Original Research

# A new robust counterpart model for uncertain linear programming problems

Hamid, Amiri[1]. Rasoul, Shafaei[1*]

Received: 07 May 2022/ Accepted: 26 Feb 2023/ Published online: 26 Feb 2023

*Corresponding Author, shafaei@kntu.ac.ir

1-.Department of Industrial Engineering, K.N. Toosi University of Technology, Tehran, Iran.

## Abstract

Many practical decision-making problems involve a significant level of data uncertainty. In such a case, modeling the uncertainty involved is critical to making informed decisions. The set-based robust optimization approach is one of the most efficient techniques for finding optimal decisions in problems involving uncertain data. The main concern with this technique is over-conservatism. This drawback has been widely investigated, and several robust formulations have been developed in the literature to deal with it. However, research is still ongoing to obtain effective formulations to handle uncertainty. In this study, we derive a robust counterpart formulation for an uncertain linear programming problem under a new uncertainty set that is defined based on a pairwise comparison of perturbation variables. The performance of the proposed robust formulation is evaluated using numerical studies and in terms of different performance metrics. For this purpose, robust counterpart models corresponding to the production-mix sample problems are solved at different protection levels. Then, for each solution obtained, violation probability is calculated using a Monte-Carlo simulation approach. The results revealed that the proposed method outperforms the existing ones.

**Keywords** - perturbation variables; robust counterpart optimization; uncertain coefficients; uncertainty set

## INTRODUCTION

In a real-life environment, due to factors such as the random nature of the input parameters and measurement errors, most problems involve data that contain uncertainties [1]. Uncertainty in input data can lead to a significant deviation in the problem's solution. Ben-Tal et al. [2] demonstrated that the solution to an optimization problem often shows high sensitivity to changes in the input data. Hence, ignoring uncertainty may lead to a solution that is not optimal or even feasible. The set-based robust optimization approach provides a powerful modeling framework for solving decision-making problems under uncertainty. This technique works in two steps. First, an uncertainty set is defined in the uncertain space to determine the values that uncertain coefficients can take. Secondly, the best solution is obtained in such a way that feasibility is guaranteed for every realization of the uncertain parameters in the defined set. The set-based robust optimization approach is widely used to deal with various uncertain problems in different fields, such as energy management of power plants and electric vehicles [3]-[9], finance [10, 11], supply chain management [12]-[15], transportation [16, 17], statistics and estimation [18]-[20], machine learning [21, 22], production planning [23, 24], etc. The robust model proposed by Soyster [25] is one of the earliest studies in this field. Soyster [25] assumed that the actual value of each uncertain coefficient varies independently within a bounded interval. He then proposed a robust model to obtain the best solution that remains feasible for every possible realization of the uncertain parameters. Despite ensuring the solution's robustness, Soyster's approach is highly conservative and significantly reduces the quality of the solution.

To reduce the over-conservatism of Soyster's approach, El-Ghaoui and Lebert [26], El-Ghaoui et al. [27], and Ben-Tal and Nemirovski [28]-[30] independently tried to improve the solution quality by slightly decreasing robustness. They introduced robust nonlinear formulations for uncertain optimization problems. Specifically, Ben-Tal and Nemirovski [28]-[30] proposed an ellipsoidal uncertainty set followed by robust models to address uncertainty in linear and quadratic programming problems. Bertsimas and Sim [31, 32] further defined a polyhedral set in the uncertain space whose size could be adjusted using a budget parameter. They then derived a robust model with the flexibility to control the robustness level of the solution accordingly. The advantage of the polyhedral uncertainty set over the ellipsoidal uncertainty set is that its corresponding robust counterpart model is linear, making it computationally less challenging to solve.

Li et al. [1] introduced several other uncertainty sets (i.e., pure polyhedral; pure ellipsoidal; adjustable box; combined polyhedral, ellipsoidal, and interval set), and then, based on each of these sets, they derived a robust counterpart model for LP and MILP problems. Mulvey et al. [33] investigated a situation where uncertain coefficients could have discrete values. In other words, they described the input data by a set of scenarios for their value and developed a robust model to obtain a solution that is the best with respect to all the scenarios for the input parameters.

Previous studies have assumed that input data are independently uncertain. However, in practice, uncertain parameters may be correlated. In this regard, Bertsimas and Sim [32] investigated a situation in which perturbation in the actual value of each coefficient is due to several known sources of uncertainty. They assumed that the perturbation at each coefficient's actual value is a linear combination of the impact rates of the uncertainty sources. They then proposed a robust counterpart model for such a situation. The main concern of Bertsimas's approach is that it is difficult and often impossible to identify all effective sources of uncertainty. To overcome the drawback of Bertsima's approach, Pachamanova [34] considered a situation in which the covariance matrix of uncertain parameters is available, but the uncertainty sources are unknown. She then incorporated the covariance matrix in the definition of the uncertainty set and derived a robust formulation accordingly.

Jalilvand-Nejad et al. [35] and Daneshvari and Shafaei [36] also examined uncertain linear problems with unknown sources of uncertainty. Each of these papers incorporated the correlation matrix of uncertain parameters in the definition of the uncertainty set and derived a robust formulation accordingly. The latter model was shown to outperform the former. In this paper, we first define a new uncertainty set for a case where uncertain coefficients are independent of each other. We then derive a robust counterpart model for the general form of an uncertain linear optimization problem. Monte-Carlo simulations are also performed to compare the performance of the proposed robust formulation with the most prominent robust counterpart models in the literature in terms of different performance measures. Within the above context, the new contributions of this study are as follows:

- To propose a new uncertainty set and derive a robust counterpart model accordingly.
- To compare the performance of the proposed model with the most prominent robust counterpart models introduced in the literature in terms of the violation probability, the objective function value, and the price of robustness criterion.

Fig.1 shows the details of the research process adopted in this paper.

The rest of the paper is organized as follows. Section 2 discusses the background and related works in detail. The definition of the proposed uncertainty set and the corresponding robust model are presented in Section 3. In Section 4, the proposed model is validated by Monte-Carlo simulations, and the results of the performance evaluation metrics are presented. The paper concludes with a summary of results and an outlook toward future investigations in section 5.

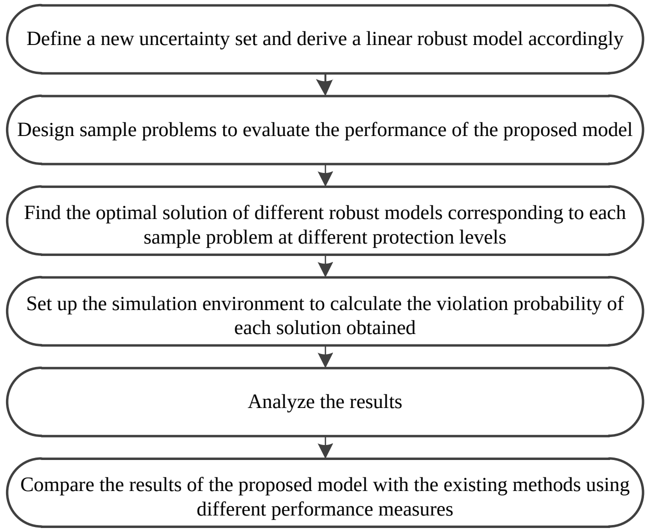


FIGURE 1
RESEARCH PROCESS

## LITERATURE REVIEW

As mentioned earlier, in set-based robust optimization, how each robust counterpart model deals with the uncertainty of the input data depends on the uncertainty set on which the model is based. So far, several uncertainty sets have been proposed for addressing data uncertainty. In the following, we review the most prominent ones. Consider the general form of an uncertain linear problem, which can be described as follows:

$$\begin{aligned} &\max \quad u^T y \\ &\text{subject to} \\ &\tilde{R} y \le v \\ &e \le y \le d \end{aligned} \tag{1}$$

Here $\tilde{R}_{m\times n}$ is a rectangular matrix representing the actual values of the constraint coefficients, $v_{m\times 1}$ is a column vector representing the right-hand side, $u_{n\times 1}$ is a column vector representing the objective function coefficients, $y_{n\times 1}$ is a column vector representing the variables, $e_{n\times 1}$ and $d_{n\times 1}$ are column vectors representing the lower and upper bounds of variables respectively, and the superscript T stands for transpose.

