Journal of Optimization in Industrial Engineering 11 (2012) 63-72

Measuring the Overall Performances of Decision-Making Units in the Presence of Imprecise Data

Hossein Azizi[a,*], Alireza Bahari[b], Rasul Jahed[c]

[a] *MSc, Parsabad Moghan Branch, Islamic Azad University, Parsabad Moghan, Iran*

[b] *MSc, Qom Branch, Islamic Azad University, Qom, Iran*

[c] *MSc, Germi Branch, Islamic Azad University, Germi, Iran*

Received 20 September, 2011; Revised 23 January, 2012; Accepted 19 February, 2012

Abstract

Data envelopment analysis (DEA) is a method for measuring the relative efficiencies of a set of decision-making units (DMUs) that use multiple inputs to produce multiple outputs. In this paper, we study the measurement of DMU performances in DEA in situations where input and/or output values are given as imprecise data. By imprecise data we mean situations where we only know that the actual values lie in certain intervals, or cases in which data are given only as ordinal relationships. In this paper, we present two distinct approaches obtaining the upper and lower bounds of efficiency which the DMU under evaluation can have with imprecise data. The optimistic approach seeks the best score among the various values of the efficiency score, while the pessimistic approach seeks the worst score. The main idea of the paper is illustrated using an example. Also, two real-world cases are presented to demonstrate how the efficiency interval is interpreted. The efficiency interval not only describes the actual situation in more detail, but also relieves the psychological pressure on all the evaluated DMUs and the decision-maker.

Keywords: Data envelopment analysis; imprecise data; optimistic efficiency interval; pessimistic efficiency interval; overall efficiency interval.

1. Introduction

Data Envelopment Analysis (DEA) is widely used for evaluation and estimation of efficiency. DEA, first developed by Charnes et al. (2000), has been extensively applied to measurement and benchmarking of relative efficiency for various decision-making entities in the public and private sectors. In recent years, numerous articles and reports have investigated the application of DEA in educational centers, industry, and so on.

DEA computes an efficiency score for each decision-making unit (DMU) under evaluation against a set of DMUs. DEA efficiency score indicates the maximum radial (proportional) decrease in all inputs (increase in all outputs) which can cause an increase in the efficiency of a DMU similar to the most efficient DMUs in the evaluated set. In other words, it chooses the most favorable weights for each DMU under evaluation. For this reason, it is said that the method proposed by Charnes et al. (2000) measures the performance of the DMUs from the optimistic point of view. The efficiency measured in this way is called the best relative efficiency or the optimistic efficiency. Its value, in the input-oriented mode, is restricted to values less than or equal to one. If the value of the optimistic efficiency of a DMU is equal to one, that DMU is said to be DEA-efficient or optimistic efficient; otherwise, it is said to be DEA-non-efficient or optimistic non-efficient. It is usually held that optimistic efficient DMUs have a better performance than optimistic non-efficient DMUs.

On the other hand, another approach has been proposed by Parkan and Wang (2000) that measures the performances of DMUs from the pessimistic point of view. This approach selects the most unfavorable weights for each DMU under evaluation. The efficiency measured from the pessimistic point of view is called the worst relative efficiency or the pessimistic efficiency. Its value, in the input-oriented mode, is restricted to values greater than or equal to one. If the value of the pessimistic efficiency of a DMU is equal to one, that DMU is said to be pessimistic inefficient or DEA-inefficient; otherwise, it is said to be pessimistic non-inefficient or DEA-non-inefficient. It is usually believed that pessimistic inefficient DMUs have a worse performance than pessimistic non-inefficient DMUs.

* Corresponding author E-mail address: hazizi@iaupmogan.ac.ir

Optimistic and pessimistic efficiencies measure two extremes of the performance of each DMU. Any evaluation method that considers only one of these two efficiencies is bound to be biased. For determination of the overall performance of each DMU, both of them should be considered simultaneously.

Entani et al. (2002) proposed a pair of DEA models with interval efficiencies that are measured from both optimistic and pessimistic points of view. Their pair of DEA models was first developed for crisp data and was then extended to interval and fuzzy data. These models are theoretically able to work with interval and fuzzy data, but they have some drawbacks. Namely, these models use only one input and one output for computation of lower bound efficiency regardless of the number of inputs and outputs in the problem. As a result, their model leads to loss of input and output information of the DMU under consideration. Furthermore, their DEA models use variable production frontiers for measurement of efficiency intervals of various DMUs. Before Entani et al. (2002), Doyle et al. (1995) were the first to study DMU performance from both optimistic and pessimistic perspectives. They obtained three pairs of models for evaluation of the upper and lower bounds for crisp data. Their models have a structure similar to Entani et al.'s (2002) models.

Wang and Yang (2007) presented a pair of bounded DEA models for crisp data. The pair of bounded DEA models makes the most use of all input and output information and measures the best and worst relative efficiencies of each DMU by including a virtual DMU called the anti-ideal DMU. The anti-ideal DMU is a DMU which consumes the most input only to produce the lowest amount of output, and when there is a zero value in each output, it has efficiency zero. As a result, their pair of DEA models faces problems in computation of the overall efficiency interval of each DMU. Recently, Azizi and Wang developed improved bounded DEA models that measure DMU efficiencies under any circumstances.

Wang et al. (2008) developed a pair of interval DEA models for dealing with crisp data. Interval DEA models determine the overall efficiency interval of each DMU using the pessimistic efficiency of a virtual DMU called anti-ideal DMU, which consumes the most amount of inputs and produces the least amount of outputs, and compute the optimistic and pessimistic efficiencies of each DMU. Azizi and Jahed (2011) pointed out that their interval DEA models face problems in determining the lower bound interval efficiency when there are zero values in each input. To fix this drawback, they developed a pair of improved interval DEA models that assess the overall efficiency of DMUs under any conditions.

