < \infty \text{ and } i = 0, 1, \dots, m.$$

Proof. The following relation exists when using l_q -norm bodies:

$$B_1(r) \subseteq B_q(r) \subseteq B_\infty(r), \quad \forall q \text{ and } 1 < q < \infty.$$

The above relation is invariant under all affine transformation $(a_i^0; b_i^0) + W^i u$, therefore:

$$U_i(1, r) \subseteq U_i(q_i, r) \subseteq U_i(\infty, r), \quad (5)$$

$$\forall q_i, 0 < q_i < \infty \text{ and } i = 0, 1, \dots, m.$$

Since the complete uncertainty set is constraint-wise of the partial ones, the proposition is proved.

Result 1. For a given r , the smallest uncertainty region among all regions defined by l_q -norms, is associated with using the l_1 -norm in the definition of the uncertainty region.

A robust counterpart formulation is obtained by replacing U_i in the problem (1) by $U_i(q, r)$. Consider the worst-case behavior of the i -th constraint: $(a_i; b_i)^T x \leq 0$, and according to the definition of the partial uncertainty set, we have:

$$\begin{aligned} \sup_{\|u\|_q \leq r} [(a_i^0; b_i^0)^T x + u^T W^i x] &\leq 0, \\ (a_i^0; b_i^0)^T x + \sup_{\|u\|_q \leq r} [u^T w^i(x)] &\leq 0. \end{aligned} \quad (6)$$

Here, we define $w^i(x) = W^i x$. Based on the definition of the dual norm, the dual of the l_q -norm is the l_p -norm, where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, see for instance Boyd and Vandenberghe [9]. Therefore:

$$\sup \{ w^i(x)^T u \mid \|u\|_q \leq r \} = r \|w^i(x)\|_p.$$

Then the robust version of the i -th constraint is:

$$(a_i^0; b_i^0)^T \hat{x} + r \|w^i(x)\|_{p_i} \leq 0. \quad (7)$$

Therefore, the corresponding deterministic model of a general robust counterpart is:

Max t
Subject to (8)

$$\begin{aligned} c^{0T} x - r_0 \|w^0(x)\|_{p_0} &\geq t \\ (a_i^0; b_i^0)^T x + r_i \|w^i(x)\|_{p_i} &\leq 0 \quad i = 1, \dots, m \text{ and } x \geq 0. \end{aligned}$$

The robust counterpart in (8) is a constraint-wise construction. We call (8), a Family Robust Counterpart (FRC) for the uncertain linear program in (1).

If the same l_q -norm is used in all the partial uncertainty sets of (1), FRC with different norms have properties as defined in the following propositions:

Definition 4. The robust counterpart feasibility set of problem (8) using l_p -norm is defined by S_p , where S_p is the intersection of all inequality constraints S_p^i for $i = 0, 1, \dots, m$ and also the nonnegative constraints. Each S_p^i is a subset related to the i -th inequality constraint.

Proposition 2. The robust counterpart feasibility sets using different norms in the constraints of FRC (8) bring the relation:

$$S_1 \subseteq S_p \subseteq S_\infty, \quad (9)$$

where S_p is the robust counterpart feasibility set of problem (8) using l_p -norm, for $\forall p$ and $p \neq 1, \infty$.

Proof. $B_p^i(t)$ is defined as a norm-body which is related to the i -th constraint of FRC (8):

$$B_p^i(t) = \left\{ t \mid r \|(W^i)^T t\|_p \leq b_i^0 \right\}.$$

Generally, there is the following relation for any l_p -norm bodies:

$$B_1^i(t) \subseteq B_p^i(t) \subseteq B_\infty^i(t), \forall p \text{ and } 1 < p < \infty.$$

Let assume $t = kx$ where $k = \frac{a_i^T x}{r \|(W^i)^T x\|_p} + 1$ is

a scalar then with above changing variable, $B_p^i(t)$ can be defined as:

$$a_i^T x + r \|(W^i)^T x\|_p \leq b_i^0,$$

which is related to the i -th constraint using l_p -norms. Therefore we have:

$S_1^i \subseteq S_p^i \subseteq S_\infty^i, i = 1, \dots, m$ and also the intersection of these m subsets brings the following relation:

$\bigcap_{i=1}^m S_1^i \subseteq \bigcap_{i=1}^m S_p^i \subseteq \bigcap_{i=1}^m S_\infty^i$ and also the nonnegative constraints are the same for all problems, therefore, $S_1 \subseteq S_p \subseteq S_\infty$.

Result 2. The above theorem shows that using l_∞ -norm in the original definition of uncertainty region, bring the smallest robust counterpart feasibility region.

Corollary 1. If there is at least one feasible solution in S_1 then any robust counterpart feasibility sets S_p will be feasible.

Proof. Corollary 1 is an immediate consequence of the proposition 2. But the converse of Corollary 1 is not true.

Proposition 3. Using different norms in problem (8) bring the inequality relation for the value of robust counterpart objective functions:

$$Z_1(x^{l_1}) \leq Z_p(x^{l_p}) \leq Z_\infty(x^{l_\infty}), \quad 1 < p < \infty, \quad (10)$$

where x^{l_p} is the optimal solution for problem (8) using l_p -norm, and $p = \{1, 2, \dots, \infty\}$ and $Z_p(x) = c^{0T}x - r \|W^{0T}x\|_p$ is the robust counterpart objective function of problem (8).

Proof:

$$D) Z_1(x^{l_1}) \leq Z_p(x^{l_p})$$

Problem (8) using l_1 -norm is a linear program (see Hanafizadeh and Seifi [15]) therefore, optimal solution becomes:

$x^{l_1} = B^{-1}b^0$, where B^{-1} is the inverse matrix which is formed by the column vectors related to basic variables in optimal solution, then:

$$Z_1(x^{l_1}) = c_B^{0T} B^{-1}b^0 - r \|W_B^{0T} B^{-1}b^0\|_1,$$

where W_B^0 is a sub matrix of W^0 and c_B is a vector which are related to basic variables of the optimal solution.

Since x^{l_p} is optimal for problem (8) using l_p -norm, we can write:

$$Z_p(x^{l_p}) \geq Z_p(x) \quad \forall x \in S_p,$$

where S_p is the feasible region for problem (8) using l_p -norm.

