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Bayesian Estimation of Shift Point in Shape Parameter of Inverse Gaussian Distribution Under Different Loss Functions

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Abstract

In this paper, a Bayesian approach is proposed for shift point detection in an inverse Gaussian distribution. In this study, the mean parameter of inverse Gaussian distribution is assumed to be constant and shift points in shape parameter is considered. First the posterior distribution of shape parameter is obtained. Then the Bayes estimators are derived under a class of priors and using various loss functions. We assumed uniform, Jeffreys, exponential, gamma and chi square distributions as prior distributions. The squared error loss function (SELF), entropy loss function (ELF), linex loss function (LLF) and precautionary loss function (PLF), are used as loss functions. We attempt to find out the best estimator for shift point under various priors and loss functions. The proposed Bayesian approach can be adapted to any similar problem for shift point detection. Simulation studies were done to investigate the performance of different loss functions. The results of simulation study denote that the Jeffrey prior distribution under PLF has the most accurate estimation of shift point for sample size of 20, and the gamma prior distribution under SELF has the most accurate estimation of shift point for sample size of 50.
Key words: Bayes estimators, shift point, inverse Gaussian distribution, loss function.

1. Introduction

When the data set of individual observations is available, a control chart can be used to detect a shift in the parameters. A likelihood ratio test approach is used to determine changes in parameters. The likelihood ratio statistic is plotted for all possible shift point values, and an appropriate Upper Control Limit (UCL) is chosen. The location of the maximum test statistic value corresponds to the maximum likelihood location of one shift point (Sullivan & Woodall, 2000).
Another approach for shift point detection is the Bayesian inference. Srivastava (2012) estimated shift point which occurs in any sequences of independent observations in Poisson model about statistical process control and computed the Bayes estimators under Asymmetric Loss Functions (ALF), Squared Error Loss Functions (SELF), Linex Loss Function (LLF), Precautionary Loss Function (PLF) and General Entropy Loss Function (GELF). He found that the asymmetric loss function was more appropriate. Some of the loss functions like linex loss function (LLF) suggested and studied by Varian (1975) and Basu and Ebrahimi (1991), general entropy loss functions (GELF) proposed by Calabria and Pulcini (1996) and precautionary loss function (PLE) studied by Norstrom (1996), are employed in this article.

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In this paper we estimated the shift point in shape parameter of inverse Gaussian distribution. The Bayesian approach is employed for change point estimation. Fallahnezhad, Rasti, and Abooie (2014) successfully applied Bayesian method for change point estimation in a sequence of exponential distribution. Kadilar and Karasoy (2007) presented another Bayes method for change point estimation. Marcos D'Angelo (2011) proposed a fault detection method in induction machine using a fuzzy-Bayesian change point estimation approach. Other approaches of change point estimation are investigated by Keramatpour, Niaki, Khedmati, and Soleymanian (2012) and Khedmati and Niaki (2013).
The definitions of various loss functions are given in Section 2. We present Likelihood, prior and posterior distributions for λ_1, λ_2 under the assumption of different priors in Section 3. The Bayes estimation of change point using different Priors under different loss functions is obtained in Section 4. The results of simulation studies are presented in Section 5. A summary of results comes in Section 6. Finally we conclude the paper in Section 7.

2. Loss Functions

A loss function represents losses incurred when we estimate the parameter θ by $\bar{\theta}$ (Naz Sindhu & Aslam, 2013). A number of asymmetric loss functions are proposed for different applications. We use the following loss functions for the change point detection.

2.1. The squared error in loss function (SELF)

In decision theory the loss criterion is specified in order to obtain a good estimator. The simplest form of the loss function is the squared error loss function (SELF) that assigns equal magnitudes to both positive and negative errors. The squared error loss function (SELF) is one of the most widely used loss functions in decision theory. The wide application of this symmetric loss function is due to its mathematical convenience. However, as this loss function is symmetric, it fails to consider the differences between overestimation and underestimation of any parameter, thus this loss function may be inappropriate in most of the estimation problems. Sometimes overestimation leads to many serious consequences. The introduction of the asymmetric linex loss function (LLF) by Varian (1975) has led to an increased discussion against the appropriateness of the SELF.
The symmetric square-error loss is widely employed in inference and it is defined as follows (Schroeder, 2011).

$$L(\bar{\theta},\theta) = (\bar{\theta}-\theta)^2 \tag{1}$$

The SELF is also often used because it does not lead to extensive numerical computation (Rao & Pandey, 2009). Pandey and Roa (2009) estimated the shape parameter of a generalized Pareto distribution under asymmetric loss function. The Bayes estimator of θ under SELF is:

$$\bar{\theta}_{\text{Self}} = E(\theta) \tag{2}$$

2.2. The general entropy loss function (GELF)

The Entropy Loss Function has been applied in estimation of parameters but its standard Mean Square Error of estimation is large (Schroeder, 2011). The entropy loss was first introduced in James and Stein (1961) for estimation of the multinomial variance-covariance matrix (Parsian & Nematollahi, 1996). Pulcini and Calabria (1996) suggested the general entropy loss function (GELF) for estimation of parameters which can be defined as:

$$L_{GELF}(\bar{\theta},\theta) = \{(\frac{\bar{\theta}}{\theta})^p - p\ln(\frac{\bar{\theta}}{\theta}) - 1\}, p \neq 0 \tag{3}$$

