Original Research

Characterizing a Subset of the PPS with Radial Projection Point on a Prespecified Hyperplane

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Abstract
In an approach proposed, Nasrabadi et al. (2014) characterized a subset of production points, the radial projection of which is located on the same facet of the production possibility set (PPS). They obtained the radial projection points by using CCR and BCC models. Some results were posited, which can help one obtain such a subset of the PPS. The sensitivity analysis of inefficient units is also provided. An interval has been achieved over which an individual input/output can be varied and, even then, its corresponding hyperplane does not change. In their proposed approach, two nonlinear programming problems need to be solved to estimate the above mentioned interval. These are, however, difficult to solve. In this paper, some new theorems have been proved so as to obtain a new formula to determine a subset of production points, the projection of which lies on the same hyperplane of the PPS. This new formula leads to the determination of the input preservation region and the output preservation region by solving two linear programming problems that have priority in calculation over the existing methods. To delineate our new approach, two numerical examples are provided at the end.

Keywords - DEA; reference hyperplane; radial projection point; preservation region; sensitivity analysis

INTRODUCTION

One of the problems which management always faces is selection of the best option among the available options or prioritizes their grading [5]. Cahrnes et al. [3] proposed one of the best methods to evaluate the relative efficiency of decision-making units, called data envelopment analysis (DEA). The preliminary DEA model (Charnes, Cooper, and Rhodes [CCR] model) considers the best set of weights for the single ratio of the weighted outputs to the weighted inputs for a particular decision-making unit (DMU), which is denoted by DMUo. The advantage of DEA over the previous methods is that the relative efficiency of DMUs can be evaluated with multiple inputs and multiple outputs under an assumption of constant or variable returns to scale (RTS) of the production technology. Not only can the evaluation of the relative efficiency of DMUs be gained by DEA, in addition, the identification of the benchmark DMU for inefficient DMUs can be another capability of DEA. This is because for each inefficient DMU, the projection point (which has been positioned on the efficient frontier) can be defined. In addition, a supporting hyperplane *H* of the reference production possibility set (PPS) can be obtained by the CCR or the BCC multiplier model (see Cooper et al. [4]), where the radial projection point of the unit under evaluation has been positioned. Podinovski [9] mentioned that this supporting hyperplane plays a major role in determining the economic rate of trade-off for each DMU. Furthermore, it presents a technology in terms of a production function, considered a reference technology for the associated DMU. On the other hand, the importance of the determination of the RTS class of DMUs has motivated many researchers to consider the hyperplanes where the DMUs are embedded or projected. In addition, the applications of the radial projection in organizations are worthy to mention, for instance, Pekka et al. [10] applied it to Helsinki School of Economics where the students had the enough opportunity to choose their personalized efficient frontier. Saati et al. [11] defined an ideal point and illustrated it work in the case study using panel data from 286 Danish district heating plants. Cooper et al. [4] mentioned that all DMUs located or projected on a specific hyperplane belong to the same class, or belong to two close classes of RTS. Eventually, Nasrabadi et al. [8] expressed the importance of determining the preservation region for maintaining the RTS status. They characterized the subset of the PPS consisting of the production points of which the radial projection points lie on the same hyperplane in two different returns to scale assumptions of the reference technology (CCR and BCC models). They proposed some models to obtain a range over which an individual input/output can vary, and, even so, the radial projection point is located on the same hyperplane. They also addressed this range as the input preservation region for inputs and the output preservation region for outputs. Their models are nonlinear and difficult to solve. In this paper, we prove some new theorems to obtain a new formula to determine a subset of production points, such that their projection lies on the same supporting hyperplane of the PPS. This new formula leads to the determination of an input or output preservation region by solving two linear programming problems, which is much easier to solve than previous nonlinear programs. Generally, we obtained the subset of the PPS by a new method, which leads to a new linear programming problem. The paper proceeds as follows: In Section 2, the preliminary DEA models are introduced. Section 3 presents the new theorems and results. Numerical examples and the conclusion are respectively presented in Sections 4 and 5.

PRELIMINARIES

Suppose we deal with n DMUs consisting of DMU_j ; $j=1,\dots,n,$ with input-output vectors (x_j, y_j) ; $j=1,\dots,n.$ Each DMU_j consume m inputs $x_j=(x_{1j},\dots,x_{mj})^t$ to produce s outputs $y_j=(y_{1j},\dots,y_{sj})^t$. The superscript "t" denotes the transpose operator. In the current paper, we assume that all inputs and outputs are positive, i.e., $x_{ij}>0, y_{rj}>0$ for all $i,j,r.$ The ith standard vector is denoted by e_i. The PPS with the production technology $\Lambda \in \{CRS, VRS\}$, including all feasible activity, is denoted by T^{Λ} and can be described as follows:

$$T^{CRS}=\left\{(x,y)\in R^{m+s} \mid x\geq \sum_{j=1}^{n}\lambda_j x_j, 0\leq y\leq \sum_{j=1}^{n}\lambda_j y_j, \lambda_j\geq 0, \forall j\right\} \quad (1)$$

and

$$T^{VRS}=\left\{(x,y)\in R^{m+s} \mid (x,y)\in T^{CRS}, \sum_{j=1}^{n}\lambda_j=1\right\} \quad (2)$$