From proposition 1, $S_1 \subseteq S_p$ and then $x^{l_1} \in S_p$. Now, the value of $Z_p(x)$ in x^{l_1} is computed as follows:

$$Z_p(x^{l_1}) = c_B^{0T} B^{-1} b^0 - r \left\| W_B^{0T} B^{-1} b^0 \right\|_p.$$

Since we generally have:

$$\left\| W_B^{0T} B^{-1} b^0 \right\|_p \leq \left\| W_B^{0T} B^{-1} b^0 \right\|_1, \text{ then } \mathbf{I} \text{ is proven.}$$

$$\mathbf{II)} \quad Z_p(x^{l_p}) \leq Z_\infty(x^{l_\infty})$$

In the same way, x^{l_∞} is optimal for problem (8) using l_∞ -norm, it means:

$$Z_\infty(x^{l_\infty}) \geq Z_\infty(x) \quad \forall x \in S_\infty,$$

where S_∞ is the feasible region for problem(8) using l_∞ -norm.

From proposition 1, $S_p \subseteq S_\infty$ and then $x^{l_p} \in S_\infty$. Now we compute the value of $Z_\infty(x)$ in x^{l_p} which is equal to:

$$Z_\infty(x^{l_p}) \geq c^{0T} x^{l_p} - r \left\| W x^{l_p} \right\|_\infty$$

Since, $Z_p(x^{l_p}) = c^{0T} x^{l_p} - r \left\| W x^{l_p} \right\|_p$ and for a given x^{l_p} , generally we have:

$$\left\| W x^{l_p} \right\|_p \geq \left\| W x^{l_p} \right\|_\infty, \text{ therefore } \mathbf{II} \text{ is proven.}$$

Result 3. For a given $r > 0$, the uncertainty set corresponding to l_1 -norm in the original problem (1) leads to the largest value for the robust counterpart objective function.

Definition 5. Any family of robust counterparts of problem (1) has a dual as defined in the following table:

Table 1. The comparison of the primal and dual robust counterparts.

Primal robust counterpart	Dual robust counterpart	Assume $p \leq q$
$\max_x Z_p(x)$	$\max_x Z_q(x)$	$S_p \subseteq S_q$
$s.t \quad x \in S_p$	$s.t \quad x \in S_q$	$Z_p(x) \leq Z_q(x)$
$U(q, r)$	$U(p, r)$	$U(p, r) \subseteq U(q, r)$

where p and q have the relation in $\frac{1}{p} + \frac{1}{q} = 1$.

Let us assume that $p \leq q$, then $U(p, r) \subseteq U(q, r)$, $S_p \subseteq S_q$ and $Z_p(x) \leq Z_q(x)$ and if the primal is feasible then the dual is also feasible, but the converse is not necessarily true. It means that the primal robust counterpart is more conservative than the dual one (however, if $p = q = 2$, primal and dual are the same).

3. The robust model of single-period portfolio selection problem

Suppose that there are n different assets in the market. The return of \$1 invested in asset j is a random variable, which is assumed to be distributed symmetrically in its domain. The problem is to allocate \$1 among the assets in order to get the highest possible total return on the selected portfolio. A model with uncertain parameters of this problem is:

$$\text{Max}_{x_j} \sum_{j=1}^n c_j x_j$$

Subject to :

$$\sum_{j=1}^n x_j = 1$$

$$x_j \geq 0 \text{ for } j = 1, \dots, n.$$

where c_j is the uncertain return of the asset j . The nominal optimal solution result: the budget should be invested in the assets which have the maximal nominal returns. Mathematically, there is always an extreme point solution that is optimal. The extreme points of the underlying feasible region are the unit vectors in \mathfrak{R}^n in this case. Therefore, there always exists a unit vector which is optimal. This solution, however, is unreliable and risky. The robust version of the uncertain LP (11) is:

$$\begin{aligned}
 & \text{Max } t \\
 & \text{Subject to} \\
 & \bar{c}^T x - r \left\| W^T x \right\|_p \geq t \tag{12} \\
 & \sum_{j=1}^n x_j = 1 \\
 & x \geq 0.
 \end{aligned}$$

This is a general model for measuring risk and we can specialize it for different optimization problems for modeling uncertainty in returns and getting different results. The model for the l_1 -norm is similar to the mean-absolute deviation (MAD) model in Konno [16] and, Konno and Yamazaki [17].

The model for l_2 -norm and l_C -norm (matrix-norm) is an SOCP problem which is similar to Markowitz [18-19] mean-variance model. The model for l_∞ -norm is similar to the minimax model introduced in Young [27]. In the later example, the effect of using different norms on the total returns will be shown.

3.1. Example 1

Consider eight risky assets from [Yahoo.com] for years 2002-2003. Model (12) needs the vector of nominal value \bar{c} and the positive definite matrix C to define the uncertainty region. Here it is important to note that $W = C^{1/2}$, $C^{1/2} = VD^{1/2}V^T$ in which D is the diagonal matrix of eigenvalues and columns of V correspond to eigen vectors.

Figure 1 shows the efficient frontiers (EF) using different norms in model (12). All cases were solved using the MATLAB ®. In this figure, the solid line with dots is related to the l_2 -norm, the dashed line with triangles is related to the l_1 -norm and the dotted line with squares is related to the l_∞ -norm solutions (these descriptions are used through out the paper). The standard deviation is used to measure risk in any l_p -norm solutions.

In Figure 2, EFs have been depicted when risk measurements are considered in different l_p -norms. The graphs in the left side of Figure 2 present the l_1 -norm risk measurements for different l_p -norm

solutions. The l_1 -EF (the dashed line with triangles) is dominating the other ones, because we directly minimized the l_1 -norm risk measurement. The graphs in the middle of Figure 2 are related to the l_2 -norm risk measurement for different l_p -norm solutions. The l_2 -EF (the solid line with dots) is usually dominating the other ones, because we directly minimized the l_2 -norm risk measurement. Lastly, the graphs in the right side of Figure 2 are related to the l_∞ -norm risk measurement for different l_p -norm solutions and here the l_∞ -norm solutions dominate others. The EFs in Figure 2 show that any norm can be applied to measure dispersion. But the question to ask is what kind of norm is appropriate for modeling risk?

In Figure 3, the value of the robust counterpart of the total return for the l_∞ -norm is the largest. This is what we obtained in proposition 4. Can we expect that we have the same result when the uncertain returns are realized?

In Figure 4, based on simulating the uncertain returns, we evaluate the real total returns for any l_p -norm solutions for each value of r (with 1000 simulated returns for each r , we compute $c_s^T x^{l_p}$ where c_s is the vector of simulated returns).