This loss function has a minimum at point $\theta = \bar{\theta}$. This loss is a general form of the entropy loss function that has been used by several authors choosing the shape parameter $p=1$. This general version allows different shapes of loss function $p>0$ for example when an overestimation causes worse consequences than an underestimation (Parsian & Nematollahi, 1996). The Bayes estimator of θ under the general entropy loss is:

$$\bar{\theta}_{GELF} = [E_{(\theta|x)}(\theta^{-p})]^{-\frac{1}{p}} \tag{4}$$

2.3. The Linex loss function (LLF)

The linex loss function is suitable for the estimation of the location parameter but not for the estimation of the scale parameter and other parametric functions (Dey, 2012). A number of asymmetric loss functions have been proposed for applications but one of the most popular one is the linex (linear-exponential) loss function (LLF). This loss function which was introduced by Varian (1975), and several others including Basu and Ebrahimi (1991), Rojo (1987), Sultan, Ellah, and Soliman (2006) and Soliman (2002), have applied this method in different estimations and prediction problems (Dey, 2012).
This loss function is convex and its shape is determined by the value of its shape parameter. The positive (negative) values of the shape parameter, gives more weight to underestimation (overestimation) (Prakash, 2013). Srivastava and Tanna (2001), Xu and Shi (2004), Prakash and Singh (2006), Singh, Prakash, and Singh (2007), Prakash and Singh (2008) and others have recently discussed the estimation procedures under LLF (Prakash, 2013). Under the assumption in which the minimal loss occurs at $\theta = \bar{\theta}$, the Linex loss function for $L_{Linex}(\bar{\theta},\theta)$ can be expressed as

$$L_{Linex}(\bar{\theta},\theta) = \{\exp(c^*(\bar{\theta}-\theta)) - c^*(\bar{\theta}-\theta) - 1\}, c^* \neq 0 \tag{5}$$

Under the Linex loss function, the Bayes estimator is defined as (Prakash, 2013):

$$\bar{\theta}_{\text{Linex}} = \frac{-1}{c^*}\ln\{E(\exp(-c^*\theta))\} \tag{6}$$

Basu and Ebrahimi (1991) considered the linex (linear-exponential) loss function and studied Bayesian estimation under this asymmetric loss function for an exponential lifetime distribution. This loss function is suitable for situations where overestimation of θ is worse than its underestimation (Pandey& Rao, 2009). Here c^* represents the shape parameter of the loss function. The behavior of the LINEX loss function changes with the choice of c^*.

2.4. The precautionary loss function (PLF)

Norstrom (1996) introduced an alternative asymmetric precautionary loss function, and also presented a general class of precautionary loss functions as a special case (Norstrom, 1996). These loss functions prevent underestimation. These estimators are very useful when underestimation may lead to the worst consequences. Precautionary loss function is defined as follows (Pandey & Rao, 2009):

$$L(\bar{\theta},\theta)=\frac{(\bar{\theta}-\theta)^2}{\bar{\theta}} \tag{7}$$

The Bayes estimator of θ under the precautionary loss function is:

$$\bar{\theta}_{PLF}=[E_{(\theta|x)}(\theta^2)]^{\frac{1}{2}} \tag{8}$$

It should be also noted that estimation with PLF is equal to the Bayes estimator under the general entropy loss function (GELF) when the GELF (shape) parameter p is equal to -2.

3. Likelihood Function, Posterior Distributions for λ_1,λ_2 under the Assumption of Different Priors

Let $x_1,x_2,...,x_n,(n\geq 3)$ be a sequence of observation that comes from an inverse Gaussian distribution with probability density function defined as follows:

$$f(x;\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}}\exp\left[\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right] \tag{9}$$

Where $\mu>0$ is the mean and $\lambda>0$ is the shape parameter. As λ tends to infinity, the inverse Gaussian distribution becomes more similar to a normal (Gaussian) distribution.

Let m be the change point in shape parameter of the inverse Gaussian distribution that leads to the event that two sequences $x_1,x_2,...,x_m$ and $x_{m+1},x_{m+2},...,x_n$ have different shape parameters. The probability density function of the sequence $x_1,x_2,...,x_m$ is obtained as follows:

$$f(x;\lambda_1)=\left(\frac{\lambda_1}{2\pi x^3}\right)^{\frac{1}{2}}\exp\left[\frac{-\lambda_1(x-\mu_1)^2}{2\mu_1^2 x}\right] \tag{10}$$

where $\lambda_1>0,\mu_1>0,\ x>0$. Also the probability density function of the sequence $x_{m+1},x_{m+2},...,x_n$ is as follows:

$$f(x;\lambda_2)=\left(\frac{\lambda_2}{2\pi x^3}\right)^{\frac{1}{2}}\exp\left[\frac{-\lambda_2(x-\mu_2)^2}{2\mu_2^2 x}\right];$$

$$\lambda_2>0,\ \mu_2>0, x>0 \tag{11}$$

The likelihood functions of the sequences are obtained as follows:

$$L(\lambda_1,m|\underline{x})=\left(\frac{\lambda_1}{2\pi}\right)^{\frac{m}{2}}\prod_{i=1}^{m}\left(\frac{1}{x_i^{\frac{3}{2}}}\right);$$

$$\exp\left[-\lambda_1\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}\right]$$

$$\underline{x}=x_1,x_2,...,x_m \tag{12}$$

$$L(\lambda_2,m|\underline{x})=\left(\frac{\lambda_2}{2\pi}\right)^{\frac{n-m}{2}}\prod_{i=m+1}^{n}\left(\frac{1}{x_i^{\frac{3}{2}}}\right);$$

$$\exp\left[-\lambda_2\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}\right]$$

$$\underline{x}=x_{m+1},...,x_n \tag{13}$$

We note that the means μ_1 and μ_2 are known (and possibly different). The posterior distributions under the assumption of uniform, Jeffreys, exponential, gamma and chi-square priors are presented in the next sections.