The envelopment and multiplier forms of CCR and BCC models are based upon CRS and VRS assumptions of the production technologies, and they can be expressed as follows:

Envelopment form

$$\begin{aligned}\theta_o=\ &\min\ \theta\\ &s.t.\quad (\theta x_o, y_o)\in T^{\Lambda}. \quad (A1)\end{aligned}$$

Multiplier form

$$\begin{aligned}\theta_o=\ &\max\ uy_o+u_0,\\ &s.t.\quad vx_o=1,\\ &\qquad uy_j-vx_j+u_0\leq 0, \forall j\\ &\qquad u\geq 0, v\geq 0,\\ &\qquad u_0\in\Gamma. \quad (A2)\end{aligned}$$

where $\Gamma^{CRS}=\{0\}$ and $\Gamma^{VRS}=R.$ In mentioned models, (A1) and (A2) θ_o indicates efficiency score of production point (observed or not) (x_o, y_o). Note that the above models are input-oriented CCR and BCC models. All results straightforwardly can be adapted for output-oriented models, as well. Let production point (x_o, y_o) be the point under evaluation. Then, the CCR (BCC)-radial projection point is defined as follows:

$$(\hat{x}_o, \hat{y}_o)=(\theta_o x_o, y_o) \quad (3)$$

$(\hat{x}_o, \hat{y}_o)$ is a radial efficient point which is located on the frontier of reference PPS (Cahrnes et al. [3]).Let (u^*, v^*, u_0^*) be an optimal solution in (A2) when the production point (x_o, y_o) is under evaluation, then

$H_o = \{(x,y) \mid uy - vx + u_0 = 0\}$ is a supporting hyperplane of PPS (see Cooper et al. [4]), i.e.,

- $uy - vx + u_0 \le 0, \forall (x,y) \in T^{\Lambda}$,
- $H_o \bigcap T^{\Lambda} \ne \phi$.

Assumptions and goal. We now formulate our assumption:

- a supporting hyperplane $H = \{(x,y) \mid uy - vx + u_0 = 0, v \ne 0\}$ of PPS is given;
- the PPS with assumption of reference technology T is known.

The following notation will be useful, $R_+^{m+s} = \{(x,y)^t \in R^{m+s} \mid x > 0, y > 0\}$, the positive orthant of R^{m+s}. The objective is to characterize a subset of T such that the radial projection point of each point in this subset is located on H. It is clear that this subset is not empty. In fact, the aim is to find

$P_H = \{(x,y)^t \in T \cap R_+^{m+s} \mid (\hat{x},\hat{y}) \in H\}$,

where $(\hat{x},\hat{y})$ is a radial projection point of the production point (x,y).

Lemma 1 Let $(x_o, y_o) \in R_+^{m+s}$ and $H = \{(x,y) \mid uy - vx + u_0 = 0, v \ne 0\}$ be a supporting hyperplane of

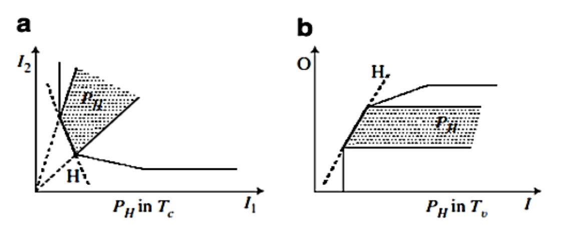

FIGURE 1
ILLUSTRATION OF P_H IN CCR AND BCC MODELS.
Source :NASRABADI ET AL.[7]

PPS. The production point (observed or not) $(x_o, y_o) \in P_H$ if and only if

$$\theta_o = \frac{uy_o + u_0}{vx_o}.$$

Proof. Assume $(x_o, y_o) \in P_H$, then $(\hat{x}_o, \hat{y}_o) = (\theta_o x_o, y_o) \in H$. This means that $uy_o - \theta_o vx_o + u_0 = 0$. Therefore, $\theta_o = \frac{uy_o + u_0}{vx_o}$, since $v \ne 0$ and $x_o \in R_+^m$. On the other hand, if $\theta_o = \frac{uy_o + u_0}{vx_o}$, then we have $uy_o - \theta_o vx_o + u_0 = 0$. This implies that $(\theta_o x_o, y_o) \in H$, and hence $(x_o, y_o) \in P_H$. This completes the proof.□

Throughout the paper, the set K_H denotes all indices of DMUs which lie on the hyperplane H

MAINTAINING THE REFERENCE YPERPLANE

The following lemma provides a necessary and sufficient characterization of P_H.