Wang and Luo (2006) combined the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), which is a technique in multiple attribute decision making, with DEA. They defined two virtual DMUs, called the ideal DMU and the anti-ideal DMU, and built two DEA models for computation of the best and the worst relative efficiencies. Combining these two distinct efficiencies, they obtained a relative closeness index which was used as a basis for ranking DMUs. Their proposed DEA models have two basic drawbacks: (1) In most cases, their DEA models use constant weights for all DMUs, and (2) when there are zero values in every input and in every output, their DEA models are infeasible.

Azizi and Fathi Ajirlu (2010) used the optimistic efficiency of the ideal DMU and the pessimistic efficiency of the anti-ideal DMU for determination of the lower bound of the overall efficiency interval for crisp data. Their DEA models also face difficulties in determining the lower bound of the overall efficiency interval when there are zero values in every input and in every output.

Amirteimoori (2007) presented an efficiency measure using two ideal and anti-ideal indices that are formed based on the efficiency and inefficient DEA frontiers. The logic of these two indices is maximization of the weighted L_1 distance from a particular DMU to the efficient and inefficient DEA frontiers.

Wang et al. (2007) proposed a geometric average efficiency measure for evaluation of the overall performance of each DMU. The geometric average efficiency combines both measures of optimistic efficiency and pessimistic efficiency for each DMU and as such, is a more comprehensive measure of performance. Recently, Wang and Chin (2009) proposed a new overall performance measure for DMU ranking. Their proposed DEA approach considers the optimistic and pessimistic efficiencies of the DMUs simultaneously. The overall performance measure defined by these authors considers not only the magnitude of the two efficiencies, but also their direction. Consequently, it appears to be more comprehensive than the geometric average efficiency of Wang et al. (2007).

According to this review of the literature, it is evident that considerable attempts should be made for assessment of the overall performance of the DMUs, since the overall performance of the DMUs should be considered in the more general case of imprecise data. Of course, Entani et al. (2002) have studied the DEA structure in the presence of interval data. However, their DEA models have some drawbacks that will be presented in Section 3. Besides, the main focus of the present paper, under the general topic of DEA, will be simultaneous consideration of crisp, ordinal, and interval data for measurement of the overall performance of the DMUs. The upper bound of overall efficiency interval is obtained from the optimistic perspective, i.e. according to the most favorable condition of each DMU and using the most favorable weights. Its lower bound is determined from the pessimistic point of view, i.e. according to the most unfavorable condition of each DMU and by using the most unfavorable weights. The overall efficiency interval provides the decision maker with all possible values of efficiency which reflect various perspectives. Three numerical examples will be used for illustrating the proposed method.

This article is organized as follows. In Section 2, we present the basic DEA models for measurement of the best and the worst relative efficiencies of the DMUs. In Section 3, first we will review Entani et al.'s (2002) DEA models and then we will present the adjusted pessimistic efficiency interval. The numerical examples will be discussed in Section 4. Section 5 will conclude the paper.

2. Interval DEA Models for Measurement of the Best and the Worst relative Efficiencies

2.1. Interval DEA models for measurement of the best relative efficiencies of DMUs

In DEA analysis, it is usually assumed that there are n production units that consume m different inputs and produce s different outputs. Specifically, the j th production unit consumes x_{ij} units of input i ($i=1,\ldots,m$) and produces y_{rj} units of output r ($r=1,\ldots,s$). In interval DEA, it is assumed that some exact values of input x_{ij} and output y_{rj} are not known. It is only known that they are in the range of the upper and lower bounds specified by intervals $[x_{ij}^L, x_{ij}^U]$ and $[y_{rj}^L, y_{rj}^U]$, and each DMU has a positive lower bound input and a positive lower bound output.

To deal with such an uncertain situation, Wang et al. (2005) presented the following pair of linear programming (LP) models that measure the best relative efficiencies of DMUs:

$$\begin{aligned} \min \quad & \theta_o^U = \sum_{i=1}^m v_i x_{io}^U \\ \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \le 0, \quad j=1,\ldots,n, \\ & \sum_{r=1}^s u_r y_{ro}^L = 1, \\ & u_r, v_i \ge \varepsilon, \quad r=1,\ldots,s; \quad i=1,\ldots,m. \end{aligned} \tag{1}$$

$$\begin{aligned} \min \quad & \theta_o^L = \sum_{i=1}^m v_i x_{io}^L \\ \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \le 0, \quad j=1,\ldots,n, \\ & \sum_{r=1}^s u_r y_{ro}^U = 1, \\ & u_r, v_i \ge \varepsilon, \quad r=1,\ldots,s; \quad i=1,\ldots,m. \end{aligned} \tag{2}$$

where DMU_o is the DMU under evaluation, v_i ($i=1,\ldots,m$) and u_r ($r=1,\ldots,s$) are decision variables, and ε is the non-Archimedean infinitesimal. θ_o^U is the best relative efficiency under the most favorable conditions and θ_o^L is the best relative efficiency under the most unfavorable conditions for DMU_o. They form the optimistic efficiency interval $[\theta_o^L, \theta_o^U]$. If there is a set of positive weights u_r^* ($r=1,\ldots,s$) and v_i^* ($i=1,\ldots,m$) that make $\theta_o^{U*}=1$, then DMU_o is called DEA-efficient or optimistic efficient; otherwise, it is called DEA-non-efficient or optimistic non-efficient.