3.1. The uniform prior

The uniform prior is assumed to be:

$$p(\lambda_1)\propto 1 \tag{14}$$

$$p(\lambda_2)\propto 1 \tag{15}$$

The posterior distribution under the assumption of the uniform prior is:

$$p(\lambda_1|\underline{x})\propto\lambda_1^{\frac{m}{2}}\exp\left[-\lambda_1\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}\right]$$

$$\lambda_1>0 \tag{16}$$

$$p(\lambda_2|\underline{x})\propto\lambda_2^{\frac{n-m}{2}}\exp\left[-\lambda_2\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}\right];$$

$$\lambda_2>0 \tag{17}$$

3.2. The Jeffreys prior

The Jeffreys prior is defined as:

$$p(\lambda_1)\propto\sqrt{|I(\lambda_1)|}=\frac{1}{\lambda_1} \tag{18}$$

$$p(\lambda_2)\propto\sqrt{|I(\lambda_2)|}=\frac{1}{\lambda_2} \tag{19}$$

The posterior distribution under the Jeffreys prior is:

$$p\left(\lambda_1|\underline{x}\right) \propto \lambda_1^{\frac{m}{2}-1} \exp\left[-\lambda_1 \sum_{i=1}^{m} \frac{\left(x_i-\mu_1\right)^2}{2\mu_1^2 x_i}\right];$$

$$\lambda_1 > 0 \tag{20}$$

$$p\left(\lambda_2|\underline{x}\right) \propto \lambda_2^{\frac{n-m}{2}-1} \exp\left[-\lambda_2 \sum_{i=m+1}^{n} \frac{\left(x_i-\mu_2\right)^2}{2\mu_2^2 x_i}\right];$$

$$\lambda_2 > 0 \tag{21}$$

3.3. The exponential prior

The exponential prior is assumed to be:

$$p\left(\lambda_1\right) \propto \exp(-k_1\lambda_1) \quad ,\lambda_1 > 0 \,, k_1 > 0 \tag{22}$$

$$p\left(\lambda_2\right) \propto \exp(-k_2\lambda_2) \quad ,\lambda_2 > 0 \,, k_2 > 0 \tag{23}$$

Where k_1, k_2 are considered as the parameters of the exponential distribution. The posterior distribution under the assumption of exponential prior is:

$$p\left(\lambda_1|\underline{x}\right) \propto \lambda_1^{\frac{m}{2}} \exp\left[-\lambda_1 \left\{\sum_{i=1}^{m} \frac{\left(x_i-\mu_1\right)^2}{2\mu_1^2 x_i} + k_1\right\}\right];$$

$$\lambda_1 > 0 \tag{24}$$

$$p\left(\lambda_2|\underline{x}\right) \propto \lambda_2^{\frac{n-m}{2}} \exp\left[-\lambda_2 \left\{\sum_{i=m+1}^{n} \frac{\left(x_i-\mu_2\right)^2}{2\mu_2^2 x_i} + k_2\right\}\right];$$

$$\lambda_2 > 0 \tag{25}$$

3.4. The gamma prior

The gamma prior is assumed to be:

$$p\left(\lambda_1\right) \propto \lambda_1^{a_1-1} e^{-\lambda_1 b_1} \,; \lambda_1 > 0 \,, a_1, b_1 > 0 \tag{26}$$

$$p\left(\lambda_2\right) \propto \lambda_2^{a_2-1} e^{-\lambda_2 b_2} \,; \lambda_2 > 0 \,, a_2, b_2 > 0 \tag{27}$$

Where a_1, b_1, a_2, b_2 and are parameters of gamma distribution. The posterior distribution under the assumption of the gamma prior is:

$$p\left(\lambda_1|\underline{x}\right) \propto \lambda_1^{\frac{m}{2}+a_1-1} \exp\left[-\lambda_1 \left\{\sum_{i=1}^{m} \frac{\left(x_i-\mu_1\right)^2}{2\mu_1^2 x_i} + b_1\right\}\right];$$

$$\lambda_1 > 0 \tag{28}$$

$$p\left(\lambda_2|\underline{x}\right) \propto \lambda_2^{\frac{n-m}{2}+a_2-1} \exp\left[-\lambda_2 \left\{\sum_{i=m+1}^{n} \frac{\left(x_i-\mu_2\right)^2}{2\mu_2^2 x_i} + b_2\right\}\right]$$

$$;\lambda_2 > 0 \tag{29}$$

3.5. The chi-square prior

The general chi-square prior is assumed to be:

$$p\left(\lambda_1\right) \propto \lambda_1^{\frac{h_1}{2}-1} e^{-\frac{\lambda_1}{2}} \,; \lambda_1 > 0 \,; h_1 > 0 \tag{30}$$

$$p\left(\lambda_2\right) \propto \lambda_2^{\frac{h_2}{2}-1} e^{-\frac{\lambda_2}{2}} \,; \lambda_2 > 0 \,; h_2 > 0 \tag{31}$$