The result of Figure 4 is that, the l_∞ -norm yields almost the best value for total returns but the values of total returns for different norms decreases for increasing radius of uncertainty. The graph may be divided in three parts in terms of the radius value: being small, medium and large. When the radius r is less than 0.2, the values of total returns are close to each other with less than 0.008 differences. If radius r is between 0.2 and 0.9, the difference between total returns of the l_∞ and of the l_1 -norms in some parts is more than 0.045. For the radius r greater than 0.9, the value of total returns of the l_∞ -norm is still the best and the difference from that of the l_1 -norm is about 0.012. It stays constant for the rest of the graph.

It can be concluded that when the uncertainty is small (r less than 0.2), there is no significant difference in using different norms (between primal and dual) but when the uncertainty is large (r more than 0.5) the l_∞ -norm is the best choice.

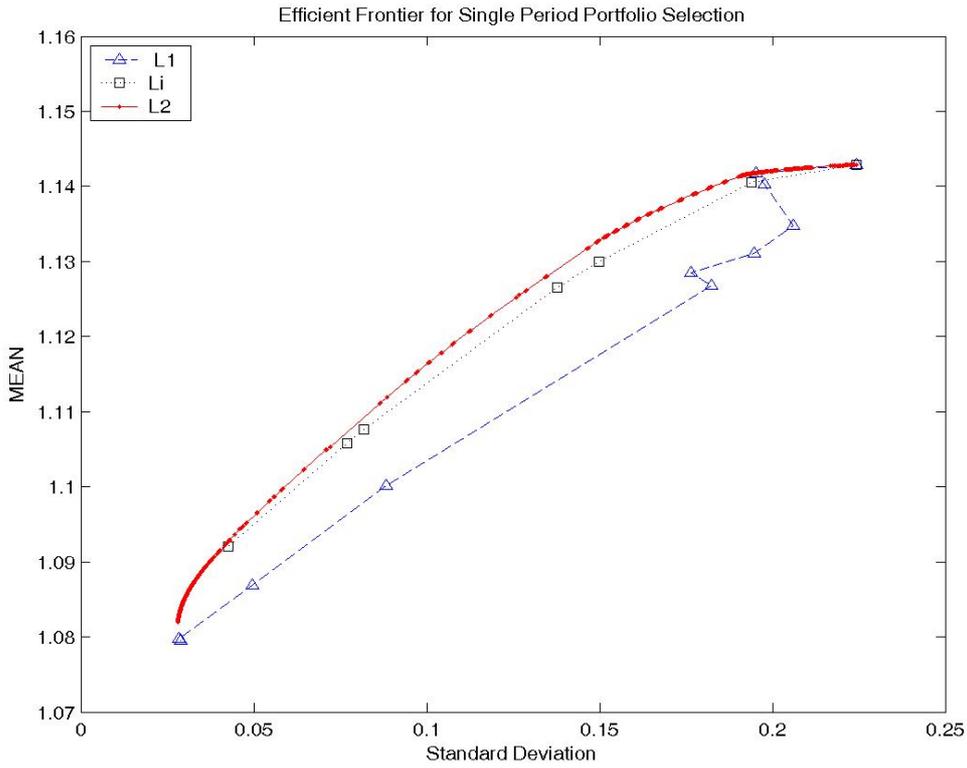


Figure 1. The comparison of Efficient Frontier of 8 stocks using standard deviation as a risk measurement in single-period portfolio selection.

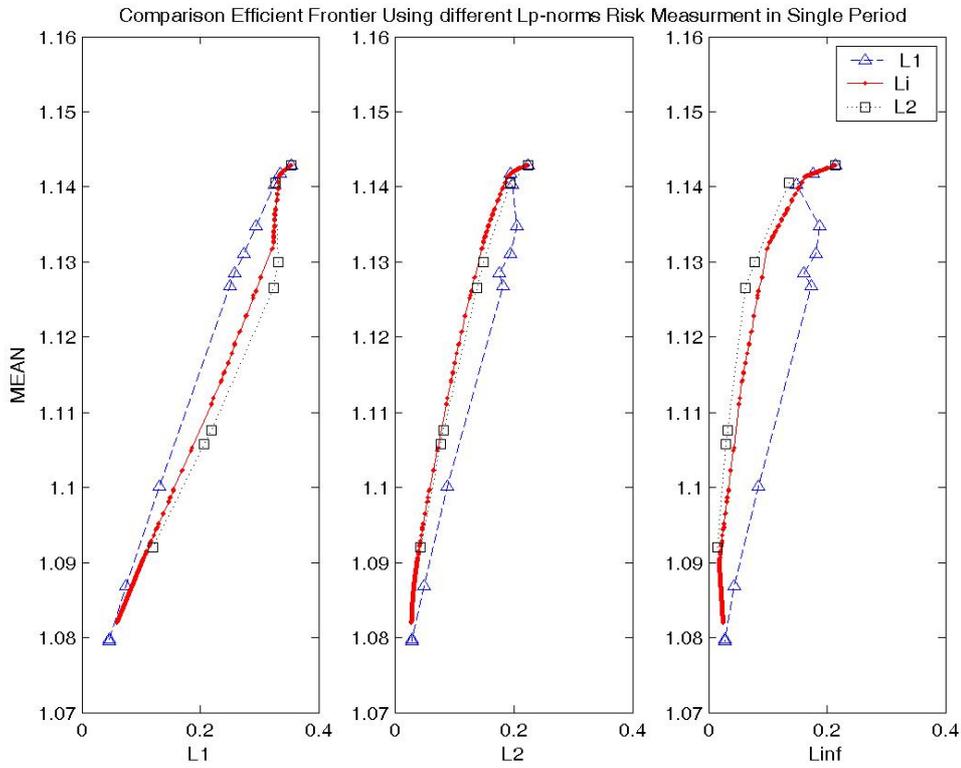


Figure 2. The comparison of efficient frontier of 8 stocks using different l_p -norms as risk measurements in single-period portfolio selection.