2.2. Interval DEA models for measurement of the worst relative efficiencies of DMUs

The input-oriented framework, which is based on the set of input requirement and its inefficiency frontier, tries to increase input values as much as possible, while keeping the output at most at its current level. This emphasizes the fact that output is kept constant and input values are increased proportionally, until the inefficient production frontier is obtained. DEA estimator for inefficient production possibility set is called the pessimistic efficiency or the worst relative efficiency. For a particular DMU, such as DMU_o, relative efficiencies can be calculated form the following pessimistic DEA models Azizi et al (2011):

$$\begin{aligned} \max \quad & \varphi_o^L = \sum_{i=1}^m v_i x_{io}^L \\ \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \ge 0, \quad j=1,\ldots,n, \\ & \sum_{r=1}^s u_r y_{ro}^U = 1, \\ & u_r, v_i \ge \varepsilon, \quad r=1,\ldots,s; \quad i=1,\ldots,m. \end{aligned} \tag{3}$$

$$\begin{aligned} \max \quad & \varphi_o^U = \sum_{i=1}^m v_i x_{io}^U \\ \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \ge 0, \quad j=1,\ldots,n, \\ & \sum_{r=1}^s u_r y_{ro}^L = 1, \\ & u_r, v_i \ge \varepsilon, \quad r=1,\ldots,s; \quad i=1,\ldots,m. \end{aligned} \tag{4}$$

In models (3) and (4), φ_o^L is the worst relative efficiency under the most unfavorable conditions and φ_o^U is the worst relative efficiency under the most favorable conditions for DMU_o. They give the pessimistic efficiency interval $[\varphi_o^L, \varphi_o^U]$ for DMU_o. When there is a set of positive weights u_r^* ($r = 1,\ldots,s$) and v_i^* ($i = 1,\ldots,m$) that satisfy $\varphi_o^{L*} = 1$, we say that DMU_o is DEA-inefficient or pessimistic inefficient; otherwise, we say that DMU_o is DEA–non-inefficient or pessimistic non-inefficient.

3. The Overall Efficiency Interval

3.1. A review of Entani et al.'s DEA models

To provide an overall efficiency interval for each DMU, Entani et al. (2002) proposed the following mathematical programming model for determination of the upper bound of the overall efficiency interval of DMU_o:

$$\begin{aligned} \max \quad & \Theta_o^U = \max_{y_{rj}, x_{ij}} \frac{\sum_{i=1}^m v_i x_{io} / \sum_{r=1}^s u_r y_{ro}}{\max_j \left\{ \sum_{i=1}^m v_i x_{ij} / \sum_{r=1}^s u_r y_{rj} \right\}} \\ \text{s.t.} \quad & u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m, \end{aligned} \tag{5}$$

where $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$. To obtain the optimal value of model (5), Entani et al. (2002) simplified model (5) to model (6):

$$\begin{aligned} \max \quad & \Theta_o^U = \sum_{i=1}^m v_i x_{io}^U / \sum_{r=1}^s u_r y_{ro}^L \\ \text{s.t.} \quad & \max \left\{ \begin{array}{l} \max_{j \neq o} \left\{ \sum_{i=1}^m v_i x_{ij}^L / \sum_{r=1}^s u_r y_{rj}^U \right\}, \\ \sum_{i=1}^m v_i x_{io}^U / \sum_{r=1}^s u_r y_{ro}^L \end{array} \right\} = 1, \\ & u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m, \end{aligned} \tag{6}$$

The upper bound of overall efficiency interval for DMU_o can be found using the following LP model:

$$\begin{aligned} \max \quad & \Theta_o^U = \sum_{i=1}^m v_i x_{io}^U \\ \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \geq 0, \quad j = 1,\ldots,n; j \neq o \\ & \sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U \geq 0 \\ & \sum_{r=1}^s u_r y_{ro}^L = 1, \\ & u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m. \end{aligned} \tag{7}$$

In models (6) and (7), the upper bounds of input intervals x_{io}^U and the lower bounds of output intervals y_{ro}^L are used for DMU_o, and lower bounds of input intervals x_{ij}^L and the upper bounds of output intervals y_{rj}^U are used for other DMUs. The main drawback of using different sets of constraints for efficiency measurement of DMUs is the lack of possibility of comparison between efficiencies, since different production frontiers have been used in the process of efficiency measurement. We use LP model (4) for obtaining the upper bound of overall efficiency interval for each DMU.
To obtain the lower bound of overall efficiency interval for DMU_o, Entani et al. (2002) proposed the following mathematical programming model for DMU_o:

$$\begin{aligned} \min \quad & \phi_o^L = \min_{y_{rj}, x_{ij}} \frac{\sum_{i=1}^m v_i x_{io} / \sum_{r=1}^s u_r y_{ro}}{\max_j \left\{ \sum_{i=1}^m v_i x_{ij} / \sum_{r=1}^s u_r y_{rj} \right\}} \\ \text{s.t.} \quad & u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m, \end{aligned} \tag{8}$$

For obtaining the optimal value of Model (8), Entani et al. (2002) converted model (8) into model (9):

$$\begin{aligned} \min \quad & \phi_o^L = \sum_{i=1}^m v_i x_{io}^L / \sum_{r=1}^s u_r y_{ro}^U \\ \text{s.t.} \quad & \max \left\{ \begin{array}{l} \max_{j \neq o} \left\{ \sum_{i=1}^m v_i x_{ij}^U / \sum_{r=1}^s u_r y_{rj}^L \right\}, \\ \sum_{i=1}^m v_i x_{io}^L / \sum_{r=1}^s u_r y_{ro}^U \end{array} \right\} = 1, \\ & u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m, \end{aligned} \tag{9}$$

In model (9), the lower bounds of input intervals x_{io}^{L} and the upper bounds of output intervals y_{ro}^{U} have been used for DMU_o, while the upper bounds of input intervals x_{ij}^{U} and the lower bounds of output intervals y_{rj}^{L} have been used for other DMUs. Model (9) cannot be converted into an LP model. To obtain the optimal value of model (9), Entani et al. (2002), after assuming $\sum_{i=1}^{m} v_i x_{ij}^{U} / \sum_{r=1}^{s} u_r y_{rj}^{L} = 1$ for each DEA-inefficient DMU, divided model (9) into k_1 sub-optimization problems $j = J_1,\ldots,J_{k_1}$, where k_1 is the number of DEA-inefficient units, and $J_1,\ldots,J_{k_1}$ are units that are DEA-inefficient:

$$
\begin{aligned}
\min \quad & \phi_{oj}^{L} = \sum_{r=1}^{s} u_r y_{ro}^{L} / \sum_{i=1}^{m} v_i x_{io}^{U} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj}^{U} / \sum_{i=1}^{m} v_i x_{ij}^{L} = 1, \\
& u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m,
\end{aligned} \tag{10}
$$