In model (1), it is assumed that uncertainty exists only on the left-hand side of the constraints. Note that if the objective uncertainty or right-hand side uncertainty exists in the problem, (1) can be reformulated as (2) so that the uncertainty reappears only on the left-hand side of the constraints [32]:

$$\begin{aligned} &\max \quad x \\ &\text{subject to} \\ &x - \tilde{u}^T y \le 0 \\ &y_0 \tilde{v} + \tilde{R} y \le 0 \\ &y_0 = -1 \\ &e \le y \le d \end{aligned} \tag{2}$$

Each entry $\tilde{r}_{ij}$ of the matrix $\tilde{R}$ in (1) denotes the actual value of the coefficient and is defined by $\tilde{r}_{ij} = r_{ij} + \xi_{ij}\hat{r}_{ij}$, where $r_{ij}$ indicates the nominal value of the coefficient, $\hat{r}_{ij}$ represents the maximum positive perturbation, and $\xi_{ij}$ is the perturbation variable that has an unknown but symmetric distribution and varies in the interval $[-1, 1]$.

The robust formulation proposed by Soyster [25] was one of the earliest works in robust optimization. Soyster proposed the following robust model for (1):

$$\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \sum_{j \in T_i} \hat{r}_{ij} w_j \le v_i \qquad \forall i \\
&-w_j \le y_j \le w_j \qquad \forall j \in T_i \\
&e \le y \le d \\
&w \ge 0
\end{aligned} \tag{3}$$

Soyster's model allows all the perturbation variables (i.e., $\xi_{ij}$s) to vary in the interval $[-1, 1]$; therefore, the solution obtained remains feasible for every possible value of the uncertain coefficients. It should be noted that there is an inverse relationship between the robustness of a solution and its quality. Thus, though Soyster's approach ensures the robustness of the solution, it is too conservative and significantly decreases the solution quality.

Ben-Tal and Nemirovski [28] argued that it is unlikely that all uncertain coefficients have their extreme simultaneously. Thus, to decrease the over-conservatism of Soyster's approach, they tried to improve the solution's quality by slightly reducing the size of the uncertainty set. To this end, they defined an ellipsoidal set in the uncertain space, which is formulated as follows:

$$U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \,\middle|\, \sum_{j \in T_i} \xi_{ij}^2 \le \Omega_i^2 \quad \forall i,\ \left|\xi_{ij}\right| \le 1 \quad \forall i, \forall j \in T_i \right\} \tag{4}$$

Where $T_i$ indicates the index set of the uncertain coefficients of $i^{th}$ constraint, and $\Omega_i$ is a parameter that is used to adjust the size of the uncertainty set (it should be noted that $\Omega_i \le (|T_i|)^{1/2}$, where $|T_i|$ is the number of elements in the set $T_i$).

Based on the ellipsoidal uncertainty set, Ben-Tal and Nemirovski proposed the following robust formulation for (1):

$$\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \sum_{j \in T_i} \hat{r}_{ij} w_{ij} + \Omega_i \sqrt{\sum_{j \in T_i} \left( \hat{r}_{ij} t_{ij} \right)^2} \le v_i \qquad \forall i \\
&-w_{ij} \le y_j - t_{ij} \le w_{ij} \qquad \forall i, \forall j \in T_i \\
&e \le y \le d \\
&W \ge 0
\end{aligned} \tag{5}$$

In the above model, the robustness of the solution can be controlled by changing the value of the parameter $\Omega$. It is worth noting that the ellipsoidal-based robust formulation is a nonlinear programming problem. Nonlinear programming problems are inherently more challenging to solve than linear ones; therefore, a linear formulation would be more desirable for any practical problem.

As already mentioned, the main drawback of the ellipsoidal uncertainty set is that the corresponding robust formulation is nonlinear. To overcome this problem, Bertsimas and Sim [32] defined a new set in the uncertain space such that the corresponding robust counterpart model is linear. In addition, the level of the solution's robustness can also be adjusted using a budget parameter. The polyhedral uncertainty set defined by Bertsimas and Sim is formulated as follows:

$$U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \,\middle|\, \sum_{j \in T_i} \left|\xi_{ij}\right| \le \Gamma_i \quad \forall i,\ \left|\xi_{ij}\right| \le 1 \quad \forall i, \forall j \in T_i \right\} \tag{6}$$

Where $T_i$ indicates the index set of the uncertain coefficients of the $i^{th}$ constraint, and $\Gamma_i$ represents a parameter used to adjust the size of the uncertainty set (note that $\Gamma_i \le |T_i|$, where $|T_i|$ represents the number of elements in the set $T_i$). Based on the above set, Bertsimas and Sim derived the following robust counterpart model for (1):

$$
\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \sum_{j \in T_i} p_{ij} + \Gamma_{\mathrm{i}} Q_i \le v_i \qquad \forall i \\
&Q_i + p_{ij} \ge \hat{r}_{ij} w_j \qquad \forall i, \forall j \in T_i \\
&-w_j \le y_j \le w_j \qquad \forall j \\
&e \le y \le d \\
&w_j \ge 0 \qquad \forall j \\
&Q_i \ge 0 \qquad \forall i \\
&p_{ij} \ge 0 \qquad \forall i, \forall j \in T_i
\end{aligned} \tag{7}
$$

In model (7), parameter $\Gamma$ adjusts the solution's robustness. That is, the larger the value of $\Gamma$ is, the more robust the solution is, and vice versa. Li et al. [1] further defined an adjustable-box set in the uncertain space as follows:

$$U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \,\middle|\, \left|\xi_{ij}\right| \le \Psi_{\mathrm{i}} \quad \forall i, \forall j \in T_i \right\} \tag{8}$$

Where $T_i$ represents the index set of the uncertain coefficients of the $i^{th}$ constraint, and $\Psi_i$ indicates a parameter that is used to adjust the size of the uncertainty set (note that $0 \le \Psi_i \le 1$). For each constraint $i$, the box uncertainty set covers all perturbation variables that vary in the interval $[-\Psi_i, \Psi_i]$. The corresponding robust formulation, under the adjustable box set, is as follows:

$$
\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \Psi_{\mathrm{i}} \left[ \sum_{j \in T_i} \hat{r}_{ij} w_j \right] \le v_i \qquad \forall i \\
&-w_j \le y_j \le w_j \qquad \forall j \in T_i \\
&e \le y \le d \\
&W \ge 0
\end{aligned} \tag{9}
$$

Li et al. [1] also introduced several other uncertainty sets (i.e., pure polyhedral; pure ellipsoidal; and combined polyhedral, ellipsoidal, and interval set), which are discussed below.