Where h_1, h_2 are taken as the parameters of chi-square distribution. The posterior distribution under the chi-square prior is:

$$p\left(\lambda_1|\underline{x}\right) \propto \lambda_1^{\frac{m}{2}+\frac{h_1}{2}-1} \exp\left[-\lambda_1 \left\{\sum_{i=1}^{m} \frac{\left(x_i-\mu_1\right)^2}{2\mu_1^2 x_i} + \frac{1}{2}\right\}\right]$$

$$;\lambda_1 > 0 \tag{32}$$

$$p\left(\lambda_2|\underline{x}\right) \propto \lambda_2^{\frac{n-m}{2}+\frac{h_2}{2}-1} \exp\left[-\lambda_2 \left\{\sum_{i=m+1}^{n} \frac{\left(x_i-\mu_2\right)^2}{2\mu_2^2 x_i} + \frac{1}{2}\right\}\right]$$

$$;\ \lambda_2 > 0 \tag{33}$$

4. Estimation of change point with Bayesian Inference

4.1. Uniform prior

We take the marginal prior distribution of m discrete uniform over the set {1, 2, 3 & (n -1)}. Therefore, the Joint probability distribution function of λ_1, λ_2, m for uniform prior is obtained as equation (34).

The marginal distribution function of m is denoted by $P(m|\underline{x})$ that is obtained as equation (35).

Therefore, the equation (36) is obtained:

$$\begin{aligned} f\left(\lambda_1,\lambda_2,m|\underline{x}\right) &\propto L\left(\lambda_1,\lambda_2,m|\underline{x}\right)p\left(\lambda_1\right)p\left(\lambda_2\right)p\left(m\right) = L\left(\lambda_1,m|\underline{x}\right)L\left(\lambda_2,m|\underline{x}\right)p\left(\lambda_1\right)p\left(\lambda_2\right)p\left(m\right) \\ &\propto \lambda_1^{m/2} \exp\left(-\lambda_1 \sum_{i=1}^{m} \frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}\right)\lambda_2^{\frac{n-m}{2}} \exp\left(-\lambda_2 \sum_{i=m+1}^{n} \frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}\right) \end{aligned} \tag{34}$$

$$P(m|\underline{x}) \propto \iint f\left(\lambda_1,\lambda_2,m\right) d\lambda_1 d\lambda_2 \tag{35}$$

$$P(m|\underline{x}) = \frac{\dfrac{\Gamma(\frac{m}{2}+1)}{(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i})^{\frac{m}{2}+1}} \cdot \dfrac{\Gamma(\frac{n-m}{2}+1)}{(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i})^{\frac{n-m}{2}+1}}}{\sum_{k=1}^{n}\left(\dfrac{\Gamma(\frac{k}{2}+1)}{(\sum_{i=1}^{k}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i})^{\frac{k}{2}+1}} \cdot \dfrac{\Gamma(\frac{n-k}{2}+1)}{(\sum_{i=k+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i})^{\frac{n-k}{2}+1}}\right)} \quad (36)$$

Srivastava (2012) estimated the shift point under different loss functions in sequences of Poisson distribution. The Bayes estimate of m under SELF is given as:

$$m^* = E(m) = \sum_{m=1}^{n-1} mP(m|\underline{x}) \quad (37)$$

And the Bayes estimation of m under PLF is given as:

$$m^* = (\sum_{i=1}^{n-1} m^2 P(m|\underline{x}))^{\frac{1}{2}} \quad (38)$$

Also the Bayes estimation of m under GELF is given as:

$$m^* = (\sum_{m=1}^{n-1} m^{-p} P(m|\underline{x}))^{-\frac{1}{p}} \quad (39)$$

Where p is the shape parameter that is assumed to be equal one. The Bayes estimation of m under LLF is given as

$$m^* = -\frac{1}{c} Ln(\sum_{m=1}^{n-1} e^{-cm} P(m|\underline{x})) \quad (40)$$

Where c is the shape parameter that is assumed to be equal one.

4.2. Jeffrey prior

The Joint probability distribution function of λ_1, λ_2, m for Jeffrey prior is given as:

$$f(\lambda_1, \lambda_2, m) = \lambda_1^{\frac{m}{2}-1} \exp(-\lambda_1 \sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i})\lambda_2^{\frac{n-m}{2}-1} \exp(-\lambda_2 \sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}) \quad (41)$$

The marginal distribution function of shift point m is obtained as:

$$P(m|\underline{x}) \propto \iint f(\lambda_1, \lambda_2, m) d\lambda_1 d\lambda_2 \quad (42)$$

Therefore equation (43) is as following.

Now the estimation of the shift point under different loss functions can be obtained.