Model (10) can be simplified by converting into k_1 LP models as follows:

$$
\begin{aligned}
\min \quad & \phi_{oj}^{L} = \sum_{r=1}^{s} u_r y_{ro}^{L} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj}^{U} - \sum_{i=1}^{m} v_i x_{ij}^{L} = 0, \\
& \sum_{i=1}^{m} v_i x_{io}^{U} = 1, \\
& u_r, v_i \geq 0, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m,
\end{aligned} \tag{11}
$$

Assume that ϕ_{oj}^{L*} is the optimal objective function value for model (11). Therefore, the lower-bound efficiency of DMU_o is finally computed as follows:

$$
\phi_{o}^{L*} = 1 \wedge \min_{j \neq o} \{\phi_{oj}^{L*}\} \tag{12}
$$

where $a \wedge b = \min\{a,b\}$. Accordingly, the overall efficiency interval for DMU_o is denoted as $[\phi_o^{L*}, \Theta_o^{U*}]$, where Θ_o^{U*} is the optimal value of the upper-bound model (7).

Model (10) has only one fractional constraint. Therefore, regardless of the number of inputs and outputs in the problem under consideration, only two decision variables can be non-zero, one for the input weight and the other for the output weight. As such, Entani et al.'s (2002) DEA models measure the optimistic efficiency of each DMU by taking into account only one input and one output. Furthermore, an important feature of measurement of the optimistic efficiency of DMUs is identification of DEA-efficient DMUs, which have the best performance among the DMUs from the optimistic point of view and form the efficiency frontier. Consequently, the decision maker can know which DMUs are DEA-efficient and which DMUs are not. In this regard, model (11) is not able to accurately identify DEA-efficient DMUs and the efficiency frontier. Compared with model (11), model (2) is able to accurately identify the optimistic efficient units and the efficiency frontier.

The pessimistic efficiency score is the opposite of the optimistic efficiency score. It is a score that each DMU obtains in its most unfavorable situation (or the most favorable situation) using *the most unfavorable weights*. Theoretically, the best and the worst relative efficiencies should be calculated in a common range and should form an interval for each DMU. For example, they can be measured in the interval $[\beta, 1]$, where $\beta > 0$ is a parameter. In the nest section, we will find a suitable value for β.

3.2. Adjustment of the worst relative efficiencies

Theoretically, the best and the worst relative efficiencies should form an interval. For this purpose, the best relative efficiencies obtained from model (1) and (2) must be adjusted. Suppose that β ($0 < \beta \leq 1$) is the adjustment factor. Then the adjusted best relative efficiencies can be written as $\beta\theta_j^* = \beta[\theta_j^{L*}, \theta_j^{U*}] = \hat{\theta}_j^* = [\hat{\theta}_j^{L*}, \hat{\theta}_j^{U*}]$ ($j = 1,\ldots,n$), which should satisfy $\hat{\theta}_j^* = \beta\theta_j^* = [\hat{\theta}_j^{L*}, \hat{\theta}_j^{U*}] \leq \varphi_j^* = [\varphi_j^{L*}, \varphi_j^{U*}]$ ($j = 1,\ldots,n$), or $\beta \leq \min_{j=1,\ldots,n} \{\varphi_j^{L*} / \theta_j^{U*}\}$. Then assuming $\theta_{\max}^{U*} = \max_{j=1,\ldots,n} \{\theta_j^{U*}\}$ and $\varphi_{\min}^{L*} = \min_{j=1,\ldots,n} \{\varphi_j^{L*}\}$, we have $\min_{j=1,\ldots,n} \{\varphi_j^{L*} / \theta_j^{U*}\} \geq \min_{j=1,\ldots,n} \{\varphi_j^{L*}\} / \max_{j=1,\ldots,n} \{\theta_j^{U*}\} = \varphi_{\min}^{L*} / \theta_{\max}^{U*}$. If we set $\beta = \varphi_{\min}^{L*} / \theta_{\max}^{U*}$, then we will be guaranteed that $\beta \leq \min_{j=1,\ldots,n} \{\varphi_j^{L*} / \theta_j^{U*}\}$. Since the value of β is not zero, we can compute the best performance of DMUs in the range of the interval $[\beta, 1]$ using the following models:

$$\min \quad \psi_o^U = \frac{\sum_{i=1}^{m} v_i x_{io}^U}{\sum_{r=1}^{s} u_r y_{ro}^L}$$

$$\text{s.t.} \quad \frac{\sum_{i=1}^{m} v_i x_{ij}^L}{\sum_{r=1}^{s} u_r y_{rj}^U} \geq \beta, \quad j = 1,\ldots,n, \tag{13}$$

$$u_r, v_i \geq \varepsilon, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m.$$

$$\min \quad \psi_o^L = \frac{\sum_{i=1}^{m} v_i x_{io}^L}{\sum_{r=1}^{s} u_r y_{ro}^U}$$

$$\text{s.t.} \quad \frac{\sum_{i=1}^{m} v_i x_{ij}^L}{\sum_{r=1}^{s} u_r y_{rj}^U} \geq \beta, \quad j = 1,\ldots,n, \tag{14}$$

$$u_r, v_i \geq \varepsilon, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m.$$

Models (13) and (14) can be converted into the following two LP models:

$$\min \quad \psi_o^U = \sum_{i=1}^{m} v_i x_{io}^U$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r (\beta y_{rj}^U) - \sum_{i=1}^{m} v_i x_{ij}^L \leq 0, \quad j = 1,\ldots,n, \tag{15}$$

$$\sum_{r=1}^{s} u_r y_{ro}^L = 1$$

$$u_r, v_i \geq \varepsilon, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m.$$

$$\min \quad \psi_o^L = \sum_{i=1}^{m} v_i x_{io}^L$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r (\beta y_{rj}^U) - \sum_{i=1}^{m} v_i x_{ij}^L \leq 0, \quad j = 1,\ldots,n, \tag{16}$$

$$\sum_{r=1}^{s} u_r y_{ro}^U = 1$$

$$u_r, v_i \geq \varepsilon, \quad r = 1,\ldots,s; \quad i = 1,\ldots,m.$$

Let ψ_o^{U*} and ψ_o^{L*} be the optimal values of models (15) and (16), respectively. Then they form a optimistic efficiency interval, which we denote by $[\psi_o^{L*}, \psi_o^{U*}]$. By repeating models (15) and (16) for each DMU, we can obtain the best performance of n DMUs. We denote their optimistic efficiency interval by $[\psi_j^{L*}, \psi_j^{U*}]$ ($j = 1,\ldots,n$).