The mathematical formulation of the pure polyhedral uncertainty set is as follows:

$$U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \,\middle|\, \textstyle\sum_{j \in T_i} \left|\xi_{ij}\right| \le \Gamma_{\mathrm{i}} \quad \forall i \right\} \tag{10}$$

It should be noted that the pure polyhedral set is not suitable for a problem with bounded uncertainty. Because the corresponding robust model either provides the same solution as Bertsimas's approach or gives a conservative solution that is also robust to perturbations outside the uncertain space. Hence, the pure polyhedral set is suitable for situations where uncertainty is unbounded. The robust model based on the pure polyhedral set is shown below.

$$
\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \Gamma_{\mathrm{i}} Q_i \le v_i \qquad \forall i \\
&Q_i \ge \hat{r}_{ij} w_j \qquad \forall i, \forall j \in T_i \\
&-w_j \le y_j \le w_j \qquad \forall j \\
&e \le y \le d \\
&w_j \ge 0 \qquad \forall j \\
&Q_i \ge 0 \qquad \forall i
\end{aligned} \tag{11}
$$

Li et al. [1] defined the pure ellipsoidal uncertainty set as follows:

$$U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \;\middle|\; \sum_{j \in T_i} \xi_{ij}^2 \le \Omega_{\mathrm{i}}^2 \quad \forall i \right\} \qquad (12)$$

Note that, like the previous set, the pure ellipsoidal set is not also suitable for a case where uncertainty is bounded. Because the corresponding robust model either provides the same solution as Ben-Tal's approach or gives a conservative solution that is also robust to perturbations outside the uncertain space. The robust model based on the pure ellipsoidal set is mathematically formulated as follows:

$$\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \Omega_{\mathrm{i}} \sqrt{\sum_{j \in T_i} \left( \hat{r}_{ij} y_j \right)^2} \le v_i \qquad \forall i \\
&e \le y \le d
\end{aligned} \qquad (13)$$

Li et al. [1] also defined a combined set in the uncertain space, which is formulated as follows:

$$U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \;\middle|\; \sum_{j \in T_i} \xi_{ij}^2 \le \Omega_{\mathrm{i}}^2 \;\; \forall i, \sum_{j \in T_i} \left|\xi_{ij}\right| \le \Gamma_{\mathrm{i}} \;\; \forall i, \left|\xi_{ij}\right| \le 1 \;\; \forall i, \forall j \in T_i \right\} \qquad (14)$$

They derived the following robust model based on the above set:

$$\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \sum_{j \in T_i} p_{ij} + \Gamma_{\mathrm{i}} Q_i + \Omega_{\mathrm{i}} \sqrt{\sum_{j \in T_i} \left( \hat{r}_{ij} w_{ij} \right)^2} \le v_i \qquad \forall i \\
&-p_{ij} \le t_{ij} \le p_{ij} \qquad \forall i, \forall j \in T_i \\
&-z_i \le \hat{r}_{ij} y_j - t_{ij} - w_{ij} \le z_i \qquad \forall i, \forall j \in T_i \\
&e \le y \le d \\
&w_{ij} \ge 0 \qquad \forall i, \forall j \in T_i \\
&Q_i \ge 0 \qquad \forall i \\
&z_i \ge 0 \qquad \forall i \\
&p_{ij} \ge 0 \qquad \forall i, \forall j \in T_i
\end{aligned} \qquad (15)$$

It is worth noting that (15) is a nonlinear optimization problem. As we know, nonlinear problems are inherently more challenging to solve than linear ones. In the sequel, we review the robust models derived based on the assumption that uncertain coefficients are correlated. One of the earliest studies to derive robust formulations for problems with uncertain correlated parameters was conducted by Bertsimas and Sim [32]. They assumed that there were several known sources of uncertainty that impacted all the input data. In other words, they considered the uncertainty model as follows:

$$\tilde{r}_{ij} = r_{ij} + \sum_{u \in U_i} \tilde{\eta}_{iu} h_{uj} \qquad \forall j \in T_i \qquad (16)$$

Where $U_i$ is the set of uncertainty sources that impact the data in row $i$, and $\tilde{\eta}_{iu}$ is a variable with an unknown symmetric distribution that varies in the interval $[-1, 1]$. Under the above definition of uncertainty, Bertsimas and Sim presented the following robust model:

$$
\begin{aligned}
& \max \quad u^T y \\
& \text{subject to} \\
& \sum_j r_{ij} y_j + \sum_{u \in U_i} t_{iu} + \Gamma_i z_i \le v_i \qquad \forall i \\
& z_i + t_{iu} \ge x_{iu} \qquad \forall i, \forall u \in U_i \\
& -x_{iu} \le \sum_{j \in T_i} h_{uj}\, y_j \le x_{iu} \qquad \forall i, \forall u \in U_i \\
& e \le y \le d \\
& x_{iu} \ge 0 \qquad \forall i, \forall u \in U_i \\
& z_i \ge 0 \qquad \forall i \\
& t_{iu} \ge 0 \qquad \forall i, \forall u \in U_i
\end{aligned} \tag{17}
$$

As mentioned earlier, one of the main limitations of the above approach is that it is difficult to identify all the possible sources of uncertainty that impact the data. To overcome this drawback, another class of models has been proposed in the literature. Two key assumptions underpin these models. First, uncertainty sources are unknown; second, a correlation/covariance matrix of uncertain parameters is available. The research conducted by Pachamanova [34] was one of the earliest studies based on these assumptions. Pachamanova defined the uncertainty set as follows:

$$
U = \left\{ \tilde{R} \;\middle|\; \left\| \Sigma^{-\frac{1}{2}} \left( Vec(\tilde{R}) - Vec(\hat{R}) \right) \right\|_1 \le \Gamma \right\} \tag{18}
$$

Where $\tilde{R}$ is the matrix of actual values of the parameters, $\hat{R}$ indicates the matrix of expected values of the parameters, $\Sigma$ indicates the covariance matrix of the uncertain parameters, and finally $Vec(\tilde{R})$ and $Vec(\hat{R})$ indicate column vectors that are produced by stacking the rows of $\tilde{R}$ and $\hat{R}$ on top of one another, respectively. Pachamanova then derived the following robust model accordingly:

$$
\begin{aligned}
& \max \quad u^T y \\
& \text{subject to} \\
& Y_i^T Vec(\hat{R}) + u^i.\Gamma \le v_i \qquad \forall i \\
& u^i.e^T \ge \Sigma^{\frac{1}{2}} Y_i \qquad \forall i \\
& u^i.e^T \ge -\Sigma^{\frac{1}{2}} Y_i \qquad \forall i \\
& u^i \ge 0 \qquad \forall i
\end{aligned} \tag{19}
$$

Where $e_{(m.n)\times 1}$ is a vector of one, and $(Y_i)_{(m.n)\times 1}$ represents a vector includes $y$ in entries $(i.n - n + 1)$ through $(i.n)$, and 0 in all other places. The robust formulation of Jalilvand-Nejad et al. [35] is another model based on the availability of the correlation matrix of uncertain parameters and the unknown sources of uncertainty. They defined the uncertainty set as follows:

$$
U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\, \hat{r}_{ij} \;\middle|\; |\xi_{ij}| + \sum_{k \ne j} \left[ \left( 1 - \left( \frac{n - \Gamma_i}{n-1} \right) |\rho_{ijk}| \right) |\xi_{ik}| \right] \le \Gamma_i \quad \forall i, \forall j \in T_i \right\} \tag{20}
$$

Where $\rho_{ijk}$ indicates the correlation coefficient between the uncertain parameters $\tilde{r}_{ij}$ and $\tilde{r}_{ik}$. Jalilvand-Nejad et al. then derived the following robust model by incorporating the correlation matrix in the definition of the uncertainty set:

$$
\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \sum_j \Gamma_i\, p_{ij} + \sum_j z_{ij} \le v_i \qquad \forall i \\
&p_{ij} + \sum_{k\neq j}\left[\left(1-\left(\frac{n-\Gamma_i}{n-1}\right)|\rho_{ijk}|\right)p_{ik}\right] \ge \hat{r}_{ij}\, x_j \quad \forall i, \forall j \in T_i \\
&e \le y \le d \\
&-x_j \le y_j \le x_j \qquad \forall j \\
&z_{ij} \ge 0 \qquad \forall i, \forall j \\
&p_{ij} \ge 0 \qquad \forall i, \forall j \\
&x_j \ge 0 \qquad \forall j
\end{aligned} \tag{21}
$$

Daneshvari and Shafaei [36] further defined another set in the uncertain space based on the correlation matrix of uncertain parameters. In particular, they considered the uncertainty set as follows:

$$
U = \left\{ \tilde{r}_{ij} = r_{ij} + \xi_{ij}\,\hat{r}_{ij} \middle| \begin{array}{c} \sum_{k\neq j} |\xi_{ij}|\left(1-\beta\Gamma_i - (2-2|\rho_{ijk}|)(1-\beta)\right) \ge \\ \sum_{k\neq j}(2|\rho_{ijk}|-1)\left(|\xi_{ik}| - \beta\Gamma_i - (1-\beta)(2-2|\rho_{ijk}|)\right),\ \forall i, \forall j \end{array} \right\} \tag{22}
$$