Lemma2 Let $H = \{(x,y) \mid uy - vx + u_0 = 0, v \ne 0\}$ be a supporting hyperplane on T. Furthermore, assume that $(x', y') \in T \cap R_+^{m+s}$. Then, $(x', y') \in P_H$ if and only if a positive multiplication of (u, v, u_0) is an optimal solution of (A2) when evaluating (x', y').

Let H be a defining hyperplane with normal vector (u,v) as already defined, we define sets Q_H and R_H as follows:

$Q_H = \{r \mid u_r = 0\}$, $R_H = \{i \mid v_i = 0\}$.

The next theorem gives the subset of the PPS consisting of the production points whose radial projection points lie on H, i.e., P_H. It is worthy to note that the following subsets are presented much more easier than the subsets in the previous research [8] .

Theorem1 If $H = \{(x, y) \mid uy - vx + u_0 = 0, v \ne 0\}$ is a supporting hyperplane of PPS, Then

- If $T = T^{CRS}$, then we have

$$P_H = \left\{ (x,y) \mid \theta x = \sum_{j \in K_H} \lambda_j x_j + \sum_{k \in R_H} \mu_k e_k, y = \sum_{j \in K_H} \lambda_j y_j - \sum_{t \in Q_H} \mu'_t e'_t \right\},$$

where in

$\theta = \frac{uy}{vx}, \theta \le 1, y \ge 0, \lambda_j \ge 0; j \in K_H, \mu_k \ge 0; k \in R_H, \mu'_t \ge 0; t \in Q_H$.

- If $T = T_v$, then we have

$$P_H = \left\{ (x,y) \mid \theta x = \sum_{j \in K_H} \lambda_j x_j + \sum_{k \in R_H} \mu_k e_k, y = \sum_{j \in K_H} \lambda_j y_j - \sum_{t \in Q_H} \mu'_t e'_t, \sum_{j=1}^{n} \lambda_j = 1 \right\}$$

wherein

$\theta = \frac{uy + u_0}{vx}, \theta \le 1, y \ge 0, \sum_{j \in K_H} \lambda_j = 1, \lambda_j \ge 0;$
$j \in K_H, \mu_k \ge 0; k \in R_H, \mu'_t \ge 0; t \in Q_H$.

Proof. We set

$$A = \left\{ (x,y) \mid \theta x = \sum_{j \in K_H} \lambda_j x_j + \sum_{k \in R_H} \mu_k e_k, y = \sum_{j \in K_H} \lambda_j y_j - \sum_{t \in Q_H} \mu'_t e'_t \right\} (4)$$

where

$\theta=\frac{uy}{vx},\theta\le 1, y\ge 0,\lambda_j\ge 0; j\in K_H,\mu_k\ge 0; k\in R_H,\mu`_t\ge 0; t\in Q_H$.

To prove the theorem, it is sufficient to show that $A=P_H$.

If $(x,y)\in A,$ then there exist some (λ,μ,μ') such that

$$\theta x=\sum_{j\in K_H}\lambda_j x_j+\sum_{k\in R_H}\mu_k e_k\ge\sum_{j\in K_H}\lambda_j y_j,$$
$$y=\sum_{j\in K_H}\lambda_j y_j-\sum_{t\in Q_H}\mu'_t e'\le\sum_{j\in K_H}\lambda_j y_j \quad (5)$$

From $\theta\le 1,$ we have $\theta x\le x.$ This implies that $(x,y)\in T^{CRS}$. Now, we show that $(\theta x,y)\in H$. Since $(x,y)\in A,$ we have

$$uy-v\theta x=\sum_{j\in K_H}\lambda_j(uy_j-vx_j)-\sum_{t\in Q_H}\mu'_t ue'_t-\sum_{k\in R_H}\mu_k ve_k=0.$$

The last equality is equal to zero follows from $(x_j,y_j)\in H;$ for each $j\in K_H$, and $ue'_t=u_t=0;$ for each $t\in Q_H$ and $ve_k=v_k=0;$ for each $k\in R_H$. This implies that $(\hat{x},\hat{y})=(\theta x,y)\in H$. Now, we show that θ in Equation(4) is the efficiency score of the production point (x,y). By contradiction, assume that θ^* is the efficiency score of (x,y) , and $\theta^*<\theta.$ We have

$uy-v\theta^* x>uy-v\theta x=0,$

since $\theta^* x<\theta x$ and $v\ne 0.$ This contradicts the assumption that H is supporting hyperplane of T^{CRS}. This implies that θ is the efficiency score of the production point (x,y). In summary, if $(x,y)\in A,$ then