DMUs can be evaluated relatively from various perspectives, and any method that considers only one of the optimistic and pessimistic viewpoints will be a one-sided approach. In order to obtain more trustworthy results, we must consider both perspectives simultaneously for scoring a problem. Thus, by integrating the optimistic efficiency interval and the pessimistic efficiency interval of the DMUs, we obtain a new interval of efficiency called *the overall efficiency interval*, in which the upper and lower bounds, i.e. the extreme values, are from different perspectives. As a result, for DMU_j, the overall efficiency interval is defined as $[\psi_j^{L*}, \varphi_j^{U*}]$ ($j = 1,\ldots,n$). The overall efficiency interval represents all possible evaluations from various perspectives.

Regarding the overall efficiency interval $[\psi_o^{L*}, \varphi_o^{U*}]$, we have the following definitions.

Definition 1. DMU_o is called DEA-inefficient or pessimistic inefficient, if $\varphi_o^{U*} = 1$, otherwise it is called DEA–non-inefficient.

Definition 2. DMU_o is called DEA-efficient or optimistic efficient, if $\psi_o^{L*} = \beta$, otherwise it is called DEA–non-efficient.

Definition 3. DMU_o is called DEA-unspecified if and only if it is neither DEA-efficient nor DEA-inefficient.

Regarding DEA-unspecified units, we could say that they are always circumscribed between the efficient and inefficient frontiers Entani et al. (2002).

4. Illustrative Examples

In this section, we present three numerical examples taken from the literature in order to illustrate the simplicity and usefulness of the DEA method with efficient and inefficient frontiers in evaluating DMU performances. In all three examples, the value of the non-Archimedean infinitesimal is assumed to be $\varepsilon = 10^{-10}$.

Example 1. Consider a problem involving performance measurement in the manufacturing industry in which seven manufacturing businesses from various cities are evaluated for performance. The DEA inputs are *capital* and *labor*, and the DEA output is *gross output value*. All data are imprecise and therefore estimated. They are given as bounds, which are shown in Table 1. Data for this analysis are from Wang et al. (2005).

Table 1
Data for seven DMUs with two inputs and one output.

DMU	Inputs		Output
	Capital	Labor	Gross Output Value
1	[564403, 621755]	[674111, 743281]	[806549, 866063]
2	[614371, 669665]	[685943, 742345]	[917507, 985424]
3	[762203, 798427]	[762207, 805677]	[1117142, 1195562]
4	[862016, 937044]	[779894, 846496]	[1206179, 1 261031]
5	[1016898, 1082662]	[799714, 877137]	[1381315, 1462543]
6	[1164350, 1267970]	[807172, 889416]	[1497679, 1652787]
7	[1731916, 1816008]	[818090, 895746]	[1702249, 1812655]

Using interval DEA models (1)–(4) and (15) and (16), we obtain the scoring results listed in Table 2. From Table 2, it can be seen four DMUs, namely DMU_2, DMU_3, DMU_6, and DMU_7, are DEA-efficient according to model (1). These four DEA-efficient units collectively form the efficiency frontier. Also, from the pessimistic efficiency perspective, two DMUs, namely DMU_1 and DMU_7, are DEA-inefficient. Together, they form an inefficiency frontier. Furthermore, units DMU_4 and DMU_5 are DEA-unspecified units. Using Table 2, the value of β is obtained as $\beta = \varphi_{\max}^{L*} / \theta_{\min}^{U*} = 0.7325/1.2365 = 0.5924$.

Table 2
Interval efficiencies for the seven DMUs.

DMU	Optimistic efficiency interval ($[\theta_j^{L*},\theta_j^{U*}]$)	Pessimistic efficiency interval ($[\varphi_j^{L*},\varphi_j^{U*}]$)	Adjusted optimistic efficiency interval ($[\psi_j^{L*},\psi_j^{U*}]$)	Overall efficiency interval ($[\psi_j^{L*},\theta_j^{U*}]$)
1	[1.0453, 1.2365]	[0.8450, 1.0000]	[0.6192, 0.7325]	[0.6192, 1.0000]
2	[1.0000, 1.1689]	[0.7835, 0.9143]	[0.5924, 0.6925]	[0.5924, 0.9143]
3	[1.0000, 1.1230]	[0.7632, 0.8589]	[0.5924, 0.6653]	[0.5924, 0.8589]
4	[1.0406, 1.1821]	[0.7849, 0.8915]	[0.6164, 0.7003]	[0.6164, 0.8915]
5	[1.0185, 1.1571]	[0.7562, 0.8620]	[0.6033, 0.6855]	[0.6033, 0.8620]
6	[1.0000, 1.2052]	[0.7325, 0.8839]	[0.5924, 0.7139]	[0.5924, 0.8839]
7	[1.0000, 1.1542]	[0.8956, 1.0000]	[0.5924, 0.6837]	[0.5924, 1.0000]

The upper bound of the overall efficiency interval is computed from the optimistic point of view, i.e. according to the most favorable conditions for each DMU and based on the most favorable weights. The lower bound of the overall efficiency interval is computed from the pessimistic point of view, i.e. according to the most unfavorable conditions for each DMU and based on the most unfavorable weights. The overall efficiency interval comprises all possible evaluations from various perspectives. As such, the overall efficiency interval provides the decision maker with all possible values of efficiency that reflect various perspectives.