4.3. Exponential prior

The Joint probability distribution function of λ_1, λ_2, m for exponential prior is given as:

$$P(m|\underline{x}) = \frac{\dfrac{\Gamma(\frac{m}{2})}{(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i})^{\frac{m}{2}}} \cdot \dfrac{\Gamma(\frac{n-m}{2})}{(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i})^{\frac{n-m}{2}}}}{\sum_{k=1}^{n}\left(\dfrac{\Gamma(\frac{k}{2})}{(\sum_{i=1}^{k}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i})^{\frac{k}{2}}} \cdot \dfrac{\Gamma(\frac{n-k}{2})}{(\sum_{i=k+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i})^{\frac{n-k}{2}}}\right)} \quad (43)$$

$$f(\lambda_1, \lambda_2, m) = \lambda_1^{\frac{m}{2}} \exp(-\lambda_1(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i} + k_1))\lambda_2^{\frac{n-m}{2}-1} \exp(-\lambda_2(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i} + k_2)) \quad (44)$$

The marginal distribution function of shift point m is:

$$P(m|\underline{x}) \propto \iint f(\lambda_1, \lambda_2, m) d\lambda_1 d\lambda_2 \quad (45)$$

Therefore,

$$P(m|\underline{x}) = \frac{\dfrac{\Gamma(\frac{m}{2}+1)}{\left(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+k_1\right)^{\frac{m}{2}+1}} \cdot \dfrac{\Gamma(\frac{n-m}{2}+1)}{\left(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+k_2\right)^{\frac{n-m}{2}+1}}}{\sum_{k=1}^{n}\left(\dfrac{\Gamma(\frac{k}{2}+1)}{\left(\sum_{i=1}^{k}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+k_1\right)^{\frac{k}{2}+1}} \cdot \dfrac{\Gamma(\frac{n-k}{2}+1)}{\left(\sum_{i=k+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+k_2\right)^{\frac{n-k}{2}+1}}\right)} \tag{46}$$

Now the estimation of the shift point under different loss functions can be obtained.

4.4. Gamma prior

The Joint probability distribution function of λ_1, λ_2, m for gamma prior is given as:

$$f(\lambda_1,\lambda_2,m) = \lambda_1^{\frac{m}{2}+a_1-1}\exp\left(-\lambda_1\left(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+b_1\right)\right)\lambda_2^{\frac{n-m}{2}+a_2-1}\exp\left(-\lambda_2\left(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+b_2\right)\right) \tag{47}$$

The marginal distribution function of shift point m is:

$$P(m|\underline{x}) \propto \iint f(\lambda_1,\lambda_2,m)\,d\lambda_1 d\lambda_2 \tag{48}$$

Thus,

$$P(m|\underline{x}) = \frac{\dfrac{\Gamma(\frac{m}{2}+a_1)}{\left(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+b_1\right)^{\frac{m}{2}+a_1}} \cdot \dfrac{\Gamma(\frac{n-m}{2}+a_2)}{\left(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+b_2\right)^{\frac{n-m}{2}+a_2}}}{\sum_{k=1}^{n}\left(\dfrac{\Gamma(\frac{k}{2}+a_1)}{\left(\sum_{i=1}^{k}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+b_1\right)^{\frac{k}{2}+a_1}} \cdot \dfrac{\Gamma(\frac{n-k}{2}+a_2)}{\left(\sum_{i=k+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+b_2\right)^{\frac{n-k}{2}+a_2}}\right)} \tag{49}$$

Now the estimation of the shift point under different loss functions can be obtained.

4.5. Chi-square prior

The Joint probability distribution function of λ_1, λ_2, m for gamma prior is given as:

$$f(\lambda_1,\lambda_2,m) = \lambda_1^{\frac{m}{2}+\frac{h_1}{2}-1}\exp\left(-\lambda_1\left(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+\frac{1}{2}\right)\right)\lambda_2^{\frac{n-m}{2}+\frac{h_2}{2}-1}\exp\left(-\lambda_2\left(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+\frac{1}{2}\right)\right) \tag{50}$$

The marginal distribution function of shift point m is:

$$P(m|\underline{x}) \propto \iint f(\lambda_1,\lambda_2,m)\,d\lambda_1 d\lambda_2 \tag{51}$$

Hence,

$$P(m|\underline{x}) = \frac{\dfrac{\Gamma(\frac{m}{2}+\frac{h_1}{2})}{(\sum_{i=1}^{m}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+\frac{1}{2})^{\frac{m}{2}+\frac{h_1}{2}}} \cdot \dfrac{\Gamma(\frac{n-m}{2}+\frac{h_2}{2})}{(\sum_{i=m+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+\frac{1}{2})^{\frac{n-m}{2}+\frac{h_2}{2}}}}{\sum_{k=1}^{n}\left(\dfrac{\Gamma(\frac{k}{2}+\frac{h_1}{2})}{(\sum_{i=1}^{k}\frac{(x_i-\mu_1)^2}{2\mu_1^2 x_i}+\frac{1}{2})^{\frac{k}{2}+\frac{h_1}{2}}} \cdot \dfrac{\Gamma(\frac{n-k}{2}+\frac{h_2}{2})}{(\sum_{i=k+1}^{n}\frac{(x_i-\mu_2)^2}{2\mu_2^2 x_i}+\frac{1}{2})^{\frac{n-k}{2}+\frac{h_2}{2}}}\right)} \quad (52)$$

Now the estimation of the shift point under different loss functions can be obtained.

5. Simulation Study

In this section, we present simulation studies of change point estimation under different priors and different loss functions. To find out the optimum estimator, we test various values for parameters of prior distribution and also different sample size. Data generation for simulation is performed in 'Matlab' software. A sensitivity analysis is performed on the parameters of priors. The simulation results for 100 runs are presented in the following Tables. The numbers in parenthesis are MSE of estimations. As it can be seen in the following Tables, the standard deviation of estimations is equal to $\frac{MSE}{\sqrt{n}} = \frac{MSE}{10}$ and its value is sufficiently small thus gathering data from 100 runs is sufficient.

