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ORIGINAL ARTICLE

New Exact Traveling Wave Solution of Fisher Kolmogorov-Petrovskii-Piskunov Equation for Favorite Genes Spreading by $(1/G)$-expansion Method

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ABSTRACT: Although designing and developing a mathematical model is extremely important in the mathematics but finding solution for designing model is essential as well. Thus one cannot propose a model without offering its solutions. In the mathematical modeling, there are many models based on nonlinear partial differential equations. In such models, there is no general method for solving any problem. However, numerical methods, approximate methods or analytical methods are available for some problems. It is clear that among the methods for solving a model based on partial differential equations, analytical methods are preferred, but for all problems, it is not possible to provide an exact solution. In this case, some methods can provide a class of solutions. In such methods, techniques that lead to more solutions are more important, but the use of different methods can provide a wide class of solutions. For this reason, various methods are used to find the possible solution of nonlinear partial differential equations. One of these methods is the $(1/G)-$expansion method. Since one of the well-known equations with wide application in genetics and gene mutation is the Fisher Kolmogorov-Petrovskii-Piskunov (Fisher KPP) equation, we applied $(1/G)-$expansion method, for finding exact traveling wave solutions which are based on the solutions of Bernoulli ordinary differential equation.

INTRODUCTION

In the biological sciences, mathematical models are important to develop various hypotheses about biological processes. It provides a powerful tool for analyzing biological problems. On the other hand, mathematical models allow developing and testing hypotheses which can lead to a better understanding of the biological process. Fisher Kolmogorov–Petrovsky–Piskunov (Fisher KPP) equation is the partial differential equation which imposes relations between the various partial derivatives of a multivariable function. The Fisher KPP model has been applied in many biological fields such as spatial spreading of

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invasive species in the environment, the spatial spreading of invasive cell populations, modelling of wound healing, tissue engineering, tumor growth and cancer treatment [1]. A mutation is a change in a DNA sequence of the genome of an organism, virus or extrachromosomal DNA. Mutations result either during DNA replication or from exposure to ultraviolet light, X-rays, particle radiation or to environmental reactive chemicals. Because mutations are random changes, they are expected to be mostly harmful, but some may be beneficial in certain environments. In general, mutation is the main source of genetic variation, which provides the raw material for evolution by natural selection [2]. Fisher proposed his model for the spreading of a gene throughout a population. Since 1937, this model has been continuously growing.

Suppose $u(x,t)$ is the proportion of people in a population at a point x and time t who have the favorite gene. Fisher introduced the following equation

$$\frac{\partial u}{\partial t} = au(u-1), \qquad a>0$$

where a is a parameter [3, 4]. He supposes that the offspring of a person at x with favorite genes would not remain in the same region but would be randomly scattered in the x neighborhood. Therefore, he used the heat diffusion equation to modify the equation and introduced the following equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + au(u-1).$$

In the same year, independently from Fisher, other scientists (Kolmogorov, Petrovskii, and Piskunov (KPP)) were investigated the favorite gene diffusion. Their model was presented form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \qquad (1)$$

which $u(x,t)$ is the frequency of the favorite gene in point x and time t. For the function $f(u)$, several assumptions were considered [4, 5].

An equation in the form (1), where $D>0$, is called the Fisher Kolmogorov-Petrovskii-Piskunov (Fisher KPP).[6].

Since there is no general method for solving nonlinear partial differential equations, so different numerical techniques and different analytical methods are considered to solve such equations. Methods such as, Sub-ODE [7,8], Tanh-expansion [9], (G'/G)-expansion [10-12], $(1/G')$-expansion [13] can solve a class of equations.

Numerical methods are also used whenever there are no analytical and existing methods. In this case, the volume of calculations is very important.

Recently, some exact solution methods, such as the Sub-ODE method, have been developed base on auxiliary differential equations. In this method, ordinary differential equations with lower order and known as Riccati and Bernoulli, and so on are used as auxiliary differential equations [7, 8, 12 and 14]. By applying this method, exact solutions are found.