Example 2. Consider the example discussed by Cooper et al. (1999). We have five DMUs that use two inputs, one crisp and the other interval, and produce two outputs, one crisp and the other ordinal. The data set is shown in Table 3.

Table 3
Imprecise data and ordinal data converted for five DMUs.

DMU	Inputs		Outputs		Converted ordinal data
	x_{1j} (exact)	x_{2j} (interval)	y_{1j} (exact)	y_{2j} (ordinal[1])	
1	100	[0.6, 0.7]	2000	4	[0.3456, 0.8333]
2	150	[0.8, 0.9]	1000	2	[0.2400, 0.5787]
3	150	[1, 1]	1200	5	[0.4147, 1.0000]
4	200	[0.7, 0.8]	900	1	[0.2000, 0.4823]
5	200	[1, 1]	600	3	[0.2880, 0.6944]

1 ranking, such that 5 ≡ highest rank, …, 1 ≡ lowest rank ($y_{23} > y_{21} > \ldots > y_{24}$).

For conversion of ordinal preference information into interval data, we used the approach proposed by Wang et al. (2005). For this example, the preference intensity parameter and the ratio parameter about the strong ordinal preference information were determined (or estimated) as $\chi_2 = 1.2$ and $\sigma_2 = 0.2$, respectively. Using the technique described in Wang et al. (2005), we can obtain an interval estimate for the second output of each DMU, which is shown in the last column of Table 3.

For the input and output data of Table 3, interval DEA models (1) and (2) are executed for each DMU, to obtain their optimistic efficiency interval. The results are shown in Table 4. In Table 4, it is evident that only one DMU, i.e. DMU_1, is DEA-efficient and determines the efficiency frontier. Also, by running interval DEA models (3) and (4) for each DMU, we obtain the pessimistic efficiency interval for the five DMUs. From the pessimistic point of view, two DMUs, i.e. DMU_4 and DMU_5, are DEA-inefficient. Furthermore, we determine the adjusted optimistic interval efficiencies of the five DMUs by determining the value of β and by running interval DEA models (15) and (16) for each DMU. Using the pessimistic efficiency intervals and the adjusted optimistic efficiency intervals of the five DMUs, we obtain the overall performance score, i.e. the overall efficiency interval, of each DMU. The results are shown in Table 4.

Table 4
Interval efficiencies for the five DMUs.

DMU	Optimistic efficiency interval ($[\theta_j^{L*}, \theta_j^{U*}]$)	Pessimistic efficiency interval ($[\varphi_j^{L*}, \varphi_j^{U*}]$)	Adjusted optimistic efficiency interval ($[\psi_j^{L*}, \psi_j^{U*}]$)	Overall efficiency interval ($[\psi_j^{L*}, \theta_j^{U*}]$)
1	[1.0000, 1.0000]	[0.2033, 0.5064]	[0.0422, 0.0422]	[0.0422, 0.5064]
2	[1.9199, 3.0000]	[0.4800, 0.9549]	[0.0810, 0.1266]	[0.0810, 0.9549]
3	[1.2500, 2.5000]	[0.5000, 0.6608]	[0.0527, 0.1055]	[0.0527, 0.6608]
4	[2.0157, 2.9630]	[0.6667, 1.0000]	[0.0851, 0.1250]	[0.0851, 1.0000]
5	[2.0000, 4.8223]	[1.0000, 1.0000]	[0.0844, 0.2035]	[0.0844, 1.0000]

It should be noted that Entani et al. (2002) have developed an approach for finding efficiency intervals for crisp data, interval data, and fuzzy data. However, they have not described the method of computation of the overall efficiency interval. Besides, they have not considered the overall efficiency interval for a mixture of crisp data, interval data, and fuzzy data. Furthermore, their upper and lower bound DEA models are not able to accurately identify DEA-efficient and DEA-inefficient units.

Example 3. Consider the problem of performance measurement of a set of 20 branches of a commercial bank in Taiwan (DMUs). Each branch was evaluated for three inputs (*total deposits*, *interest expenses*, and *non-interest expenses*) and three outputs (*total loans*, *interest income*, and *non-interest income*). The data set for this analysis was borrowed from Kao and Liu (2004).

Tables 5 and 6 show the interval inputs and the interval outputs for these DMUs. Furthermore, Table 7 presents overall efficiency interval scores, optimistic efficiency intervals, pessimistic efficiency intervals, and adjusted optimistic efficiency intervals for these DMUs according to models (1)–(4) and (15) and (16).

When we evaluate bank branches from the optimistic point of view, 11 DMUs achieve the efficiency score of 100% under the best conditions. These 11 DMUs are classified as optimistic efficient and are considered to have the best performance. (If they are in the best production conditions, they are DEA-efficient; otherwise they are DEA–non-efficient.) However, when the bank branches are evaluated form the pessimistic point of view, 4 DMUs obtain the smallest efficiency scores under the worst conditions. These 4 DMUs are classified as pessimistic inefficient and are considered to have the worst performance. (If they are in the worst production conditions, they are DEA-inefficient; otherwise they are DEA–non-inefficient.) These 4 DMUs are candidates for bankruptcy. Evaluation of the investment risk is an important issue for financial institutes or business investors in bank branches. Consequently, financial institutes or individual investors must definitely evaluate the performance of bank branches before investing in the banking industry.

Table 5
Inputs and outputs data for 24 bank branches.