5.1. Uniform prior distribution

The results for uniform prior distribution are presented in Table 1. As it can be seen in the results, the performance of different loss functions is not similar to each other, where PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

5.2. Jeffrey prior distribution

The results for Jeffrey prior distribution are presented in Table 2. It is concluded that PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

5.3. Exponential prior distribution

We performed the sensitivity analysis on parameters of prior exponential distribution for determining the best set of the parameters. Table 3 shows the Bayes estimation under exponential prior with parameters $k_1=k_2= 0.1$ using different loss functions. As it can be seen in Table3, PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.
We changed the parameters of exponential prior distribution to investigate their effects. Table 5.3.2 shows Bayes estimation under exponential prior with parameters $k_1=k_2= 0.5$ using different loss functions. As it can be seen in Table 4, PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

Table 1
Bayes estimation under uniform prior using different loss functions

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	7.8740 (5.2258)	10.6135 (3.6594)	6.7312 (39.7182)	2.0819 (8.9509)
50	23	18.5296 (11.1499)	27.5037 (11.1433)	4.9607 (32.5563)	1.7851 (21.2273)

Table 2
Bayes estimation under Jeffrey prior using different loss functions

Sample size	Shift point	Shift point Estimation($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	7.9208 (4.5403)	11.0745 (3.3585)	2.1376 (8.9472)	1.7648 (9.2439)
50	23	18.7110 (10.1495)	28.0427 (10.8908)	1.9617 (21.0606)	1.6294 (21.3743)

Table 3
Bayes estimation under exponential prior using different loss function (k1=k2= 0.1)

Sample size	Shift point	Shift point Estimation($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	8.0289 (4.9402)	10.8367 (3.3874)	2.9364 (8.6081)	2.0424 (8.9760)
50	23	18.6585 (10.9333)	27.7050 (11.0221)	3.3549 (22.6877)	1.7757 (21.2344)

In this case, we increased the parameters of exponential prior distribution. Table 5 shows Bayes estimation under exponential prior with parameters $k_1=k_2=1$ using different loss functions. It can be seen that PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

Table 6 shows Bayes estimation under exponential prior with parameters $k_1=k_2=2$ using different loss functions. As it can be seen in Table6, PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

With changing the parameters of exponential prior distribution, it is concluded that the performance of different prior exponential distributions are relatively similar to each other. Table 7 shows the total results for Bayes estimation of the change point under exponential prior with different parameters using different loss functions. It is observed from Table 7 that when $k_1=k_2=0.1$ then the PLF has the best estimation of shift point in sample size of 20 and the SELF has the best estimation of shift point in sample size of 50 when parameters of exponential prior distribution are $k_1=k_2=2$.

5.4. Gamma prior distribution

The results for gamma prior distribution are presented in the following Tables. The values of $(a_1,b_1)=(a_2,b_2)=(0.1,0.1)$ are assumed as the parameters of gamma prior distribution .As it can be seen in Table 5.4.1, PLF has the best performance in determining the shift point for sample size of 20, SELF has best performance in determining the shift point for sample size of 50.

Table 9 shows the Bayes estimation for gamma prior distribution with parameters $(a_1,b_1)=(a_2,b_2)=(0.1,0.5)$. As you can see, PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

Table 4
Bayes estimation under exponential prior using different loss functions (k1=k2= 0.5)

Sample size	Shift point	Shift point Estimation($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	8.9500 (3.7844)	11.8696 (2.5104)	2.8373 (8.3953)	2.0756 (8.9371)
50	23	20.8482 (9.0507)	30.3078 (10.6490)	2.4079 (20.6445)	1.8353 (21.1694)

Table 5
Bayes estimation under exponential prior using different loss functions (k1=k2=1)

Sample size	Shift point	Shift point Estimation($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	9.1875 (3.1130)	12.1584 (2.1712)	2.6231 (8.4389)	2.0633 (8.9435)
50	23	21.1736(7.8341)	30.8171 (10.2883)	2.2481 (20.7696)	1.8093 (21.1938)

Table 6
Bayes estimation under exponential prior using different loss functions (k1=k2=2)

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	9.2407 (2.6626)	12.2459 (1.9424)	2.4956 (8.5260)	2.0421 (8.9618)
50	23	21.8979 (6.7887)	31.5718 (10.3479)	0.0022 (0.0021)	1.8142 (21.1880)

Table 7
The loss functions with the best performances for exponential prior

No. of Table	Sample size		Loss function		Parameters of prior distribution
	20	50	For sample size=20	For sample size=50	
6.3.1	10.8367	18.6585	PLF	SELF	$k_1=k_2=0.1$
6.3.2	11.8696	20.8482	PLF	SELF	$k_1=k_2=0.5$
6.3.3	12.1584	21.1736	PLF	SELF	$k_1=k_2=1$
6.3.4	12.2459	21.8979	PLF	SELF	$k_1=k_2=2$

Table 8
Bayes estimation under gamma prior using different loss functions ((a1,b1)=(a2,b2)=(0.1,0.1))

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	8.0494 (4.3626)	11.2081 (3.1512)	2.1104 (8.9181)	1.7871 (8.9760)
50	23	18.8050 (10.0230)	28.1687 (10.7884)	1.9850 (21.0370)	1.6439 (21.2344)

Table 9
Bayes estimation under gamma prior using different loss functions ((a1,b1)=(a2,b2)=(0.1,0.5))

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	8.9091 (2.9830)	12.2130 (2.1373)	2.1675 (8.8521)	1.8390 (8.9371)
50	23	21.0094 (7.9488)	30.7143 (10.4827)	2.0679 (20.9528)	1.7101 (21.1694)

Table 10 shows the Bayes estimation for gamma prior distribution with parameters $(a_1,b_1)=(a_2,b_2)=(0.1,1)$. As it is shown, PLF has the best performance in determining the shift point for sample size 20 and SELF has the best performance in determining the shift point for sample size of 50.