Where $\rho_{ijk}$ indicates the correlation coefficient between the uncertain parameters $\tilde{r}_{ij}$ and $\tilde{r}_{ij}$, $\Gamma_i$ adjusts the solution's robustness, and $\beta$ represents the risk aversion of the decision-maker that varies in the interval $[0, 1]$. They then derived the following robust model accordingly:

$$
\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_j r_{ij} y_j + \sum_j \sum_{k\neq j} |\rho_{ijk}|\left(\beta\Gamma_i + (1-\beta)|1-\rho_{ijk}|\right) M_{ij} + \sum_j N_{ij} + \sum_j \Gamma_i\, p_{ij} \le v_i \quad \forall i \\
&\sum_{k\neq j} (\beta\Gamma_i - 1 + (1-\beta)|1-\rho_{ijk}|) M_{ij} + p_{ij} + \\
&\sum_{k\neq j}\left(1-\left(\frac{n-\Gamma_i}{n-1}\right)|\rho_{ijk}|\right)p_{ik} - \sum_{k\neq j}|\rho_{ijk}| M_{ik} + N_{ij} \ge \hat{r}_{ij}\, x_j \quad \forall i, \forall j \\
&e \le y \le d \\
&-x_j \le y_j \le x_j \qquad \forall j \\
&M_{ij} \ge 0 \qquad \forall j, j \\
&N_{ij} \ge 0 \qquad \forall j, j \\
&p_{ij} \ge 0 \qquad \forall j, j \\
&x_j \ge 0 \qquad \forall j
\end{aligned} \tag{23}
$$

In the next section, first, a new uncertainty set is defined in the uncertain space. Then a new robust formulation for linear problems with independent uncertain parameters is derived accordingly.

## THE PROPOSED APPROACH

In this section, we derive a new robust model for a linear optimization problem with independent uncertain parameters. The proposed model could improve the quality of the solution without much impact on its robustness. The details are discussed below.

- *Definition of the new uncertainty set*

As mentioned earlier, it is unlikely that all the uncertain parameters of a problem have their extreme values simultaneously. In other words, there are some points in the uncertainty space with a low probability of occurring that their coverage by the uncertainty set results in over-conservatism. Therefore, to decrease over-conservatism, the new uncertainty set is defined based on a pairwise comparison of perturbation variables. In particular, the proposed uncertainty set is formulated as follows:

$$U = \{\tilde{r}_{ij} = r_{ij} + \xi_{ij}\,\hat{r}_{ij} \mid |\xi_{ik}| + |\xi_{is}| \leq \theta_i \quad \forall i, \forall s, k \in T_i \quad s < k, |\xi_{ij}| \leq 1 \quad \forall i, \forall j \in T_i\} \qquad (24)$$

Where $T_i$ indicates the index set of the uncertain parameters of the $i^{th}$ constraint, and $\theta_i$ is a parameter used to adjust the size of the uncertainty set (note that $0 \leq \theta_i \leq 2$). The boundaries of the proposed uncertainty set are determined by the following constraints:

$$\begin{array}{ll} \xi_{ik} - \xi_{is} \leq \theta_i & \forall i, \forall s, k \in T_i \quad s < k \\ -\xi_{ik} + \xi_{is} \leq \theta_i & \forall i, \forall s, k \in T_i \quad s < k \\ \xi_{ik} + \xi_{is} \leq \theta_i & \forall i, \forall s, k \in T_i \quad s < k \\ -\xi_{ik} - \xi_{is} \leq \theta_i & \forall i, \forall s, k \in T_i \quad s < k \\ |\xi_{ij}| \leq 1 & \forall i, \forall j \in T_i \end{array} \qquad (25)$$

Defining the uncertainty set, as described above, removes the perturbation vectors with a low probability of occurrence from the uncertainty set. Hence, the quality of the solution could be improved without much impact on robustness. Note that if there are only two uncertain parameters in a row, the newly defined set and the polyhedral set are identical.

*II. The proposed model*

Here, we derive a robust linear formulation based on the defined uncertainty set for (1). To this end, we first derive a robust nonlinear formulation for the problem and then present an equivalent linear model. Consider the general form of an uncertain linear programming problem in (1). By defining the uncertainty with $\tilde{r}_{ij} = r_{ij} + \xi_{ij}\,\hat{r}_{ij}$ the $i^{th}$ constraint of (1) can be reformulated as follows:

$$\sum_{j \notin T_i} r_{ij}\, y_j + \sum_{j \in T_i} (r_{ij} + \xi_{ij}\,\hat{r}_{ij}) y_j \leq v_i \qquad (26)$$

Since in the robust counterpart optimization technique under a predefined uncertainty set, we are willing to find a solution that guarantees feasibility for every realization of the uncertain parameters in the uncertainty set, so the robust counterpart formulation related to the constraint (26) could be defined as follows:

$$\sum_{j} r_{ij} y_j + \left[ \max_{\xi \in U} \left( \sum_{j \in T_i} \xi_{ij}\,\hat{r}_{ij}\, y_j \right) \right] \leq v_i \qquad (27)$$

Consequently, the robust counterpart formulation for model (1) is derived as shown below:

$$\begin{array}{l} \max \quad u^T y \\ \text{subject to} \\ \displaystyle\sum_j r_{ij} y_j + \max_{\{\zeta_{ij} \,||\xi_{ik}| + |\xi_{is}| \leq \theta_i \quad \forall s,k \in T_i \;\; s<k, |\xi_{ij}| \leq 1 \;\; \forall j \in T_i\}} \left\{ \sum_{j \in T_i} |\xi_{ij}|\,\hat{r}_{ij}\,|y_j| \right\} \leq v_i \quad \forall i \\ e \leq y \leq d \end{array} \qquad (28)$$

Since the above robust counterpart model is a nonlinear programming problem, we derive an equivalent linear model for it using the following theorem.

**Theorem**: The robust nonlinear optimization model in (28) can be equivalently recast as the following linear programming problem:

$$
\begin{aligned}
&\max \quad u^T y \\
&\text{subject to} \\
&\sum_{j=1}^{n} r_{ij} y_j + \sum_{s=1}^{n-1} \sum_{k=s+1}^{n} \theta_i M_{isk} + \sum_{j=1}^{n} Q_{ij} \le v_i \quad \forall i \\
&\sum_{k=2}^{n} M_{i1k} + Q_{i1} \ge \hat{r}_{i1}\, w_1 \quad \forall i \\
&\sum_{s=1}^{n-1} M_{isn} + Q_{in} \ge \hat{r}_{in}\, w_n \quad \forall i \\
&\sum_{s=1}^{j-1} M_{isj} + \sum_{k=j+1}^{n} M_{ijk} + Q_{ij} \ge \hat{r}_{ij}\, w_j \quad \forall i, \forall j \quad j \ne 1, n \\
&e \le y \le d \\
&-w_j \le y_j \le w_j \qquad \forall j \\
&M_{isk} \ge 0 \qquad \forall i, \forall s, k \quad s < k \\
&Q_{ij} \ge 0 \qquad \forall i, j \\
&w_j \ge 0 \qquad \forall j
\end{aligned}
\tag{29}
$$

**Proof**: Let $Y^*$ denote the optimal solution of model (28), then the protection function of the $i^{th}$ constraint is defined as follows:

$$
\sigma_i(Y^*, \theta_i) = \max_{\left\{\zeta_{ij} \,\left|\, |\xi_{ik}| + |\xi_{is}| \le \theta_i \quad \forall s,k \in T_i \;\; s<k, |\xi_{ij}| \le 1 \quad \forall j \in T_i \right.\right\}} \left\{ \sum_{j \in T_i} |\xi_{ij}|\, \hat{r}_{ij}\, |y_j^*| \right\} \tag{30}
$$

Where it can be recast as the nonlinear optimization problem in (31):

$$
\begin{aligned}
&\sigma_i(Y^*, \theta_i) = \max \sum_{j \in T_i} |\xi_{ij}|\, \hat{r}_{ij}\, |y_j^*| \\
&\text{subject to} \\
&\quad |\xi_{is}| + |\xi_{ik}| \le \theta_i \qquad \forall s, k \in K_i \quad s < k \\
&\quad |\xi_{ij}| \le 1 \qquad \forall j \in T_i
\end{aligned}
\tag{31}
$$

Model (31) can also be transformed into the following linear programming problem by introducing new variables $P_{ij} = |\xi_{ij}|$:

$$
\sigma_i(Y^*, \theta_i) = \max \sum_{j \in T_i} P_{ij}\, \hat{r}_{ij}\, |y_j^*| \tag{32}
$$

subject to

$$
P_{is} + P_{ik} \le \theta_i \qquad \forall s, k \in T_i \quad s < k \tag{33}
$$

$$
0 \le P_{ij} \le 1 \qquad \forall j \in T_i \tag{34}
$$

For simplicity and without loss of generality, from here on, we assume that all the coefficients are subject to uncertainty. It should be noted that if some of the coefficients have deterministic values, we can consider them uncertain parameters with a maximum positive perturbation equal to zero.