- $(x,y)\in T^{CRS}$,
- θ is the efficiency score of the production point (x,y).
- $(\theta x,y)\in H.$

These in turn, imply that $(x,y)\in P_H$, and therefore $A\subseteq P_H$. On the other side, if $(x,y)\in P_H$, then $(\hat{x},\hat{y})=(\theta^* x,y)\in T^{CRS}$. Note that θ^* is the radial efficiency of the production unit (x,y). Therefore, there exists some optimal solution $(\lambda^*,\theta^*)\ge 0$ and slack variables $(s^-,s^+)\ge 0$ for Model (A1) such that

$$\theta^* x=\sum_{j=1}^{n}\lambda_j^* x_j+\sum_{k=1}^{m}s_k^- e_k,\quad y=\sum_{j=1}^{n}\lambda_j^* y_j-\sum_{t=1}^{s}s_t^+ e_t$$

Since $(\theta^* x,y)\in H,$ and H is a supporting hyperplane at T, it is clear that $\theta^*=\frac{uy}{vx}\le 1.$ Notice that when $\lambda_j^*>0,$ then $j\in K_H$. Also, if $s_k^->0$ for some $k\in\{1,\ldots,m\}$ or $s_t^+>0,$ for some $t\in\{1,\ldots,s\},$ then $v_k=0$ or $u_t=0,$ based on the complementary slackness conditions (see Bazarra et al. [2]). Thus, we have

$$\theta^* x=\sum_{j\in K_H}\lambda_j^* x_j+\sum_{k\in R_H}s_k^- e_k,\quad y=\sum_{j\in K_H}\lambda_j^* y_j-\sum_{t\in Q_H}s_t^+ e_t$$

This means that $(x,y)\in A.$ Therefore $P_H\subseteq A,$ and the proof of part 1 is completed. The proof of part 2 is similar and is hence omitted. □

SENSITIVITY ANALYSIS

After determining the set P_H, the sensitivity analysis of inefficient DMUs can be performed. They assume that the DMU_o under evaluation is inefficient (i.e., $\theta_o^*<1$ in Model (A1)) , and also that its radial projection is located on Hyperplane H. They also provide some models to obtain an "input preservation region" and an " output preservation region," where the corresponding radial projection point is located on the same hyperplane. Their models are nonlinear and difficult to solve. In this paper, the above mentioned range is obtained by solving two new linear programming problems. A range can be obtained over which each individual input or output of the DMU_o can be varied, without changing its hyperplane, H. For instance, to determine the range over which an input of the DMU_o can be varied, and yet the radial projection point is located on H, the next two problems have to be solved:

$$\begin{aligned}\bar{\delta}_t = \;& \max_{\delta_t\ge 0}\ \delta_t\\ & s.t.\quad (x_{1o},\ldots,\delta_t x_{to},\ldots,x_{mo},y_{1o},\ldots,y_{so})\in P_H.\end{aligned}\quad (6)$$

$$\begin{aligned}\underline{\delta}_t = \;& \min_{\delta_t\ge 0}\ \delta_t\\ & s.t.\quad (x_{1o},\ldots,\delta_t x_{to},\ldots,x_{mo},y_{1o},\ldots,y_{so})\in P_H.\end{aligned}\quad (7)$$

Note that the above models are feasible, since $(x_o,y_o)\in P_H$, $\delta_t=1$ is a feasible solution for. (6), (7) Also, it is worthwhile to note that in the case $v_t=0,$ we have $\bar{\delta}_t=+\infty$ (see[6], [7]).Based upon the definition of P_H the following nonlinear models give the range of δ_t.

$$
\begin{aligned}
\bar{\delta}_t = \quad & \max \quad \delta_t \\
s.t. \quad & \sum_{j \in K_H} \lambda_j x_{ij} + \mu_i = \tfrac{uy_o + u_0}{vx_o + (\delta_t - 1)v_t x_{to}} x_{io}, \quad i \in R_H, i \neq t \\
& \sum_{j \in K_H} \lambda_j x_{ij} = \tfrac{uy_o + u_0}{vx_o + (\delta_t - 1)v_t x_{to}} x_{io}, \quad i \notin R_H, i \neq t \\
& \sum_{j \in K_H} \lambda_j x_{tj} \leq \tfrac{uy_o + u_0}{vx_o + (\delta_t - 1)v_t x_{to}} \delta_t x_{to}, \quad r \in Q_H, \\
& \sum_{j \in K_H} \lambda_j y_{rj} - \mu'_r = y_{ro}, \\
& \sum_{j \in K_H} \lambda_j y_{rj} = y_{ro}, \quad r \notin Q_H, \\
& \sum_{j \in K_H} \lambda_j = 1, \\
& \lambda_j \geq 0; \forall j \in K_H, \ \mu_i \geq o; \\
& \forall i \in R_H, \mu'_r \geq 0; \forall r \in Q_H, \qquad (8)
\end{aligned}
$$