Table 11 shows the Bayes estimation for gamma prior distribution with parameters $(a_1,b_1)=(a_2,b_2)=(0.1,2)$. PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.

Table 12 shows the summarized results for Bayes estimation of the change points under gamma prior using different loss functions. It is seen that when $(a_1,b_1)=(a_2,b_2)=(0.1,0.1)$ then PLF has the best performance to estimate the change points based on the sample size of 20 and the SELF is the best estimator for shift point in sample size of 50 when parameters $(a_1,b_1)=(a_2,b_2)=(0.1,2)$ are assumed for gamma prior distribution.

5.5. Chi-square prior distribution

The results for chi-square prior distribution are presented in the following Tables. The values of $h_1=h_2=0.1$ are assumed as the parameters of chi-square prior distribution. As it comes in Table 13, PLF has the best performance in determining the shift point for sample size of 20; also PLF has the best performance in determining the shift point for sample size of 50.

We changed the parameters of chi-square prior distribution to determine the best estimator. Table 14 shows Bayes estimation under chi-square prior with parameters $h_1=h_2=0.5$ using different loss functions. As you can see in Table14, PLF has the best performance in determining the shift point for sample size of 20 and the sample size of 50.

Table 10
Bayes estimation under gamma prior using different loss functions ((a1,b1)=(a2,b2)=(0.1,1))

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	9.1603 (2.4919)	12.4642 (1.9924)	2.1450 (8.8650)	1.8455 (8.9435)
50	23	21.3193 (6.9787)	31.1537 (10.2412)	1.9973 (21.0089)	1.6998 (21.1938)

Table 11
Bayes estimation under gamma prior using different loss functions ((a1,b1)=(a2,b2)=(0.1,2))

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	9.2024 (2.3488)	12.5068 (1.9506)	2.1160 (8.8898)	1.8386 (8.9618)
50	23	22.0313 (5.9906)	31.8620 (10.3783)	0.0020 (0.0021)	1.7125 (21.1880)

Table 12
The loss functions with the best performances for gamma prior

No. of Table	Sample size		Loss function		Parameters of prior distribution
	20	50	For sample size=20	For sample size=50	
6.41	11.2081	18.8050	PLF	SELF	$(a_1,b_1)=(a_2,b_2)=(0.1,0.1)$
6.4.2	12.2130	20.8482	PLF	SELF	$(a_1,b_1)=(a_2,b_2)=(0.1,0.5)$
6.4.3	12.4642	21.3193	PLF	SELF	$(a_1,b_1)=(a_2,b_2)=(0.1, 1)$
6.4.4	12.5068	22.0313	PLF	SELF	$(a_1,b_1)=(a_2,b_2)=(0.1,2)$

Table 13
Bayes estimation under chi-square prior using different loss functions (h1=h2=0.1)

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	6.7320 (4.9917)	10.3960 (2.6122)	2.6178 (8.4071)	2.0200 (8.9847)
50	23	11.1175 (12.9614)	21.8358 (6.9127)	3.3991 ()19.6396	2.1976 (20.8057)

Table 14
Bayes estimation under chi-square prior using different loss functions (h1=h2=0.5)

Sample size	Shift point	Shift point Estimation ($\lambda_1=5$, $\lambda_2=20$, $\mu_1=\mu_2=1$)			
		SELF	PLF	GELF	LLF
20	11	7.8749 (3.8935)	11.3940 (1.9534)	2.6090 (8.4398)	2.0163 (8.9902)
50	23	15.7367 (9.6295)	26.4884 (7.3439)	2.8427 (20.1939)	2.0286 (20.9744)

Table 15 shows Bayes estimation under chi-square prior with parameters h_1=h_2=1 using different loss functions. As you can see in Table15, PLF has the best performance in determining the shift point for sample size of 20 and SELF has the best performance in determining the shift point for sample size of 50.
Table 16 shows Bayes estimation under chi-square prior with parameters of h_1=h_2=2 using different loss functions. As it can be seen in this Table, SELF has the best performance in determining the shift point for sample size of 20 and 50.
Table 17 shows the summarized results for Bayes estimation of the change point under chi-square prior using different loss functions. It can be concluded that when h_1=h_2=2 the SELF comes up with the best estimation for shift point in sample size of 20 and the PLF has best performance to estimate the shift point in sample size of 50 when parameter h_1=h_2=0.1 are assumed for the chi-square prior distribution.

6. Results

The summarized results of simulation studies for all prior distributions and loss functions SELF, PLF, GELF and LLF have been shown in Table18, based on which, the following conclusions are made:

1. For sample size of 20, the Jeffrey prior distribution under PLF is the most accurate shift point estimator.
2. For sample size of 50, the gamma prior distribution under SELF is the most accurate shift point estimator when the parameters of prior distribution are (a_1,b_1)=(a_2,b_2)=(0.1,2).
3. Also it is shown that the loss function GELF and LLF do not estimate the shift point accurately therefore they are not useful to be applied in the problem of change point detection for shape parameter of the Inverse Gaussian distribution.