Let $M_{isk}$ and $Q_{ij}$ be the dual variables corresponding to constraints (33) and (34), respectively, then the dual formulation of the sub-problem (32)-(34) is as follows:

$$
\begin{aligned}
&\min \sum_{s=1}^{n-1}\sum_{k=s+1}^{n}\theta_i M_{isk}+\sum_{j=1}^{n}Q_{ij}\\
&\text{subject to}\\
&\quad \sum_{k=2}^{n}M_{i1k}+Q_{i1}\geq \hat{r}_{i1}\,|y_1^*|\\
&\quad \sum_{s=1}^{n-1}M_{isn}+Q_{in}\geq \hat{r}_{in}\,|y_n^*|\\
&\quad \sum_{s=1}^{j-1}M_{isj}+\sum_{k=j+1}^{n}M_{ijk}+Q_{ij}\geq \hat{r}_{ij}\,\left|y_j^*\right| \quad \forall j \quad j\neq 1,n\\
&\quad M_{isk}\geq 0 \qquad \forall s,k \quad s<k\\
&\quad Q_{ij}\geq 0 \qquad \forall j
\end{aligned}
\tag{35}
$$

If the primal sub-problem in (32)-(34) has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the dual problem (35). Therefore, according to the strong duality theorem, the objective functions of these two problems are equal. Consequently, we can obtain our desired result by substituting $\sigma_i(Y^*,\theta_i)$ in (28) with the dual problem (35). In general, the proposed model could find superior solutions in terms of the violation probability and the objective function's value. In the following, numerical studies are conducted to evaluate the performance of the proposed model.

## NUMERICAL RESULTS

In this section, we consider a production-mix problem with left-hand side uncertainty to assess the performance of the new robust model. The problem is described below.

### *I. Production planning problem with uncertain coefficients*

Consider a production-mix problem in which it is aimed to decide on the production quantity of $P$ different products during a mid-term scheduling horizon. It is assumed that there are $K$ different machines, and each machine has a certain capacity (i.e., the number of hours of processing time) available per period. It is also assumed that the processing time of each product on each machine is uncertain. The goal of the problem is to determine the mix of production quantities that maximizes profit while respecting the limited capacity of each machine. This uncertain production-mix problem is formulated as follows:

.

$$
\begin{aligned}
&\max \sum_{j=1}^{P}u_j y_j\\
&\text{subject to}\\
&\sum_{j=1}^{P}\tilde{t}_{ij}y_j\leq v_i \qquad i=1,2,\dots,K\\
&y_j\geq 0 \qquad\qquad j=1,2,\dots,P
\end{aligned}
\tag{36}
$$

Where $u_j$ denotes the unit profit for product $j$, $\tilde{t}_{ij}$ denotes the actual value of processing time on machine $i$ per unit of product $j$, $v_i$ indicates processing time available per period on machine $i$, and $y_j$ represents the decision variable.

Note that problem (36) is identical to the one utilized by Jalilvand-Nejad et al. [35], with the difference that the uncertain coefficients in this problem are assumed to be uncorrelated. In the following, we use this problem to study the performance of the newly derived robust formulation. To do this, we consider 8 sample problems. Table 1 shows the details of the sample problems. In these problems, the expected values of processing times and the objective function coefficients are randomly generated from uniform distributions with a range of $[20, 29]$ and $[50, 79]$, respectively.

It is also assumed that the available time for each machine is equal to 1500 hours, and the actual values of the processing times have a maximum of 10% perturbation around their nominal values. The same level of protection is applied to all the constraints.

TABLE 1
SAMPLE-PROBLEM DETAILS

| Problem number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| No. of machine | 3 | 5 | 4 | 4 | 3 | 10 | 5 | 10 |
| No. of product | 10 | 10 | 15 | 20 | 30 | 10 | 30 | 20 |
| No. of UP* | 30 | 50 | 60 | 80 | 90 | 100 | 150 | 200 |

*Uncertain Parameter

*II. Performance evaluation of the proposed approach*

In this section, the performance of the proposed formulation is evaluated using various performance metrics. For this purpose, first, different robust formulations corresponding to each sample problem are solved at different protection levels. Then, the simulation model is repeated 10,000 times to calculate the violation probability of each solution obtained. The results are then used to compare the proposed formulation with three other widely used methods. These methods include robust formulations based on the polyhedral, ellipsoidal, and box uncertainty sets.

It is worth noting that linear formulations were solved using the CPLEX solver, and nonlinear formulations were solved using the MOSEK solver of GAMS 24.8 software. The actual values of the uncertain parameters in the Monte-Carlo simulation model are also produced using the following formula:

$$\chi_i(r) = \chi_i(0) + \kappa_s \zeta_i(r) \qquad (37)$$

Where $\chi_i(r)$ indicates the $i^{th}$ random value produced corresponding to the $r^{th}$ uncertain parameter, $\chi_i(0)$ is a random seed, and $\zeta_i(r)$ indicates the noise factor. The value of the multiplier $\kappa_s$ in the above formula determines the degree of correlation among the $\chi_i(r)$s, where $\kappa_s = 0.8$ results in a correlation matrix whose components vary in the range of $(0, 0.06)$.

Fig. 2 and Fig. 3 show the objective function values and the violation probabilities for solutions obtained from the different robust formulations at different protection levels, respectively. The results in Fig. 2 and Fig. 3, demonstrate how the objective function value and violation probability of different robust formulations change with respect to the corresponding adjustable parameters. However, to compare the performance of the four robust counterpart models studied,

Fig. 4 shows changes in the objective function values and violation probabilities at different protection levels simultaneously. In other words, Fig. 4 shows the results related to the performance of different robust counterpart models at different protection levels. Note that the X-axis indicates the objective function value, and the Y-axis shows the violation probability.

As can be seen in Fig 4, both the objective function value and violation probability increase with a decrease in protection level. The results in Fig. 4 also show that the proposed model provides solutions with a superior objective value and a lower violation probability than those of the other three models.

In the sequel, we compare the proposed formulation with the other three models in terms of another metric, the price of robustness, presented by Bertsimas and Sim [32]. This measure offers a trade-off between the quality of the solution and its robustness.

Let $F_{Det}$ denote the objective function value for the solution obtained from the deterministic model, $F_{Rob}$ denotes the objective function value for the solution obtained from the robust counterpart model corresponding to a certain level of protection, and PoR stands for the price of robustness criterion. The value of PoR is then computed as follows:

$$\mathrm{PoR} = \frac{F_{Det} - F_{Rob}}{F_{Det}} \times 100 \qquad (38)$$

It is worth noting that at a certain level of protection, a model that can provide solutions with a lower price of robustness is preferred to the other models. Here, the PoR values for 0.99-protected solutions of the proposed formulation are compared to those of the other three models. The results are given in Table 2. The corresponding plots are also shown in Fig. 5. Note that, here, a solution is considered α-protected solution if it is the best solution that has a violation probability less than $(1 - \alpha)$.