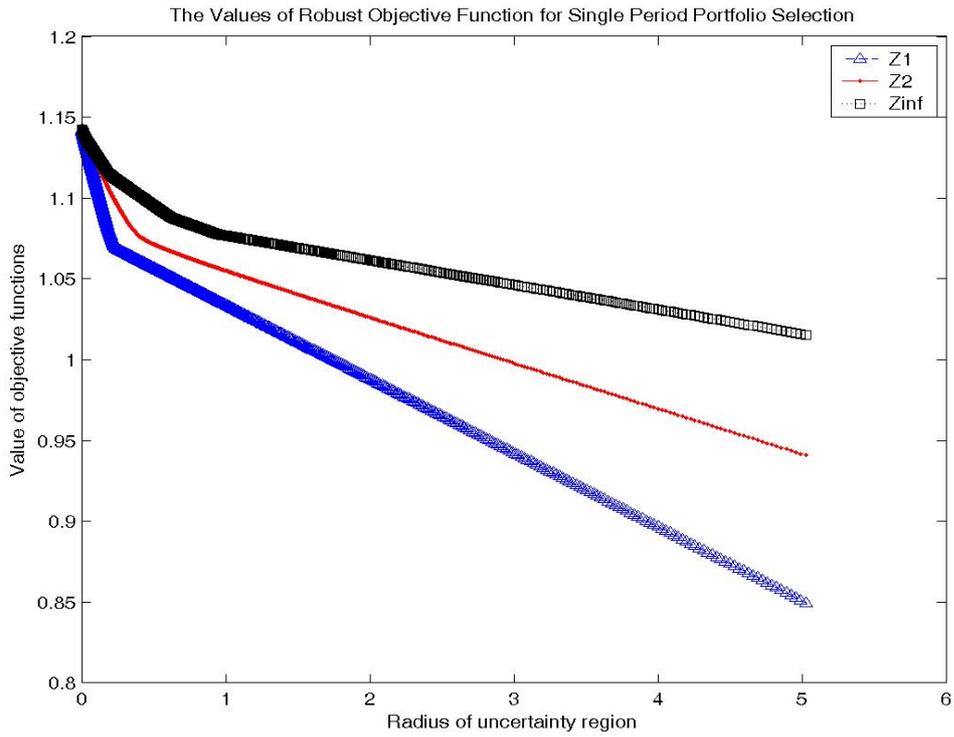


Figure 3. The value of optimal robust counterpart objective functions for different l_p -norms in single-period portfolio selection.

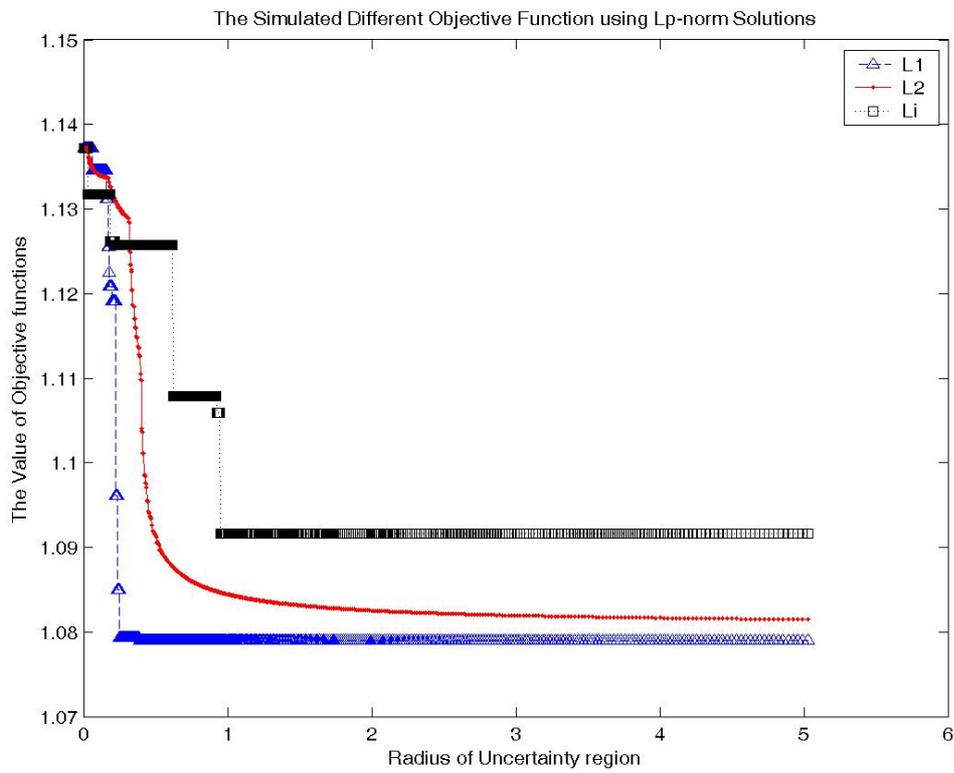


Figure 4. The comparison of real total returns in simulation with different l_p -norm solutions in the single-period portfolio selection.

4. The robust model of multi-period portfolio selection problem

Consider an investor who currently holds the following portfolio: $x^0 \in R^n$, where x_j^0 denotes the number of shares of asset j in the portfolio for $j = 1, \dots, n$. Also, let x_0^0 denote the investor's cash holding. The investor is trying to determine how to adjust his/her portfolio in the next L investment periods to maximize his/her total wealth at the end of final period L . The following decision variables are used to model this multi-period portfolio selection problem: b_j^l denotes the number of additional shares of asset j bought at the beginning of period l and s_j^l denotes the number of shares of asset j sold at the beginning of period l , for $j = 1, \dots, n$ and $l = 1, \dots, L$.

Let P_j^l denote the price of a share of asset j in period l . For initial prices, without loss of generality, we choose $P_j^0 = 1$ for all $j = 0, 1, \dots, n$, we can always normalize the x^0 quantities if necessary. We make the assumption that the cash account earns no interest so that $P_0^l = 1, \forall l$.

It is assumed that proportional transaction costs are paid on each purchase and sale and denote them with α_j^l and β_j^l for sale and purchase respectively for asset j and period l . It is assumed that α_j^l 's and β_j^l 's are all known at the beginning of period 0, although they can vary from period to period and from asset to asset. Transaction costs are paid from the investor's cash account (see Figure 5).

Since the objective function involves uncertain parameters P_j^l , the problem is formulated such that all the uncertainty is moved in the constraints:

Max $_{x,s,b,t}$ t

$$t \leq \sum_{j=0}^n P_j^L x_j^L$$

$$x_0^l \leq x_0^{l-1} + \sum_{j=1}^n (1 - \alpha_j^l) P_j^l s_j^l - \sum_{j=1}^n (1 + \beta_j^l) P_j^l b_j^l, \quad l = 1, \dots, L \quad (13)$$

$$x_j^l = x_j^{l-1} - s_j^l + b_j^l, \quad j = 1, \dots, n, \quad l = 1, \dots, L$$

$$s_j^l \geq 0, \quad j = 1, \dots, n, \quad l = 1, \dots, L$$

$$b_j^l \geq 0, \quad j = 1, \dots, n, \quad l = 1, \dots, L$$

$$x_j^l \geq 0, \quad j = 1, \dots, n, \quad l = 1, \dots, L.$$

This is a linear programming problem as stated by Dantzig and Infanger in [10] when the parameters are certain and can be solved easily using the simplex method or interior-point methods. The first two sets of constraints of this reformulation are the constraints that are affected by uncertainty and we would like to find a solution that satisfies these constraints for most possible realizations of the uncertain parameters P_j^l . To determine the robust version of these constraints, a general uncertainty region for P_j^l is defined as follows:

$$U^l(q, r) = \left\{ P^l : P^l = \bar{P}^l + W^l u \mid \|u\|_q \leq r \right\} \quad (14)$$

and $l = 1, \dots, L$,

where vector \bar{P}^l is the nominal for P^l and W^l is a symmetric positive definite matrix.