Different types of numerical, approximate or analytical methods have been used to solve the Fisher KPP equation but $(1/G')$-expansion method has not been used to find the solutions so far [3, 6, 8, 15 and 16].

Hence, in this article we use $(1/G')$-expansion method to find exact traveling wave solutions of the Fisher KPP equation.

This article is organized in this form. In section 2, we introduce the problem, in section 3 we explain $(1/G')$-expansion method for solving nonlinear partial differential equations, in section 4 we apply this method for finding solutions to the Fisher KPP equation and the conclusion is given at the end.

Problem statement

We assume $u(x,t)$ is a function of two variables $x \in \Re$ and $t \in [0,+\infty)$. We consider the equation

$$u_t = Du_{xx} + \beta u + \gamma u^2 + \delta u^3 \qquad (2)$$

where D,β,γ and δ are constants [3,16]. This equation can combination with the initial condition to form $u(x,0) = u_0, \quad x \in \Re$. One of the important forms of the equation is the following, which we will find the solutions with using $(1/G')$-expansion method.

$$u_t = Du_{xx} + f(u) \qquad (3)$$

where $f(u) = S\,u\,(u-1)$, that S and D are real constants.

The $(1/G')$-expansion method

Suppose that p is a polynomial of $u(x,t)$ and its derivation of different order as follows:

$$p(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \qquad (4)$$

With transform $\xi = c(x - wt)$, we will have

$$\begin{aligned} \frac{\partial u}{\partial x} &= c\frac{du}{d\xi} \\ \frac{\partial u}{\partial t} &= -cw\frac{du}{d\xi} \\ \frac{\partial^2 u}{\partial t^2} &= c^2 w^2 \frac{d^2 u}{d\xi^2} \\ \frac{\partial^2 u}{\partial x^2} &= c^2 \frac{d^2 u}{d\xi^2} \\ &\ldots \end{aligned} \qquad (5)$$

So, differential equation (4) is converted to ordinary differential equation to form

$$H(u, u', u'', \ldots) = 0 \quad . \quad (6)$$

Lets G be solution Bernoulli differential equation

$$G'' + \lambda G' + \mu = 0, \quad \lambda \neq 0 \qquad (7)$$

that λ and μ are constant numbers. Solutions of above equation

$$G(\xi) = -\frac{\mu\xi}{\lambda} + ce^{-\lambda\xi} + d.$$

That c and d are constant numbers. Then

$$\frac{1}{G'} = -\frac{\lambda}{\mu + \lambda^2 c_1(\cosh(\lambda\xi) - \sinh(\lambda\xi))}.$$

Now, if suppose that solutions of fisher KPP can be written to form a power series of $\frac{1}{G'}$, as follows

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{1}{G'(\xi)}\right)^i \qquad (8)$$

That $a_i, i = 0,1,2,\ldots,n$ are parameters that could be determined and n is given of balance more nonlinear terms and more order derivation in (6).

With substitution (8) in (6) and sort to various powers of $\frac{1}{G'}$, and vanish coefficient relater , a system of algebraic differential obtain. By solve this system, coefficients $a_i, i = 0,1,2,\ldots,n$ will have find [10].

Computation results

We consider Fisher KPP equation as follows

$$u_t = Du_{xx} + Su(u-1) \qquad (9)$$

With $\xi = c(x - wt)$ we follows ordinary differential equation will have

$$-cw = Dc^2 u'' + Su(u-1) \qquad (10)$$

By suppose that equation (10) have a solution to form (7), balancing terms of u'' and u^2 ,

$$u'' = n(n-1)a_n \left(\frac{1}{G'(\xi)}\right)^{n+2} + ...,$$
$$u^2 = a_n^2 \left(\frac{1}{G'(\xi)}\right)^{2n} + ...$$

We have $n+2 = 2n$. Therefore $n = 2$. So,

$$u(\xi) = a_0 + a_1 \frac{1}{G'(\xi)} + a_2 \left(\frac{1}{G'(\xi)}\right)^2 \qquad (11)$$