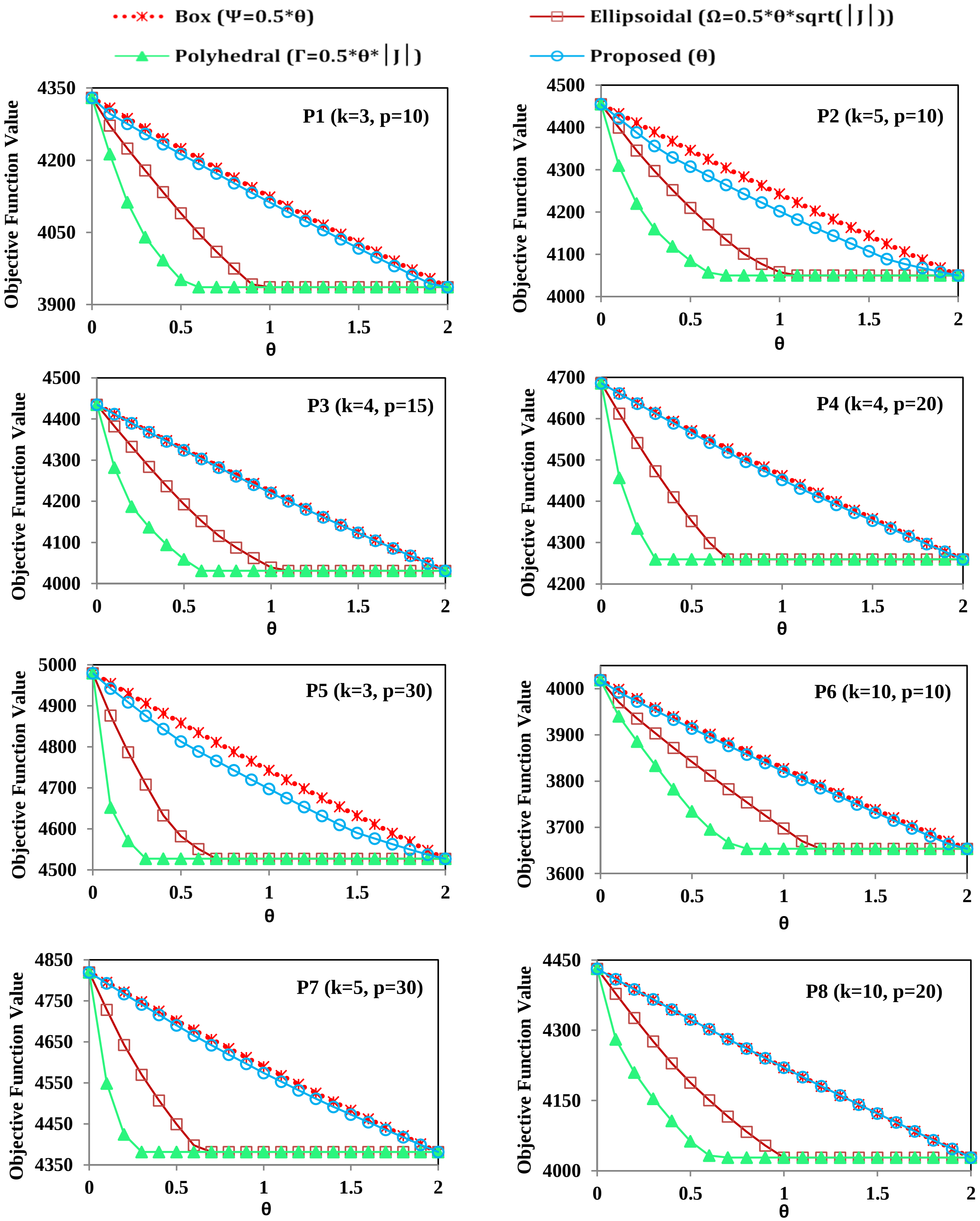


FIGURE 2
OBJECTIVE FUNCTION VALUES FOR SOLUTIONS OBTAINED FROM THE DIFFERENT ROUST FORMULATIONS

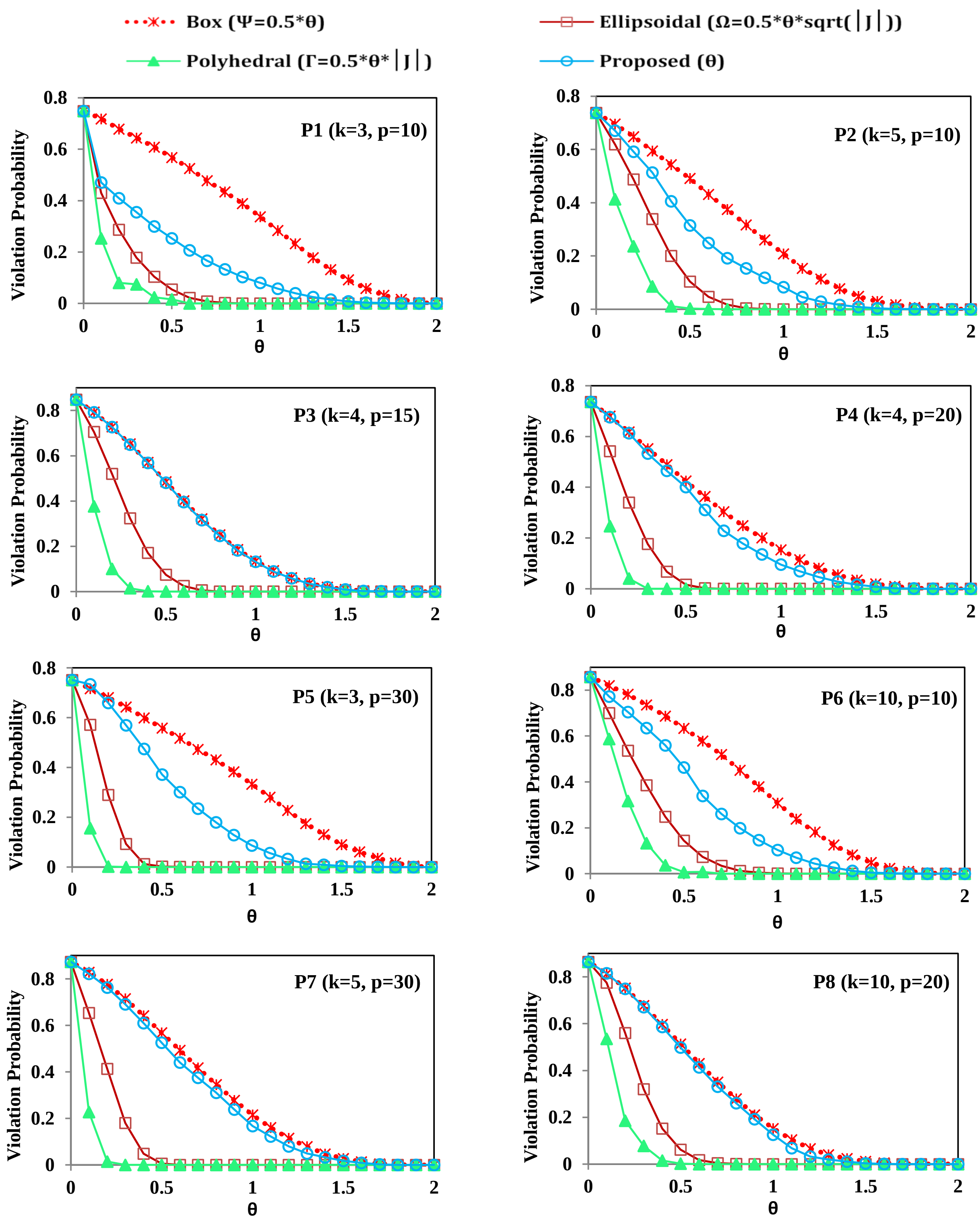


FIGURE 3
VIOLATION PROBABILITIES FOR SOLUTIONS OBTAINED FROM THE DIFFERENT ROBUST FORMULATIONS

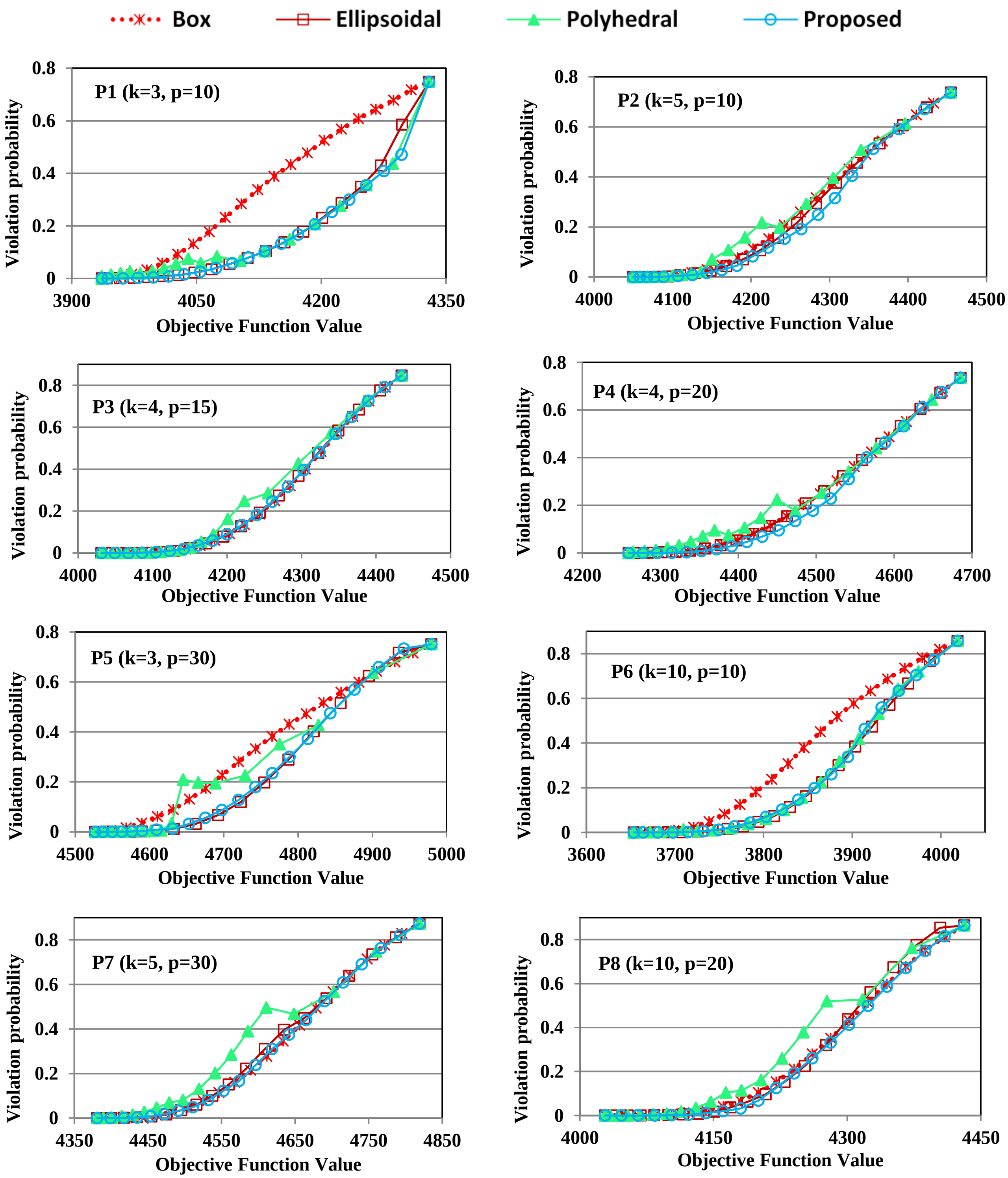


FIGURE 4
PERFORMANCE OF THE DIFFERENT ROBUST FORMULATIONS

TABLE 2
THE POR VALUES ($\alpha$ = 0.99)

| Problem number | Box | Ellipsoidal | Polyhedral | Proposed |
|---|---|---|---|---|
| 1 | 8.583% | 7.321% | 9.032% | 6.905% |
| 2 | 7.621% | 7.475% | 7.518% | 7.357% |
| 3 | 6.906% | 6.897% | 6.937% | 6.894% |
| 4 | 7.384% | 7.379% | 8.538% | 7.005% |
| 5 | 8.481% | 7.074% | 7.211% | 7.058% |
| 6 | 7.861% | 6.861% | 6.879% | 6.860% |
| 7 | 7.516% | 7.443% | 8.359% | 7.443% |
| 8 | 6.988% | 6.587% | 7.435% | 6.565% |

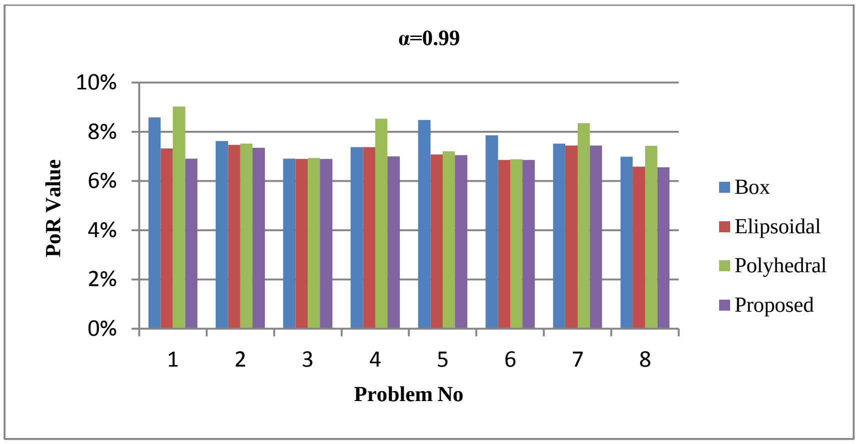


FIGURE 5
THE POR VALUES FOR DIFFERENT ROBUST FORMULATIONS

As it is evident from Table 2 and Fig. 5, the performance of the newly derived robust model is better than that of the other three models, and it provides solutions with a lower PoR value.

## CONCLUSIONS

This paper provided a robust formulation for uncertain linear problems using a new uncertainty set defined based on a pairwise comparison of perturbation variables. The definition of the uncertainty set based on the pairwise comparison of perturbation variables makes it possible to obtain high-quality solutions with a slight decrease in robustness (due to eliminating the perturbation vectors with a low probability of occurrence). The performance of the proposed method was studied by solving several uncertain sample problems at different protection levels. Monte-Carlo simulations were also performed to calculate the violation probability of each solution obtained from the robust models studied. The