Table 15
Bayes estimation under chi-square prior using different loss functions (h1=h2=1)

Sample size	Shift point	Shift point Estimation (λ_1 =5 , λ_2 = 20, μ_1= μ_2=1)			
		SELF	PLF	GELF	LLF
20	11	8.9189 (3.2913)	12.0543 (2.2497)	2.4295 (8.6289)	1.9500 (9.0575)
50	23	20.8036 (8.0622)	30.4853 (10.3188)	2.1401 (20.8753)	1.7519 (21.2510)

Table 16
Bayes estimation under chi-square prior using different loss functions (h1=h2=2)

Sample size	Shift point	Shift point Estimation (λ_1 =5 , λ_2 = 20, μ_1= μ_2=1)			
		SELF	PLF	GELF	LLF
20	11	11.2311 (3.7655)	13.2292 (3.4885)	2.0139 (9.0239)	1.7652 (9.2461)
50	23	37.7769 (20.7037)	40.8602 (20.5257)	0.0014 (0.0022)	1.2198 (21.7852)

Table 17
The loss functions with the best performances for the chi-square prior

No. of tables	Sample size		Loss function		Parameters of prior distribution
	20	50	For sample size=20	For sample size=50	
6.5.1	10.3960	21.8358	PLF	PLF	h_1=h_2=0.1
6.5.2	11.3940	26.4884	PLF	PLF	h_1=h_2=0.5
6.5.3	12.0543	20.8036	PLF	SELF	h_1=h_2=1
6.5.4	11.2311	37.7769	SELF	SELF	h_1=h_2=2

Table18
The prior distribution functions with the best performances

Prior distribution	Sample size		Loss function		Parameters of prior distribution	
	20	50	For sample size=20	For sample size=50	For sample size=20	For sample size=50
uniform	10.6135	18.5296	PLF	SELF	-	-
Jeffrey	11.0745	18.7110	PLF	SELF	-	-
exponential	10.8367	21.8979	PLF	SELF	k_1=k_2=0.1	k_1=k_2=2
gamma	11.2081	22.0313	PLF	SELF	(a_1,b_1)=(a_2,b_2)=(0.1,0.1)	(a_1,b_1)=(a_2,b_2)=(0.1,2)
Chi-square	11.2311	21.8358	SELF	PLF	h_1=h_2=2	h_1=h_2=0.1

6.1 Discussion

In the Bayes estimation of an integer valued parameter, such as the shift point m , the loss function must be defined only for integer values both for the possible point estimator and for the unknown value of the parameter: m, $m = 1,2,...,n-1$. Indeed, as written in the book of Martz and Waller (1982) "in the case of Bayesian point estimation, the action space A consists of the possible point estimates of the parameter and thus is a subset of the parameter space". As a consequence, the posterior expected loss $E\left\{L(\overline{m},m)\right\}$ is not a continuous function and is not differentiable. For example, the SELF relative to m is no longer a real valued function, but is given by: $L(\overline{m},m) = (\overline{m} - m)^2$, $\overline{m}, m = 1,2,...,n-1$. Then, the Bayes estimator of m under the SELF is no longer the posterior mean. Generally, the Bayes estimator of m under any error loss function should be obtained by numerically minimizing the corresponding posterior expected loss which is, in its turn, defined only for integer values of the possible point estimator. As a consequence, all the Bayes estimates in Equations (4.1.4)-(4.1.7) seem to be not acceptable. A simple way to obtain the Bayes estimate of m under a given (integer-valued) loss function is to treat initially m as a continuous (real-valued) parameter and to obtain the (real) value of m, say m^*, that minimizes the continuous expected loss. Then, the Bayes estimate of the shift point, m is given by the integer part of m^* or of m^*+1, depending on which of them provides the smaller posterior expected loss.

Also it is seen that the range of investigation in this paper is very limited and the parameters of the loss functions are not analyzed. It is necessary to mention that selection of optimal parameters for loss functions depends on the application, and optimality in general case is meaningless. We have shown that parameter of prior distribution affects the final results and also a method for determining the optimal prior along with optimal loss function has been proposed. Moreover, comparing the statistical performances of the Bayes estimators of a quantity of interest under different loss functions has no sense. Indeed, the loss function should not be chosen on the basis of its statistical performances, but on the basis of costs considerations, that is, on the basis of the effect of over-estimating or under-estimating the quantity of interest. For example, if over-estimation produces more serious consequences than underestimation, the SELF should not be chosen, whereas the General Entropy loss function with positive shape parameter is a suitable choice. Likewise, the prior distribution should be selected depending on the prior information available to the analyst, and not on the basis of its statistical performances.

7. Conclusion

In this paper we estimated the shift point in sequences of inverse Gaussian distribution with Bayesian analysis under various loss functions. Presented Bayesian inference is based on the different prior distributions including uniform, Jeffrey, exponential, gamma and chi-square under various loss functions including SELF, ELF, LLF, and PLF. First the posterior distribution of the shape parameter was obtained using Bayesian Inference. Then the Bayes estimators were derived along with mean standard error of estimations. Simulation studies were performed to investigate the performance of different loss functions. We tried to find out the best estimator for shift point when different sample sizes were available. The results showed that the sample size was important in superiority of each loss function. The results of simulation study denoted that the Jeffrey prior distribution under PLF was the most accurate change point estimator for sample size of 20, and the gamma prior distribution under SELF was the most accurate change point estimator for sample size of 50.

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