With substitution (11) in (10) and sort statement in respect to a different power of $\frac{1}{G'(\xi)}$ and vanish coefficients, an algebraic system of equations is the result.

$$\begin{aligned}
&\left(\frac{1}{G'(\xi)}\right)^0 : && -a_0 S(a_0 - 1) = 0 \\
&\left(\frac{1}{G'(\xi)}\right)^1 : && cw\lambda a_1 + c\lambda^2 D a_1 - a_0 S a_1 - S a_1(a_0 - 1) = 0 \\
&\left(\frac{1}{G'(\xi)}\right)^2 : && cw\mu a_1 + 2cw\lambda a_2 + 2c\lambda D(a_1\mu + 2a_2\lambda) + c\mu D\lambda a_1 - a_0 S a_2 - S a_1^2 - S a_2(a_0 - 1) = 0 \qquad (12) \\
&\left(\frac{1}{G'(\xi)}\right)^3 : && 2cw\mu a_2 + 6c\lambda D a_2 \mu + 2c\mu D(a_1\mu + 2a_2\lambda) - 2S a_1 a_2 = 0 \\
&\left(\frac{1}{G'(\xi)}\right)^4 : && 6 D a_2 c \mu^2 - a_2^2 S = 0
\end{aligned}$$

To solve algebraic system equations, the coefficients $a_i, i = 0,1,2$ are will obtain. Other than the two solutions $u(x,t) = 0$ and $u(x,t) = 1$, the other solutions are as follows.

The first solution:

$$a_0 = 0, a_1 = 0, a_2 = \frac{\mu^2}{\lambda^2}, c = \frac{1}{6}\frac{S}{D\lambda^2}, w = -5D\lambda.$$

That in this case

$$u_1(x,t) = \frac{\mu^2}{\left(\mu + \lambda^2 c_1\left(\cosh\left(\frac{S(5D\lambda t + x)}{6\lambda D}\right) - \sinh\left(\frac{S(5D\lambda t + x)}{6\lambda D}\right)\right)\right)^2}$$

is an exact traveling wave solution of the Fisher KPP equation. Where $D.S, \lambda, \mu,$ and c_1 are constant numbers.

Figure 1, for $D = 1, S = 6, \lambda = -3, \mu = 1$ and $c_1 = 2$, $u_1(x,t)$ is plotted.

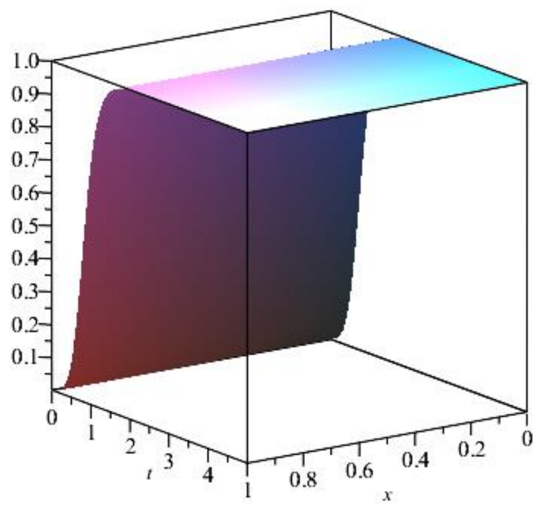

Figure 1. Plot of $u_1(x,t)$ with $D=1, S=6, \lambda=-3, \mu=1, c_1=2.$

The second solution

$a_0=1, a_1=0, a_2=-\frac{\mu^2}{\lambda^2}, c=-\frac{1}{6}\frac{5}{D\lambda^2}, w=-5D\lambda,$ And

$$u_2(x,t)=1-\frac{\mu^2}{\left(\mu+\lambda^2 c_1\left(\cosh\left(\frac{S(5D\lambda t+x)}{6\lambda D}\right)+\sinh\left(\frac{S(5D\lambda t+x)}{6\lambda D}\right)\right)\right)^2}$$

is an exact traveling wave solution of the Fisher KPP equation. Where $D, S, \lambda, \mu,$ and c_1 are constant numbers.