simulation results showed that the proposed model outperforms the other three studied robust formulations in terms of the violation probability and the objective function value. In other words, the results demonstrated that the proposed method provides solutions with a better objective value and a lower probability of violation than the other three models. To further investigate the performance of the proposed method, the price of robustness was used as a performance metric. According to this metric, a model with a lower price of robustness is more desirable because a lower value for this metric means a better objective value at a certain protection level. Comparing the proposed model with the other three formulations showed the superiority of the proposed approach in providing solutions with a lower price of robustness. Therefore, the proposed approach can be used as an efficient tool to solve different uncertain problems in various fields. The authors are currently working on robust formulations to derive offering curves for the participation of power plants in the restructured electricity markets.

## REFERENCES


[1] Li, Z., Ding, R., & Floudas, C. A. (2011). A comparative theoretical and computational study on robust counterpart optimization: I. Robust linear optimization and robust mixed integer linear optimization. *Industrial & Engineering Chemistry Research*, *50*(18), 10567–10603. https://doi.org/10.1021/ie200150p.

[2] Ben-Tal, A., El Ghaoui, L., & Nemirovski, A. (2009). *Robust optimization*. Princeton university press. https://doi.org/10.1016/s1474-6670(17)42591-2.

[3] Baringo, A., & Baringo, L. (2016). A stochastic adaptive robust optimization approach for the offering strategy of a virtual power plant. *IEEE Transactions on Power Systems*, *32*(5),3492–3504. https://doi.org/10.1109/TPWRS.2016.2633546.