The next theorem provides two LP problems to obtain the interval $[\underline{\delta}_t x_{to}, \bar{\delta}_t x_{to}]$ over which the t-th input of DMU_o can be varied and still its radial projection point is located on H.

$$
\begin{aligned}
\underline{\delta}_t = \quad & \min \quad \delta_t \\
s.t. \quad & \sum_{j \in K_H} \lambda_j x_{ij} + \mu_i = \tfrac{uy_o + u_0}{vx_o + (\delta_t - 1)v_t x_{to}} x_{io}, \quad i \in R_H, i \neq t \\
& \sum_{j \in K_H} \lambda_j x_{ij} = \tfrac{uy_o + u_0}{vx_o + (\delta_t - 1)v_t x_{to}} x_{io}, \quad i \notin R_H, i \neq t, \\
& \sum_{j \in K_H} \lambda_j x_{tj} \leq \tfrac{uy_o + u_0}{vx_o + (\delta_t - 1)v_t x_{to}} \delta_t x_{to}, \\
& \sum_{j \in K_H} \lambda_j y_{rj} - \mu'_r = y_{ro}, \quad r \in Q_H, \\
& \sum_{j \in K_H} \lambda_j y_{rj} = y_{ro}, \quad r \notin Q_H, \\
& \sum_{j \in K_H} \lambda_j = 1, \\
& \lambda_j \geq 0; \forall j \in K_H, \ \mu_i \geq o; \\
& \forall i \in R_H, \mu'_r \geq 0; \forall r \in Q_H, \qquad (9)
\end{aligned}
$$

*Theorem*2 we have

$$
\begin{aligned}
\bar{\delta}_t = \quad & \max \quad \delta_t \\
s.t. \quad & \sum_{j \in K_H} \hat{\lambda}_j x_{ij} + \hat{\mu}_i = (uy_o + u_0) x_{io}, \quad i \in R_H, i \neq t \\
& \sum_{j \in K_H} \hat{\lambda}_j x_{ij} = (uy_o + u_0) x_{io}, \quad i \notin R_H, i \neq t \\
& \sum_{j \in K_H} \hat{\lambda}_j x_{tj} \leq (uy_o + u_0) \delta_t x_{to}, \\
& \sum_{j \in K_H} \hat{\lambda}_j y_{rj} - \hat{\mu}'_r = (vx_o + (\delta_t - 1)v_t x_{to}) y_{ro}, \quad r \in Q_H, \\
& \sum_{j \in K_H} \hat{\lambda}_j y_{rj} = (vx_o + (\delta_t - 1)v_t x_{to}) y_{ro}, \quad r \notin Q_H, \\
& \sum_{j \in K_H} \hat{\lambda}_j = vx_o + (\delta_t - 1)v_t x_{to}, \\
& \delta_t \geq 0, \hat{\lambda}_j \geq 0; \forall j \in K_H, \hat{\mu}_i \geq 0; \\
& \forall i \in R_H, \hat{\mu}'_i \geq 0; \forall i \in Q_H. \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\delta}_t = \quad & \min \quad \delta_t \\
s.t. \quad & \sum_{j \in K_H} \hat{\lambda}_j x_{ij} + \hat{\mu}_i = (uy_o + u_0) x_{io}, \quad i \in R_H, i \neq t, \\
& \sum_{j \in K_H} \hat{\lambda}_j x_{ij} = (uy_o + u_0) x_{io}, \quad i \notin R_H, i \neq t, \\
& \sum_{j \in K_H} \hat{\lambda}_j x_{tj} \leq (uy_o + u_0) \delta_t x_{to}, \\
& \sum_{j \in K_H} \hat{\lambda}_j y_{rj} - \hat{\mu}'_r = (vx_o + (\delta_t - 1)v_t x_{to}) y_{ro}, \quad r \in Q_H, \\
& \sum_{j \in K_H} \hat{\lambda}_j y_{rj} = (vx_o + (\delta_t - 1)v_t x_{to}) y_{ro}, \quad r \notin Q_H, \\
& \sum_{j \in K_H} \hat{\lambda}_j = vx_o + (\delta_t - 1)v_t x_{to}, \\
& \delta_t \geq 0, \hat{\lambda}_j \geq 0; \forall j \in K_H, \hat{\mu}_i \geq 0; \\
& \forall i \in R_H, \hat{\mu}'_r \geq 0; \forall r \in Q_H. \qquad (11)
\end{aligned}
$$