Figure 2, for $D=1, S=6, \lambda=-3, \mu=1,$ and $c_1=2,$ $u_2(x,t)$ is plotted.

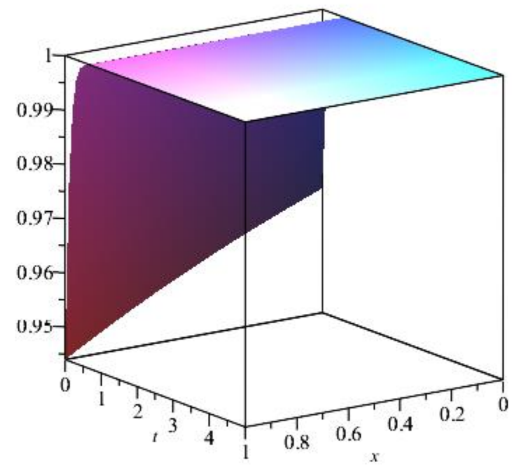

Figure 2. Plot of $u_2(x,t)$ with $D=1, S=6, \lambda=-3, \mu=1, c_1=2.$

The third solution

$$a_0=1, a_1=6\frac{\mu}{\lambda}, a_2=6\frac{\mu^2}{\lambda^2}, c=\frac{S}{D\lambda^2}, w=0.$$

And

$$u_3(x,t)=1-\frac{6\mu}{\mu+\lambda^2 c_1\left(\cosh\left(\frac{Sx}{\lambda D}\right)-\sinh\left(\frac{Sx}{\lambda D}\right)\right)}+\frac{6\mu^2}{\left(\mu+\lambda^2 c_1\left(\cosh\left(\frac{Sx}{\lambda D}\right)-\sinh\left(\frac{Sx}{\lambda D}\right)\right)\right)^2}$$

is an exact traveling wave solution of the Fisher KPP equation. Where $D, S, \lambda, \mu,$ and c_1 are constant numbers.

Figure 3, for $D=1, S=6, \lambda=-3, \mu=1,$ and $c_1=2,$ $u_3(x,t)$ is plotted.

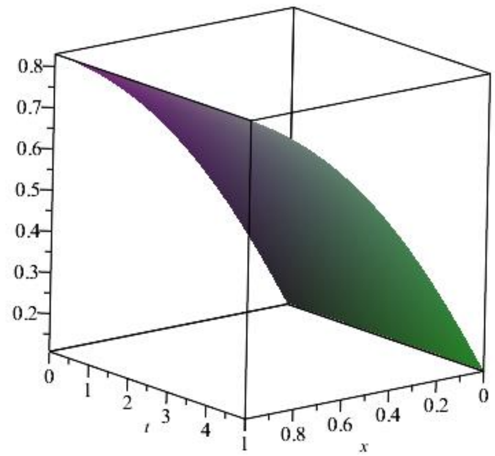

Figure 3. Plot of $u_3(x,t)$ with $D=1, S=6, \lambda=-3, \mu=1, c_1=2.$

The fourth solution

$$a_0=0, a_1=-6\frac{\mu}{\lambda}, a_2=-6\frac{\mu^2}{\lambda^2}, c=-\frac{S}{D\lambda^2}, w=0.$$

So we will have

$$u_4(x,t)=\frac{6\mu}{\mu+\lambda^2 c_1\left(\cosh\left(\frac{Sx}{\lambda D}\right)+\sinh\left(\frac{Sx}{\lambda D}\right)\right)}-\frac{6\mu^2}{\left(\mu+\lambda^2 c_1\left(\cosh\left(\frac{Sx}{\lambda D}\right)+\sinh\left(\frac{Sx}{\lambda D}\right)\right)\right)^2}$$

Where $D, S, \lambda, \mu,$ and c_1 are constant numbers. This solution, for $D=1, S=6, \lambda=-3, \mu=1,$ and $c_1=2,$ is plotted in Figure 4.