According to the definition of uncertainty region, the first constraint that is defined in (13) becomes:

$$t \leq (P^L)^T x^L, \quad P^L \in U^L(q, r)$$

Considering the worst-case behavior, we have:

$$\inf_{P^L \in U^L} [(P^L)^T x^L] \geq t$$

$$\inf_{\|u\|_q \leq r} [(\bar{P}^L)^T x^L + u^T (W^L)^T x^L] \geq t \quad (15)$$

$$(\bar{P}^L)^T x^L - \sup_{\|u\|_q \leq r} u^T (W^L)^T x^L \geq t$$

Based on the definition of the dual norm:

$$(\bar{P}^L)^T x^L - r \|(W^L)^T x^L\|_p \geq t,$$

the second constraint is also affected by uncertain parameters P_j^l . The second constraint is also written as:

$$x_0^l - x_0^{l-1} \leq (P^l)^T D_\alpha^l s^l - (P^l)^T D_\beta^l b^l, \quad (16)$$

where D_α^l and D_β^l are the diagonal matrices as follows:

$$D_\alpha^l = \begin{bmatrix} (1 - \alpha_1^l) & & \\ & \ddots & \\ & & (1 - \alpha_n^l) \end{bmatrix}$$

$$D_\beta^l = \begin{bmatrix} (1 + \beta_1^l) & & \\ & \ddots & \\ & & (1 + \beta_n^l) \end{bmatrix}.$$

Now, we define $\lambda_s^l = D_\alpha^l s^l$ and $\lambda_b^l = D_\beta^l b^l$, which are deterministic. So the constraint in (16) can be written as:

$$x_0^l - x_0^{l-1} \leq (P^l)^T (\lambda_s^l - \lambda_b^l).$$

According to the definition of uncertainty region, the robust version of the constraint may be obtained by:

$$x_0^l - x_0^{l-1} \leq (P^l)^T (\lambda_s^l - \lambda_b^l), P^l \in U^l(q, r).$$

Using the worst-case approach:

$$x_0^l - x_0^{l-1} \leq \inf_{P^l \in U^l} [(P^l)^T (\lambda_s^l - \lambda_b^l)]$$

$$x_0^l - x_0^{l-1} \leq \inf_{\|u\|_q \leq r} [(\bar{P}^L)^T (\lambda_s^l - \lambda_b^l) + u^T (W^l)^T (\lambda_s^l - \lambda_b^l)]$$

$$x_0^l - x_0^{l-1} \leq (\bar{P}^l)^T (\lambda_s^l - \lambda_b^l) - \sup_{\|u\|_q \leq r} u^T (W^l)^T (\lambda_s^l - \lambda_b^l)$$

$$x_0^l - x_0^{l-1} \leq (\bar{P}^l)^T (\lambda_s^l - \lambda_b^l) - r \|(W^l)^T (\lambda_s^l - \lambda_b^l)\|_p.$$

Therefore, the corresponding deterministic model of a general robust counterpart is:

$$\text{Max}_{x,s,b,t} t$$

$$(\bar{P}^L)^T x^L - r \|(W^L)^T x^L\|_p \geq t$$

$$\begin{aligned} & (\bar{P}^l)^T (D_\alpha^l s^l - D_\beta^l b^l) \\ & - r \|(W^l)^T (D_\alpha^l s^l - D_\beta^l b^l)\|_p \geq \\ & x_0^l - x_0^{l-1}, \quad l = 1, \dots, L \end{aligned} \quad (17)$$

$$x_j^l = x_j^{l-1} - s_j^l + b_j^l, \quad j = 1, \dots, n, l = 1, \dots, L$$

$$s_j^l \geq 0, \quad j = 1, \dots, n, l = 1, \dots, L$$

$$b_j^l \geq 0, \quad j = 1, \dots, n, l = 1, \dots, L$$

$$x_j^l \geq 0, \quad j = 1, \dots, n, l = 1, \dots, L.$$

We obtain a general robust counterpart of the uncertain model. This model can guarantee that the original constraint will "almost surely" be satisfied depending on r and norms.

The resulting problem has nonlinear constraints because of the general norm formulations. However, these constraints can be written as SOCP and LP constraints and result in SOCPs and LPs optimization problems respectively. In the next section, we derive some versions of the general robust multi-period portfolio selection models.

4.1. Robust multi-period portfolio selection model corresponding to the l_1 -norm

At the first constraint and using l_1 -norm in the constraints of general model in (17) we have:

$$\begin{aligned} \|W^L x^L\|_1 &= |w_{11}^L x_1^L + \dots + w_{1n}^L x_n^L| + \\ &\dots + |w_{n1}^L x_1^L + \dots + w_{nn}^L x_n^L|, \end{aligned}$$

where w_{kj}^L , $j = 1, \dots, n$, $k = 1, \dots, n$ are elements of W^L . Since we want to make the constraint linear, we define two auxiliary variables as follows:

$$|w_{k1}^L x_1^L + \dots + w_{kn}^L x_n^L| = z_k^{L+} + z_k^{L-}, \quad k = 1, \dots, n.$$

The above equations are equivalent to:

$$w_{k1}^L x_1^L + \dots + w_{kn}^L x_n^L = z_k^{L+} - z_k^{L-}$$

$$z_k^{L+} \geq 0 \quad \text{and} \quad z_k^{L-} \geq 0$$

$$z_k^{L+} z_k^{L-} = 0.$$

The last condition will be satisfied automatically by any basic feasible solution (e.g., an optimal solution delivered by the simplex algorithm); therefore, the first constraint becomes linear as:

$$(\bar{P}^L)^T x^L - r \sum_{k=1}^n (z_k^{L+} + z_k^{L-}) \geq t.$$

By extension, the above concept for the other constraints in (17) yields the following formulation:

Max _{x,s,b,t} t

$$(\bar{P}^L)^T x^L - r \sum_{k=1}^n (z_k^{L+} + z_k^{L-}) \geq t$$

$$(\bar{P}^l)^T (D_\alpha^l s^l - D_\beta^l b^l) - r \sum_{k=1}^n (y_k^{l+} + y_k^{l-}) \geq x_0^l - x_0^{l-1}, \quad l=1, \dots, L \quad (18)$$

$$x_j^l = x_j^{l-1} - s_j^l + b_j^l, \quad j=1, \dots, n, \quad l=1, \dots, L$$

$$s_j^l \geq 0, \quad j=1, \dots, n, \quad l=1, \dots, L$$

$$b_j^l \geq 0, \quad j=1, \dots, n, \quad l=1, \dots, L$$

$$x_j^l \geq 0, \quad j=1, \dots, n, \quad l=1, \dots, L,$$

where

$$\left| w_{k1}^l \lambda_{sb1}^l + \dots + w_{kn}^l \lambda_{sbn}^l \right| = y_k^{l+} + y_k^{l-}, \quad k=1, \dots, n,$$

and

$$\lambda_{sbj}^l = (1 - \alpha_j^l) s_j^l - (1 + \beta_j^l) b_j^l, \quad j=1, \dots, n.$$

This is a new robust formulation for the multi-period uncertain portfolio problem (13) associated with the l_1 -norm.

4.2. Robust multi-period portfolio selection model corresponding to the l_2 -norm

Choosing l_2 -norm in the first constraint of (17):

$$\|W^L x^L\|_2 = \sqrt{(w_{11}^L x_1^L + \dots + w_{ln}^L x_n^L)^2 + \dots + (w_{n1}^L x_1^L + \dots + w_{nn}^L x_n^L)^2}$$

and substituting it in the constraint, we obtain:

$$(\bar{P}^L)^T x^L - r \sqrt{(w_{11}^L x_1^L + \dots + w_{ln}^L x_n^L)^2 + \dots + (w_{n1}^L x_1^L + \dots + w_{nn}^L x_n^L)^2} \geq t$$

By extension, the above concept for the other constraints in (17) yields the following formulation:

Max _{x,s,b,t} t

$$(\bar{P}^L)^T x^L - r \sqrt{(w_{11}^L x_1^L + \dots + w_{ln}^L x_n^L)^2 + \dots + (w_{n1}^L x_1^L + \dots + w_{nn}^L x_n^L)^2} \geq t$$

$$(\bar{P}^l)^T (D_\alpha^l s^l - D_\beta^l b^l) -$$

$$r \sqrt{(w_{11}^l \lambda_{sb1}^l + \dots + w_{1n}^l \lambda_{sbn}^l)^2 + \dots + (w_{n1}^l \lambda_{sb1}^l + \dots + w_{nn}^l \lambda_{sbn}^l)^2} \geq x_0^l - x_0^{l-1}, \quad l=1, \dots, L$$

$$x_j^l = x_j^{l-1} - s_j^l + b_j^l, \quad j=1, \dots, n, \quad l=1, \dots, L \quad (19)$$

$$s_j^l \geq 0, \quad j=1, \dots, n, \quad l=1, \dots, L$$

$$b_j^l \geq 0, \quad j=1, \dots, n, \quad l=1, \dots, L$$

$$x_j^l \geq 0, \quad j=1, \dots, n, \quad l=1, \dots, L.$$

This is an SOCP model and similar to a 3-sigma approach, which was introduced in Ben-Tal and et al. [2].

4.3. Robust multi-period portfolio selection model corresponding to the l_∞ -norm

Taking l_∞ -norm and assuming

$$\|(W^L)^T x^L\|_\infty = \max_k (|w_{k1}^L x_1^L + \dots + w_{kn}^L x_n^L|) = \psi^L,$$

and substituting it in the first constraint in (17), we obtain:

$$(\bar{P}^L)^T x^L - r \psi^L \geq t$$

$$|w_{k1}^L x_1^L + \dots + w_{kn}^L x_n^L| \leq \psi^L, \quad k=1, \dots, n.$$

Similarly, proceed to the second constraint:

$$(\bar{P}^l)^T \lambda_{sb}^l - r \|(W^l)^T \lambda_{sb}^l\|_\infty \leq x_0^l - x_0^{l-1},$$

then it can be substituted in the first constraint in (17) to obtain:

$$(\bar{P}^l)^T \lambda_{sb}^l - r\tau^l \leq x_0^l - x_o^{l-1}.$$

Then, the optimization problem becomes:

$$\text{Max}_{x,s,b,t} t$$

$$(\bar{P}^L)^T x^L - r\psi^L \geq t$$

$$-\psi^L \leq w_{k1}^L x_1^L + \dots + w_{kn}^L x_n^L \leq \psi^L, k=1, \dots, n$$

$$(\bar{P}^l)^T (D_{\alpha}^l s^l - D_{\beta}^l b^l) - r\tau^l \geq x_0^l - x_o^{l-1}, l=1, \dots, L$$

$$-\tau^l \leq w_{k1}^l [(1 - \alpha_1^l) s_1^l - (1 + \beta_1^l) b_1^l] + \dots + w_{kn}^l [(1 - \alpha_n^l) s_n^l - (1 + \beta_n^l) b_n^l] \leq \tau^l, \\ l=1, \dots, L, k=1, \dots, n$$

$$x_j^l = x_j^{l-1} - s_j^l + b_j^l, j=1, \dots, n, l=1, \dots, L \quad (20)$$

$$s_j^l \geq 0, j=1, \dots, n, l=1, \dots, L$$

$$b_j^l \geq 0, j=1, \dots, n, l=1, \dots, L$$

$$x_j^l \geq 0, j=1, \dots, n, l=1, \dots, L.$$

This is a new robust formulation for multi-period uncertain portfolio problem (13) associated with the l_{∞} -norm.

4.4. Example 2

Suppose that there are eight different assets in the market and we are going to consider three periods in our planning horizon. Model (17) needs nominal values and covariance matrices for different periods which are related to the specification of uncertainty regions. Here in order to simplify, we assume that the covariance matrix does not change during the planning horizon, only the nominal values are changing period to period. Figure 6 shows the efficient frontiers (EF) using different norms in models (17) which are nonsmooth and nonconvex. This issue for multi period EF has been also reported in Mulvey [20].

In Figure 7, EFs have been depicted when risk measurements are considered with different l_p -norms. The graphs in the left side of Figure 7 are related to the l_1 -norm risk measurement for different

l_p -norm solutions. The graphs in the middle of Figure 7 are related to the l_2 -norm risk measurement for different l_p -norm solutions and the graphs in the right side of figure 7 is related to the l_{∞} -norm risk measurement for different l_p -norm solutions. The EFs show that any norm can be applicable to measure dispersion.