DMU_j	x_{1j}^L	x_{2j}^L	x_{3j}^L	y_{1j}^L	y_{2j}^L	y_{3j}^L
1	788670.598	40241.939	11811.938	724380.137	60822.392	7094.716
2	926135.923	42863.302	15496.878	786268.246	66067.139	12826.685
3	895985.403	40469.853	13030.998	770236.241	57395.587	11691.722
4	458981.787	29869.433	6267.727	418079.491	44354.534	5663.309
5	235351.052	7881.369	2820.190	169336.032	11427.471	1618.144
6	256277.540	8499.210	1163.290	200432.663	11234.126	2845.686
7	108792.763	5421.990	1405.508	80058.742	7848.875	302.146
8	78795.804	4052.711	2488.023	47904.990	4975.084	249.434
9	383560.820	27531.866	5352.499	325799.311	3534.225	5393.143
10	507635.274	22708.680	3727.914	402910.427	37609.645	3298.457
11	166251.006	8518.755	3621.040	147175.582	11443.133	1671.398
12	176709.762	8324.757	1554.942	158536.003	11591.017	710.441
13	432487.877	21002.182	2693.838	349537.634	29012.385	4799.480
14	717622.843	32432.931	5207.240	591874.449	45500.257	3017.951
15	101281.254	5491.093	4927.333	78813.646	7421.864	578.585
16	126969.320	7023.181	3063.381	122170.193	9147.275	1698.281
17	145850.899	7933.351	5981.423	127122.118	12139.733	757.213
18	143347.258	8101.257	2799.391	126680.923	11828.337	2366.530
19	190173.529	9307.438	661.977	145200.476	13106.068	2027.188
20	216899.750	9514.108	1910.872	149165.081	12403.453	4760.565
21	131203.426	6496.850	3852.749	101543.039	8989.748	1593.302
22	214511.844	10666.968	4651.894	171767.407	19395.850	3022.838
23	155200.082	9242.283	14248.464	91728.198	10657.757	1038.968
24	153476.455	7816.278	1619.780	144453.154	11601.726	918.045

Table 6
Inputs and outputs data for 24 bank branches.

DMU_j	x_{1j}^U	x_{2j}^U	x_{3j}^U	y_{1j}^U	y_{2j}^U	y_{3j}^U
1	840589.352	43683.964	12022.587	773314.721	66231.622	7623.200
2	1014339.344	49421.622	17952.668	850782.563	72605.032	14022.957
3	989805.864	43436.229	13986.150	861676.051	64209.395	13079.723
4	516654.891	32997.122	6924.034	467712.458	49620.152	6335.638
5	259995.141	8706.643	3115.498	189439.027	12784.101	1810.244
6	283112.884	9389.179	1285.101	224227.342	12567.803	3183.516
7	120184.676	5791.026	1596.834	89563.042	8612.776	338.016
8	85261.100	4430.684	2720.066	53079.753	5459.290	279.046
9	411675.225	29694.054	5744.829	353146.196	3831.942	5963.517
10	560790.800	25086.553	4118.272	441709.209	42074.533	3543.683
11	182253.777	8960.141	3886.457	167355.327	12593.500	1869.820
12	194858.332	8804.193	1644.494	177494.947	12967.063	794.782
13	477774.566	23201.364	2975.916	391033.546	32456.636	5369.258
14	770223.470	35154.161	5577.298	662139.758	50901.891	3376.232
15	111886.621	6066.077	5443.284	87607.243	8302.962	647.272
16	141594.059	7758.592	3384.154	136673.820	10233.208	1899.895
17	164181.887	8573.004	6607.750	144379.110	13437.207	8487.107
18	165099.735	8949.557	2904.323	144429.801	13232.557	2647.476
19	210086.987	10282.038	731.294	162438.180	14661.976	2267.849
20	239611.766	10510.350	2110.963	166873.449	13875.948	5016.701
21	141507.360	7041.088	4119.012	108383.652	9644.386	1718.138
22	239220.015	11895.624	5187.714	194236.441	21933.037	3418.258
23	167934.447	10000.625	15577.337	105261.867	12003.138	1190.318
24	166608.130	8307.353	1721.547	157371.729	12320.208	969.975

Now taking joint advantage of the set of upper bound of pessimistic efficiency and lower bound of optimistic efficiency, we can determine the lower bound of the overall efficiency interval for each DMU, obtaining the value $\beta = \varphi_{\min}^{L*} / \theta_{\max}^{U*} = 0.5738/1.6865 = 0.3402$. The overall efficiency intervals for the DMUs are shown in Table 7.

Consequently, our study makes it possible to provide the bank branch managers with more useful resources of information. It's for this reason that our study is necessary and desirable for dealing with imprecise data.

Table 7
Interval efficiencies of the 24 bank branches

DMU	Optimistic efficiency interval ($[\theta_j^{L*}, \theta_j^{U*}]$)	Pessimistic efficiency interval ($[\varphi_j^{L*}, \varphi_j^{U*}]$)	Adjusted optimistic efficiency interval ($[\psi_j^{L*}, \psi_j^{U*}]$)	Overall efficiency interval ($[\psi_j^{L*}, \theta_j^{U*}]$)
1	[1.0202, 1.1751]	[0.6102, 0.6976]	[0.3471, 0.3998]	[0.3471, 0.6976]
2	[1.0302, 1.2639]	[0.6491, 0.7726]	[0.3497, 0.4300]	[0.3497, 0.7726]
3	[1.0066, 1.2236]	[0.6431, 0.7947]	[0.3412, 0.4163]	[0.3412, 0.7947]
4	[1.0000, 1.2418]	[0.5951, 0.7355]	[0.3402, 0.4225]	[0.3402, 0.7355]
5	[1.0067, 1.2442]	[0.7903, 0.9767]	[0.3402, 0.4233]	[0.3402, 0.9767]
6	[1.0000, 1.2359]	[0.7547, 0.9327]	[0.3402, 0.4204]	[0.3402, 0.9327]
7	[1.1385, 1.3762]	[0.8420, 1.0000]	[0.3873, 0.4682]	[0.3873, 1.0000]
8	[1.4068, 1.6865]	[0.8390, 1.0000]	[0.4786, 0.5738]	[0.4786, 1.0000]
9	[1.0760, 1.2559]	[0.8595, 1.0000]	[0.3661, 0.4272]	[0.3661, 1.0000]
10	[1.0000, 1.2196]	[0.6635, 0.8085]	[0.3402, 0.4149]	[0.3402, 0.8085]
11	[1.0289, 1.2468]	[0.6128, 0.7574]	[0.3500, 0.4242]	[0.3500, 0.7574]
12	[1.0000, 1.2006]	[0.7618, 0.9398]	[0.3402, 0.4084]	[0.3402, 0.9398]
13	[1.0000, 1.2359]	[0.6646, 0.8214]	[0.3402, 0.4204]	[0.3402, 0.8214]
14	[1.0223, 1.2320]	[0.7734, 0.9287]	[0.3478, 0.4091]	[0.3478, 0.9287]
15	[1.1475, 1.4122]	[0.6959, 0.8601]	[0.3904, 0.4804]	[0.3904, 0.8601]
16	[1.0000, 1.2413]	[0.5738, 0.7159]	[0.3402, 0.4223]	[0.3402, 0.7159]
17	[1.0000, 1.2625]	[0.5883, 0.7843]	[0.3402, 0.4295]	[0.3402, 0.7843]
18	[1.0076, 1.2855]	[0.5813, 0.7574]	[0.3428, 0.4373]	[0.3428, 0.7574]
19	[1.0000, 1.2359]	[0.6869, 0.8489]	[0.3402, 0.4204]	[0.3402, 0.8489]
20	[1.0000, 1.1642]	[0.7807, 0.9648]	[0.3402, 0.3960]	[0.3402, 0.9648]
21	[1.1775, 1.3608]	[0.7132, 0.8223]	[0.4006, 0.4629]	[0.4006, 0.8223]
22	[1.0000, 1.2611]	[0.6077, 0.7664]	[0.3402, 0.4290]	[0.3402, 0.7664]
23	[1.3706, 1.6702]	[0.8137, 1.0000]	[0.4663, 0.5682]	[0.4663, 1.0000]
24	[1.0000, 1.1567]	[0.6535, 0.7523]	[0.3402, 0.3935]	[0.3402, 0.7523]