*Note***.** The above two models are linear and can be solved by any standard LP software. It is easy to show that if $(\lambda_j^*; j \in K_H, \mu_i^*; i \in R_H, \mu_r'^*; r \in Q_H, \bar{\delta}_t)$ is an optimal solution for (8), then the solution is $(\hat{\lambda}_j, \hat{\mu}_i, \hat{\mu}'_r, \delta_t)$ where

$$
\begin{aligned}
& \hat{\lambda}_j = (vx_o + (\delta_t - 1)v_t x_{to}) \lambda_j^*; \quad j \in K_H, \\
& \hat{\mu}_i = (vx_o + (\delta_t - 1)v_t x_{to}) \mu_i^*; \quad i \in R_H, \\
& \hat{\mu}'_r = (vx_o + (\delta_t - 1)v_t x_{to}) \mu_r'^*; \quad r \in Q_H, \\
& \delta_t = \bar{\delta}_t, \qquad (12)
\end{aligned}
$$

is a feasible solution for (10) and the objective function value related to this feasible solution is $\bar{\delta}_t$. Since the objective function is in the form of maximization, we have the optimal objective value (10) greater than or equal to $\bar{\delta}_t$. On the other side, if

$$(\hat{\lambda}_j^*; j \in K_H, \ \hat{\mu}_i^*; \ i \in R_H, \ \hat{\mu}_r'^*; r \in Q_H, \delta_t^*)$$

is an optimal solution to (10), then $(\lambda_j, \mu_i, \mu'_r, \delta_t)$ with

$$
\begin{aligned}
& \lambda_j = \frac{\hat{\lambda}_j^*}{(vx_o + (\delta_t^* - 1)v_t x_{to})}; \quad j \in K_H, \\
& \mu_i = \frac{\hat{\mu}_i^*}{(vx_o + (\delta_t^* - 1)v_t x_{to})}; \quad i \in R_H, \\
& \mu'_r = \frac{\hat{\mu}_r'^*}{(vx_o + (\delta_t^* - 1)v_t x_{to})}; \quad r \in Q_H, \\
& \delta_t = \delta_t^*, \qquad (13)
\end{aligned}
$$

is a feasible solution to (8) and the objective value corresponding to this solution is δ^*. The objective function of (8) is in the form of maximization. Therefore, $\bar{\delta}_t \geq \delta^*$. These imply that the optimal objective values of (8) and (10) are equal. With the same process, it can be shown that the optimal objective values of Models (9) and (11) are equal. The proof of this theorem is completed.

Proportional changes in inputs can be handled similarly. Concerning the above approach, we can determine a range for proportional changes in outputs, which allows DMU_o to have its radial projection point on H. To characterize the preservation range for the outputs of DMU_o to be projected on H, we have the following two mathematical programming problems:

$$\bar{\delta} = \max_{\delta\geq 0} \ \delta \quad s.t. \quad (x_o, \delta y_o) \in P_H \qquad \underline{\delta} = \min_{\delta \geq 0} \ \delta_t \quad s.t. \quad (x_o, \delta y_o) \in P_H$$

Regarding Theorem1, we have to solve the following two linear models:

$$\begin{aligned} \bar{\delta} = \ & \max \quad \delta \\ s.t. \ & \sum_{j\in K_H} \lambda_j x_{ij} + \mu_i = \tfrac{\delta u y_o + u_0}{v x_o} x_{io}, & i \in R_H, \\ & \sum_{j\in K_H} \lambda_j x_{ij} = \tfrac{\delta u y_o + u_0}{v x_o} x_{io}, & i \notin R_H, \\ & \sum_{j\in K_H} \lambda_j y_{rj} - \mu'_r = \delta y_{ro}, & r \in Q_H, \\ & \sum_{j\in K_H} \lambda_j y_{rj} = \delta y_{ro}, & r \notin Q_H, \\ & \sum_{j\in K_H} \lambda_j = 1, \\ & \lambda_j \geq 0, & j \in K_H, \\ & \tfrac{\delta u y_o + u_0}{v x_o} \leq 1, \\ & \delta \geq 0, & (14) \end{aligned}$$

$$\begin{aligned} \underline{\delta} = \ & \min \quad \delta \\ s.t. \ & \sum_{j\in K_H} \lambda_j x_{ij} + \mu_i = \tfrac{\delta u y_o + u_0}{v x_o} x_{io}, & i \in R_H, \\ & \sum_{j\in K_H} \lambda_j x_{ij} = \tfrac{\delta u y_o + u_0}{v x_o} x_{io}, & i \notin R_H, \\ & \sum_{j\in K_H} \lambda_j y_{rj} - \mu'_r = \delta y_{ro}, & r \in Q_H, \\ & \sum_{j\in K_H} \lambda_j y_{rj} = \delta y_{ro}, & r \notin Q_H, \\ & \sum_{j\in K_H} \lambda_j = 1, \\ & \lambda_j \geq 0, & j \in K_H, \\ & \tfrac{\delta u y_o + u_0}{v x_o} \leq 1, \\ & \delta \geq 0, & (15) \end{aligned}$$

In the above two models, the vector variables are $(\lambda, \mu, \mu', \delta)$.