As shown in Figure 8, the value of the robust counterpart of the total return for the l_{∞} -norm is the largest. This is what we obtained in proposition 4. It can be concluded that when the uncertainty is small (r less than 0.2), there is no significant differences among solutions using different norms but when the uncertainty is large (r greater than 0.5) the l_{∞} -norm is the best choice.

5. Conclusion

In this paper, a general definition for uncertainty sets was given based on the definition of norm bodies. The uncertainty region size can be adjusted by the radius and the degree of norms. The robust counterpart of the uncertain LP corresponding to the general uncertainty region leads us to a family of robust counterparts (FRC). For a given radius of the uncertainty region and the same l_q -norm for all partial uncertainty sets, we obtained the following properties:

1. The complete uncertainty set with the l_{∞} -norm body leads to the largest uncertainty set among solutions using different l_q -norm bodies and using the l_1 -norm results in the smallest one.
2. The largest complete uncertainty set leads to the smallest robust counterpart feasibility set and vice versa.
3. If the smallest robust counterpart feasibility set is not empty, the other robust counterpart feasibility sets using different l_p -norms are feasible.
4. The largest uncertainty set leads to the smallest value of robust counterpart objective function and vice versa.
5. Any family of robust counterpart has a dual one in which it has a smaller uncertainty region, a larger counterpart feasibility set and also a larger robust counterpart objective function provided the degree of norm in the primal robust counterpart is less than the dual one.

The above properties are tested by the application of portfolio selection with single and multi-periods.

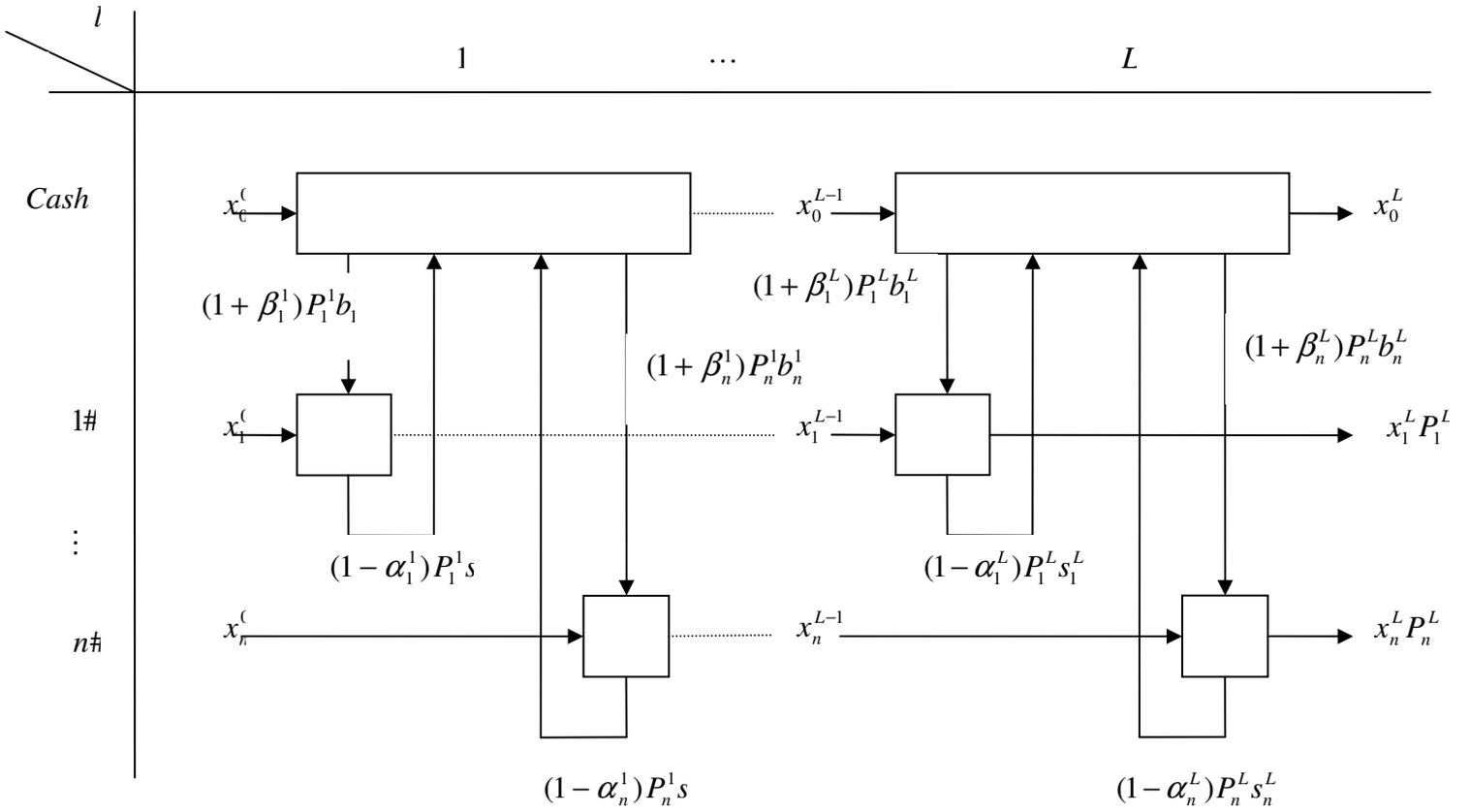


Figure 5. The cash and stock flows during the planning horizon.