5. Conclusions

Measurement of DMU efficiencies is a complicated yet important decision-making problem which requires taking into account multiple quantitative and qualitative selection criteria. In the present article, we developed a new approach for dealing with interval data, ordinal preference data, and their mixtures in DEA. This approach provides more complete features for using the conventional DEA in working with imprecise data. The proposed method measures the efficiency of each DMU from both optimistic and pessimistic perspectives. This method leads to the creation of an upper bound and a lower bound for efficiency, which we call overall efficiency interval. The overall efficiency interval represents the whole range of imprecise efficiency for each DMU. Using the overall efficiency interval, we can further prioritize DMU performances. Compared with the overall efficiency interval developed by Entani et al. (2002), our proposed efficiency interval uses two constant and unified production frontiers (the efficient frontier and the inefficient frontier) as a benchmark for measurement of the efficiency of all DMUs. This causes our overall efficiency interval to be more logical, more reliable, and more usable. The overall efficiency interval not only describes the actual situation in more detail, it also diminishes the psychological pressure upon all DMUs under consideration and the people performing the evaluations. Three numerical examples were studied to illustrate the simplicity and utility of the proposed approach for measurement of DMU efficiencies.

6. References

[1] Amirteimoori, A. (2007). DEA efficiency analysis: Efficient and anti-efficient frontier. *Applied Mathematics and Computation*, 186, 10-16.

[2] Azizi, H., & Fathi Ajirlu, S. (2010). Measurement of overall performances of decision-making units using ideal and anti-ideal decision-making units. *Computers & Industrial Engineering*, 59, 411-418.

[3] Azizi, H., & Ganjeh Ajirlu, H. (2011). Measurement of the worst practice of decision-making units in the presence of non-discretionary factors and imprecise data. *Applied Mathematical Modelling*, 35, 4149-4156.

[4] Azizi, H., & Jahed, R. (2011). Improved data envelopment analysis models for evaluating interval efficiencies of decision-making units. *Computers & Industrial Engineering*, 61, 897-901.

[5] Charnes, A., Cooper, W.W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2, 429-444.

[6] Cooper, W.W., Park, K.S., & Yu, G. (1999). IDEA and AR-IDEA: Models for dealing with imprecise data in DEA. *Management Science*, 45, 597-607.

[7] Doyle, J.R., Green, R.H., & Cook, W.D. (1995). Upper and lower bound evaluation of multi-attribute objects: Comparison models using linear programming. *Organizational Behavior and Human Decision Processes*, 64, 261-273.

[8] Entani, T., Maeda, Y., & Tanaka, H. (2002). Dual models of interval DEA and its extension to interval data. *European Journal of Operational Research*, 136, 32-45.

[9] Kao, C., & Liu, S.T. (2004). Predicting bank performance with financial forecasts: A case of Taiwan commercial banks. *Journal of Banking & Finance*, 28, 2353–2368.

[10] Parkan, C., & Wang, Y.M. (2000). *Worst Efficiency Analysis Based on Inefficient Production Frontier*. Working Paper, Department of Management Sciences, City University of Hong Kong.

[11] Wang, Y.M., & Yang, J.B. (2007). Measuring the performances of decision-making units using interval efficiencies. *Journal of Computational and Applied Mathematics*, 198, 253-267.

[12] Wang, N.S., Yi, R.H., & Wang, W. (2008). Evaluating the performances of decision-making units based on interval efficiencies. *Journal of Computational and Applied Mathematics*, 216, 328-343.

[13] Wang, Y.M., & Luo, Y. (2006). DEA efficiency assessment using ideal and anti-ideal decision making units. *Applied Mathematics and Computation*, 173, 902-915.

[14] Wang, Y.M., Chin, K.S., & Yang, J.B. (2007). Measuring the performances of decision-making units using geometric average efficiency. *Journal of the Operational Research Society*, 58, 929-937.

[15] Wang, Y.M., & Chin, K.S. (2009). A new approach for the selection of advanced manufacturing technologies: DEA with double frontiers. *International Journal of Production Research*, 47, 6663-6679.

[16] Wang, Y.M., Greatbanks, R., & Yang, J.B. (2005). Interval efficiency assessment using data envelopment analysis, *Fuzzy Sets and Systems*, 153, 347-370.