Example 1 Consider an example, used by Nasrabadi et al. [8], consisting of seven DMUs, in which each DMU consumes two inputs to produce a single constant output equal to one. The data are presented in Table 1, see also [4].

TABLE I
DATA IN EXAMPLE 1

DMUs	A	B	C	D	E	F	G
x_1	4	7	8	4	2	10	3
x_2	3	3	1	2	4	1	7
y	1	1	1	1	1	1	1

The efficient frontier under CRS assumption of the reference technology is shown graphically in Fig. 2. Suppose that the observed unit A is under evaluation. The optimal solution for Model (A2-CRS) is $(u^* = 0.8571, v_1^* = 0.1429, v_2^* = 0.1429))$, and the efficiency measure of this unit is 0.8568 . Thus, the supporting hyperplane in T^{CRS} is obtained as $H_A = \{(x_1, x_2, y): 0.8571y - 0.1429x_1 - 0.1429x_2 = 0\}$. It is observed that $K_{H_A} = \{D, E\}, R_{H_A} = \phi, Q_{H_A} = \phi$. Therefore, set P_{H_A} can be expressed as

$$\begin{aligned} P_{H_A} = \{(x_1, x_2, y): \ & \tfrac{0.8568y}{0.1429x_1 + 0.1429x_2} x_1 = 4\lambda_D + 2\lambda_E, \\ & \tfrac{0.8568y}{0.1429x_1 + 0.1429x_2} x_2 = 2\lambda_D + 4\lambda_E, \\ & y = \lambda_D + \lambda_E, \qquad \lambda_D, \lambda_E \geq 0.\} \end{aligned} \tag{16}$$

We perform the sensitivity analysis for the first input of unit A, the unit under evaluation. We wish to find the range $[\underline{\delta_1} x_{1A}, \bar{\delta}_1 x_{1A}]$, over which the first input of DMU_A varies and it's radial projection point lies on the same hyperplane H_A. Regarding Theorem2, we have to solve the following two new linear programming problems:

$$\begin{aligned} \bar{\delta}_1 = \ & \max \quad \delta \\ s.t. \ & 4\lambda_D + 2\lambda_E = 0.8568 \times 4 \times \delta, \\ & 2\lambda_D + 4\lambda_E = 0.8568 \times 2, \\ & \lambda_D + \lambda_E = (1 + (\delta - 1) \times 0.1429 \times 4) \times 1, \\ & \lambda_D, \lambda_E \geq 0, \delta \geq 0, & (17) \end{aligned}$$

$$
\begin{aligned}
\underline{\delta}_1 = \quad & \min \quad \delta \\
s.t. \quad & 4\lambda_D + 2\lambda_E = 0.8568 \times 4 \times \delta, \\
& 2\lambda_D + 4\lambda_E = 0.8568 \times 2, \\
& \lambda_D + \lambda_E = (1 + (\delta - 1) \times 0.1429 \times 4) \times 1, \\
& \lambda_D, \lambda_E \geq 0, \delta \geq 0 \qquad (18)
\end{aligned}
$$

The optimal objective values of Models (17) and (18) are $\bar{\delta} = \frac{3}{2}$ and $\underline{\delta} = \frac{3}{4}$. We found that if x_{1A} varies over range $[3,6]$, then the projection point remains on the same hyperplane, i.e., H_A.

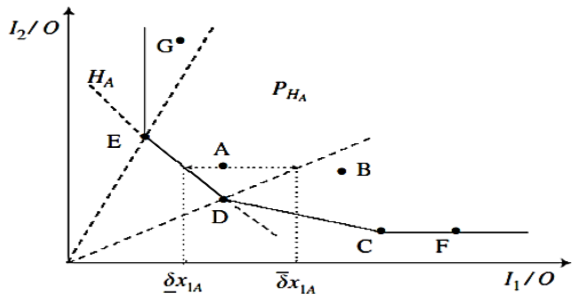

FIGURE 2
DETERMINATION OF P_{H_A} IN EXAMPLE 1.

Example 2 Consider six DMUs, with the input and output (The production frontier is illustrated in Figure3) that are listed in Table 2. To compare the results, this example is selected from Cooper et al. [4]. Assume that DMU_E is the unit under evaluation by Model (A2-VRS). The optimal solution of Model (A2-VRS) is $(u^* = \frac{1}{4}, v^* = \frac{1}{4}, u_0^* = 0)$, and hence, the projection point of DMU_E is located on the hyperplane

$$H_E = \{(x,y): \frac{1}{4}y - \frac{1}{4}x = 0\}.$$

The efficiency score of DMU_E is 0.75.