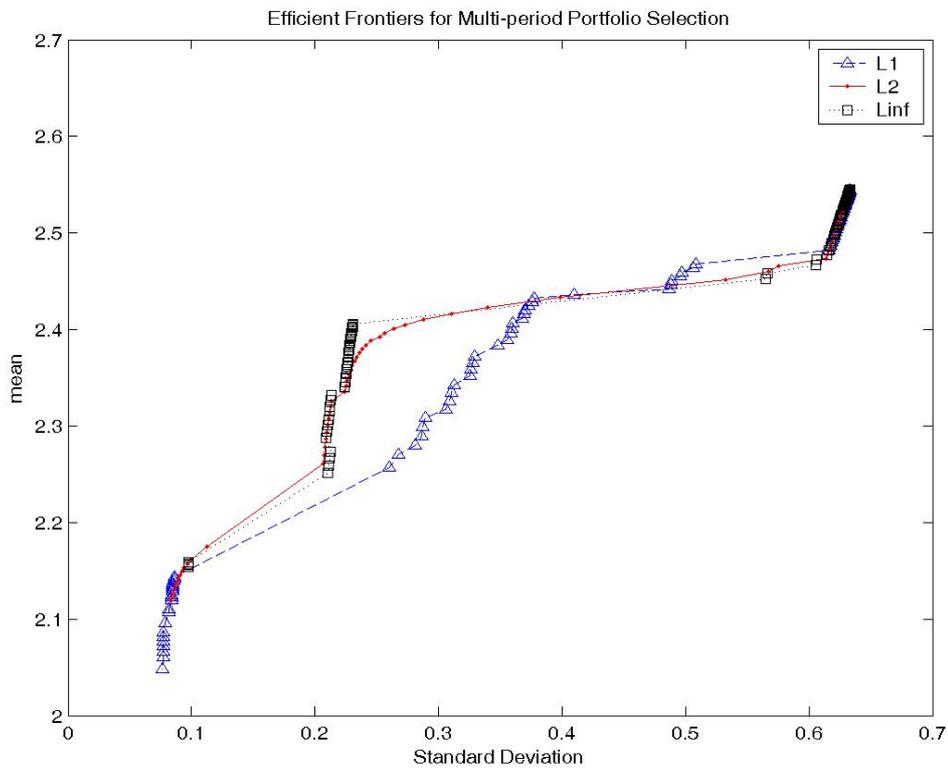


Figure 6. The comparison of efficient frontiers of 8 dependent stocks using standard deviation as a risk measurement in multi-period portfolio selection.

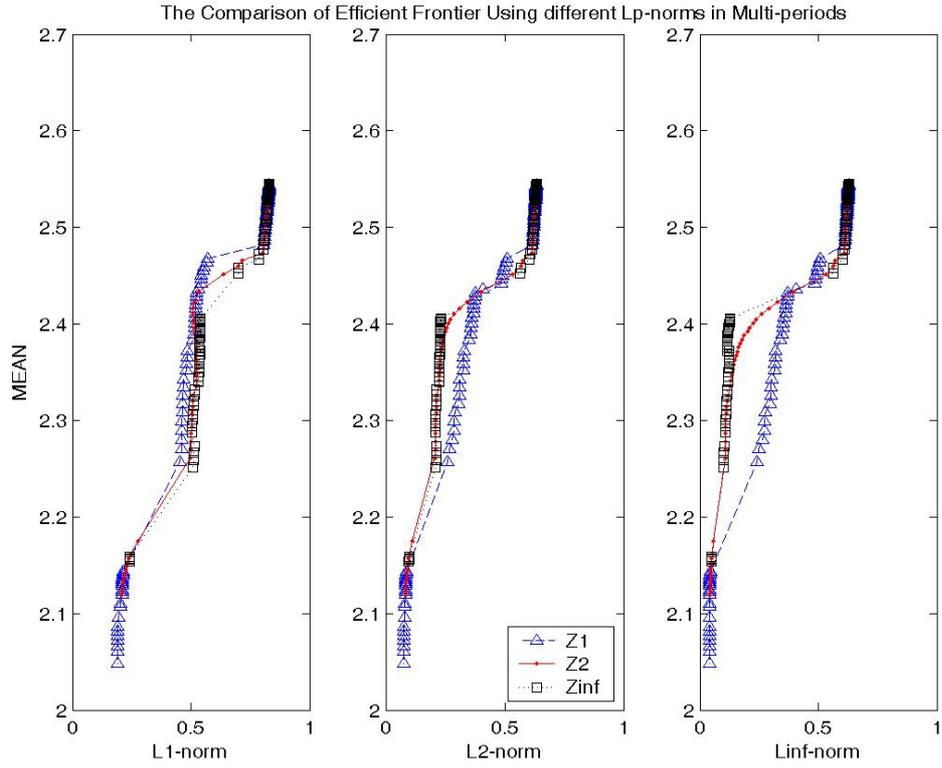


Figure 7. The comparison of efficient frontier of 8 dependent stocks using different l_p -norms as risk measurements in multi-period portfolio selection.

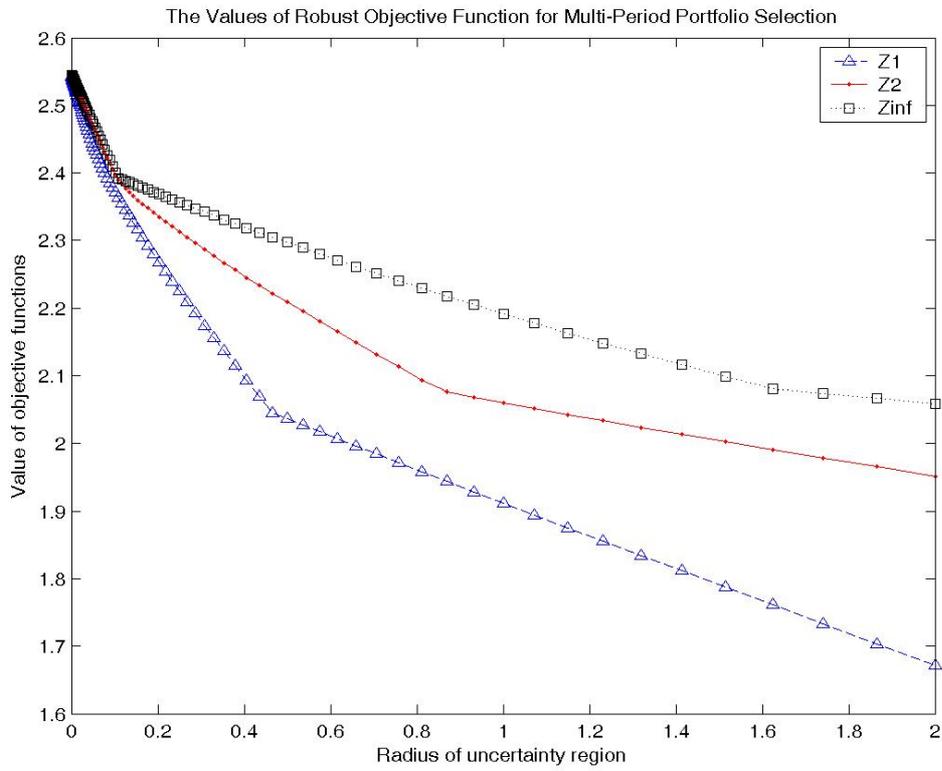


Figure 8. The value of optimal robust counterpart objective functions with different l_p -norms in multi-period portfolio selection.

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