[4] Taghizadeh, M., Bahramara, S., Adabi, F., & Nojavan, S. (2020). Optimal thermal and electrical operation of the hybrid energy system using interval optimization approach. *Applied Thermal Engineering*, *169*, 114993. https://doi.org/10.1016/j.applthermaleng.2020.114993.

[5] Yu, D., Zhang, T., He, G., Nojavan, S., Jermsittiparsert, K., & Ghadimi, N. (2020). Energy management of wind-PV-storage-grid based large electricity consumer using robust optimization technique. *Journal of Energy Storage*, *27*, 101054. https://doi.org/10.1016/j.est.2019.101054.

[6] Cao, Y., Huang, L., Li, Y., Jermsittiparsert, K., Ahmadi-Nezamabad, H., & Nojavan, S. (2020). Optimal scheduling of electric vehicles aggregator under market price uncertainty using robust optimization technique. *International Journal of Electrical Power and Energy Systems*, *117*, 105628. https://doi.org/10.1016/j.ijepes.2019.105628.

[7] Luo, Y., Wang, Z., Zhu, J., Lu, T., Xiao, G., Chu, F., & Wang, R. (2022). Multi-objective robust optimization of a solar power tower plant under uncertainty. *Energy*, *238*, 121716. https://doi.org/10.1016/j.energy.2021.121716.

[8] Yu, D., Wu, J., Wang, W., & Gu, B. (2022). Optimal performance of hybrid energy system in the presence of electrical and heat storage systems under uncertainties using stochastic p-robust optimization technique. *Sustainable Cities and Society*, *83*, 103935. https://doi.org/10.1016/j.scs.2022.103935.

[9] Parvar, S. S., & Nazaripouya, H. (2022). Optimal Operation of Battery Energy Storage Under Uncertainty Using Data-Driven Distributionally Robust Optimization. *Electric Power Systems Research*, *211*, 108180. https://doi.org/10.1016/j.epsr.2022.108180.

[10] Righetto, G. M., Morabito, R., & Alem, D. (2016). A robust optimization approach for cash flow management in stationery companies. *Computers & Industrial Engineering*, *99*, 137–152. https://doi.org/10.1016/j.cie.2016.07.010.

[11] Ashrafi, H., & Thiele, A. C. (2021). A study of robust portfolio optimization with European options using polyhedral uncertainty sets. *Operations Research Perspectives*, *8*, 100178. https://doi.org/10.1016/j.orp.2021.100178.

[12] Aalaei, A., & Davoudpour, H. (2017). A robust optimization model for cellular manufacturing system into supply chain management. *International Journal of Production Economics*, *183*, 667–679. https://doi.org/10.1016/j.ijpe.2016.01.014.

[13] Bairamzadeh, S., Saidi-Mehrabad, M., & Pishvaee, M. S. (2018). Modelling different types of uncertainty in biofuel supply network design and planning: A robust optimization approach. *Renewable Energy*, *116*, 500–517. https://doi.org/10.1016/j.renene.2017.09.020.

[14] Thevenin, S., Ben-Ammar, O., & Brahimi, N. (2022). Robust optimization approaches for purchase planning with supplier selection under lead time uncertainty. *European Journal of Operational Research*. https://doi.org/10.1016/j.ejor.2022.03.029.

[15] Krishnan, R., Arshinder, K., & Agarwal, R. (2022). Robust optimization of sustainable food supply chain network considering food waste valorization and supply uncertainty. *Computers & Industrial Engineering*, *171*, 108499.

[16] Chen, L., Gendreau, M., Hà, M. H., & Langevin, A. (2016). A robust optimization approach for the road network daily maintenance routing problem with uncertain service time. *Transportation Research Part E: Logistics and Transportation Review*, *85*, 40–51. https://doi.org/10.1016/j.tre.2015.11.006.

[17] Ziaei, Z., & Jabbarzadeh, A. (2021). A multi-objective robust optimization approach for green location-routing planning of multi-modal transportation systems under uncertainty. *Journal of Cleaner Production*, *291*, 125293. https://doi.org/10.1016/j.jclepro.2020.125293.

[18] Banerjee, O., El Ghaoui, L., & d'Aspremont, A. (2008). Model selection through sparse maximum likelihood estimation for multivariate Gaussian or binary data. *The Journal of Machine Learning Research*, *9*, 485–516. https://doi.org/10.1145/1390681.1390696.

[19] Candès, E. J., Li, X., Ma, Y., & Wright, J. (2011). Robust principal component analysis? *Journal of the ACM (JACM)*, *58*(3), 1–37. https://doi.org/10.1145/1970392.1970395.

[20] Xu, H., Caramanis, C., & Mannor, S. (2010). Robust regression and lasso. *IEEE Transactions on Information Theory*, *56*(7), 3561–3574. https://doi.org/10.1109/TIT.2010.2048503.

[21] Xu, H., Caramanis, C., & Mannor, S. (2009). Robustness and Regularization of Support Vector Machines. *Journal of Machine Learning Research*, *10*(7).

[22] Wang, C., Peng, X., Shang, C., Fan, C., Zhao, L., & Zhong, W. (2021). A deep learning-based robust optimization approach for refinery planning under uncertainty. *Computers and Chemical Engineering*, *155*, 107495. https://doi.org/10.1016/j.compchemeng.2021.107495.

[23] Singh, S., & Biswal, M. P. (2021). A robust optimization model under uncertain environment: An application in production planning. *Computers and Industrial Engineering*, *155*, 107169. https://doi.org/10.1016/j.cie.2021.107169.

[24] Zhang, L., Yuan, Z., & Chen, B. (2022). Adjustable Robust Optimization for the Multi-period Planning Operations of an Integrated Refinery-Petrochemical Site under Uncertainty. *Computers and Chemical Engineering*, *160*, 107703. https://doi.org/10.1016/j.compchemeng.2022.107703

[25] Soyster, A. L. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, *21*(5), 1154–1157. https://doi.org/10.1287/opre.21.5.1154.

[26] El Ghaoui, L., & Lebret, H. (1997). Robust solutions to least-squares problems with uncertain data. SIAM Journal on Matrix Analysis and Applications, 18(4), 1035–1064. https://doi.org/10.1137/S0895479896298130.

[27] El Ghaoui, L., Oustry, F., & Lebret, H. (1998). Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, *9*(1), 33–52. https://doi.org/10.1137/S1052623496305717.

[28] Ben-Tal, A., & Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research*, *23*(4), 769–805. https://doi.org/10.1287/moor.23.4.769.

[29] Ben-Tal, A., & Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research Letters*, *25*(1), 1–13. https://doi.org/10.1016/S0167-6377(99)00016-4.

[30] Ben-Tal, A., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, *88*(3), 411–424. https://doi.org/10.1007/PL00011380.

[31] Bertsimas, D., & Sim, M. (2003). Robust discrete optimization and network flows. *Mathematical Programming*, *98*(1), 49–71. https://doi.org/10.1007/s10107-003-0396-4.

[32] Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations Research*, *52*(1), 35–53. https://doi.org/10.1287/opre.1030.0065.

[33] Mulvey, J. M., Vanderbei, R. J., & Zenios, S. A. (1995). Robust optimization of large-scale systems. *Operations Research*, *43*(2), 264–281. https://doi.org/10.1287/opre.43.2.264.

[34] Pachamanova, D. A. (2002). *A robust optimization approach to finance*. Massachusetts Institute of Technology.

[35] Jalilvand-Nejad, A., Shafaei, R., & Shahriari, H. (2016). Robust optimization under correlated polyhedral uncertainty set. *Computers & Industrial Engineering*, *92*, 82–94. https://doi.org/10.1016/j.cie.2015.12.006.

[36] Daneshvari, H., & Shafaei, R. (2021). A new correlated polyhedral uncertainty set for robust optimization. *Computers & Industrial Engineering*, *157*, 107346. https://doi.org/10.1016/j.cie.2021.107346.