TABLE 2
DATA IN EXAMPLE 2

DMUs	A	B	C	D	E	F
x	2	4	6	8	4	6
y	2	4	5	5	3	4

As seen $K_{H_E} = \{A, B\}$, and hence the set P_{H_E} can be expressed as

$$
\begin{aligned}
P_{H_A} = \{(x,y): \ & \tfrac{0.25y}{0.25x} x = 2\lambda_A + 4\lambda_B, \\
& y = 2\lambda_A + 4\lambda_B, \\
& 1 = \lambda_A + \lambda_B, \ \lambda_A, \ \lambda_B \geq 0\}.
\end{aligned}
$$

Now, we perform output sensitivity analysis for DMU_E. Regarding Models (14) and (15), we have to solve the following two linear programming problems:

$$
\begin{aligned}
\bar{\delta} = \quad & \max \quad \delta \\
s.t. \quad & 2\lambda_A + 4\lambda_B = 3\delta, \\
& 2\lambda_A + 4\lambda_B = 3\delta, \\
& \lambda_A + \lambda_B = 1, \\
& \delta \leq \tfrac{4}{3}, \\
& \delta, \lambda_A, \lambda_B \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
\underline{\delta} = \quad & \min \quad \delta \\
s.t. \quad & 2\lambda_A + 4\lambda_B = 3\delta, \\
& 2\lambda_A + 4\lambda_B = 3\delta, \\
& \lambda_A + \lambda_B = 1, \\
& \delta \leq \tfrac{4}{3}, \\
& \delta, \lambda_A, \lambda_B \geq 0.
\end{aligned}
$$

The optimal objective values of the above two models are $\bar{\delta} = \frac{4}{3}$ and $\underline{\delta} = \frac{2}{3}$, respectively. This implies if y varies over the range $[2,4]$, the projection point still remains on the hyperplane H_E. Now, we consider another DMU named as DMU_F. The optimal solution of Model (A2-VRS) is $(u^* = \frac{1}{3}, v^* = \frac{1}{6}, u_0^* = -\frac{2}{3})$, when DMU_F is under evaluation, and the efficiency score of unit F is 0.66. Thus, the projection point of DMU_F lies on the following supporting hyperplane:

$$H_F = \{(x,y): \frac{1}{3}y - \frac{1}{6}x - \frac{2}{3} = 0\}$$

We have $K_{H_F} = \{B, C\}$. Therefore, the set P_{H_F} can be represented as

$$
\begin{aligned}
P_{H_F} = \{(x,y): \ & \tfrac{0.33y - 0.66}{0.16667x} x = 4\lambda_B + 6\lambda_C, \\
& y = 4\lambda_B + 5\lambda_C, \\
& 1 = \lambda_B + \lambda_C, \ \lambda_B, \ \lambda_C \geq 0\}.
\end{aligned}
$$

By performing sensitivity analysis in the outputs of

DMU_F , we get the interval $[4,5]$, in which the outputs can be varied and the projection point still remains on the hyperplane H_F.

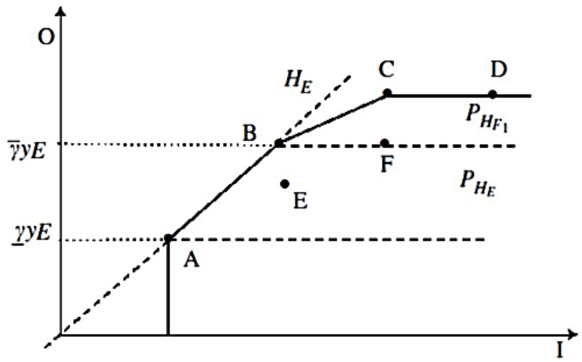

FIGURE 3
THE PRODUCTION FRONTIER OF EXAMPLE 2.

CONCLUSION

In this paper, the set of all projection points in a subset of the PPS is obtained by the prespecified hyperplane. Some new theorems are presented here to show the radial projection points of the subset of the PPS, consisting of the production points, which lie on the prespecified hyperplane. Also, the production possibility set can be partitioned into a finite number of discrete sets. In this study, we found the set of all production points: The input preservation region and the output preservation region are obtained by solving a linear programming problem. We also proved that a range can be obtained over which each individual input or output of DMUo can be varied without changing its hyperplane. By presenting a new theorem, we provided two LP problems to obtain the interval in which the input of DMUo can be varied and yet its radial projection point is located on the prespecified hyperplane. Finally, we should mention that in methods that were introduced earlier, these preservation regions for preserving the RTS classification of the DMUs have been reached by nonlinear models, which are difficult to solve. This subject has the variety of applications and as a future research, the suggested models can be used specially in project selection with regard to their economic efficiency (inspired from [12]).

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