The fifth solution

$$a_0=1, a_1=2\frac{\mu}{\lambda}, a_2=\frac{\mu^2}{\lambda^2}, c=-\frac{1}{6}\frac{S}{D\lambda^2}, w=5D\lambda.$$

So,

$$u_5(x,t)=1-\frac{2\mu}{\mu+\lambda^2 c_1\left(\cosh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)-\sinh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)\right)}+\frac{\mu^2}{\left(\mu+\lambda^2 c_1\left(\cosh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)-\sinh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)\right)\right)^2}$$

Where $D, S, \lambda, \mu,$ and c_1 are constant numbers. This solution, for $D=1, S=6, \lambda=-3, \mu=1,$ and $c_1=2,$ is plotted in Figure 5.

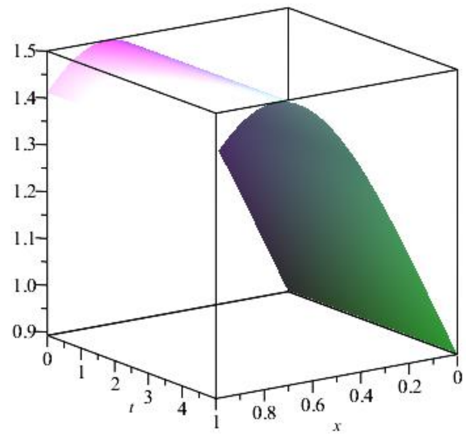

Figure 4. Plot of $u_4(x,t)$ with $D=1, S=6, \lambda=-3, \mu=1, c_1=2.$

The sixth solution:

$$a_0=0, a_1=-2\frac{\mu}{\lambda}, a_2=-\frac{\mu^2}{\lambda^2}, c=-\frac{1}{6}\frac{S}{D\lambda^2}, w=5D\lambda.$$

Therefore

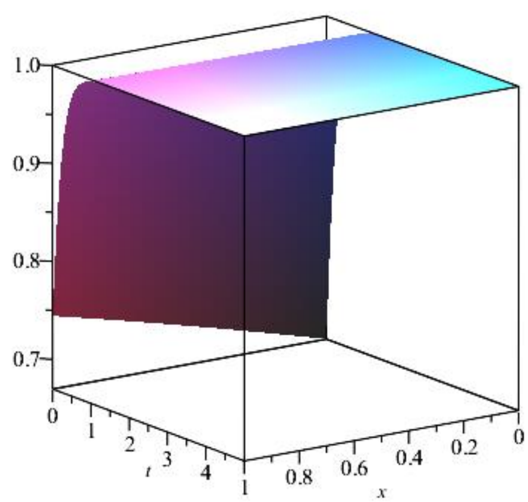

Figure 5. Plot of $u_5(x,t)$ with $D=1, S=6, \lambda=-3, \mu=1, c_1=2.$

$$u_6(x,t)=\frac{2\mu}{\mu+\lambda^2 c_1\left(\cosh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)+\sinh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)\right)}-\frac{\mu^2}{\left(\mu+\lambda^2 c_1\left(\cosh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)+\sinh\left(\frac{S(-5D\lambda t+x)}{6\lambda D}\right)\right)\right)^2}.$$

Where $D, S, \lambda, \mu,$ and c_1 are constant numbers. This solution, for $D=1, S=6, \lambda=-3, \mu=1,$ and $c_1=2$, is plotted in Figure 6.

In reference [17], the Fisher KPP with initial condition

$$\begin{cases} u_t = u_{xx} + u(1-u) \\ u(x,0) = \dfrac{1}{\left(1+e^x\right)^2} \end{cases} \qquad (13)$$

with use of homotopy method is solved and approximate solution

$$u(x,t) = \frac{1}{\left(1+e^x\right)^2} + \frac{10e^x}{\left(1+e^x\right)^3}t + \frac{25e^x\left(-1+e^{2x}\right)}{\left(1+e^x\right)^4}t^2 + \frac{125\left(-1+7e^x-4e^{2x}\right)}{3\left(1+e^x\right)^5}t^3 + \Lambda .$$

obtained. This series converge to exact solution in the form

$$u(x,t) = \frac{1}{\left(1+e^{x-5t}\right)^2}.$$

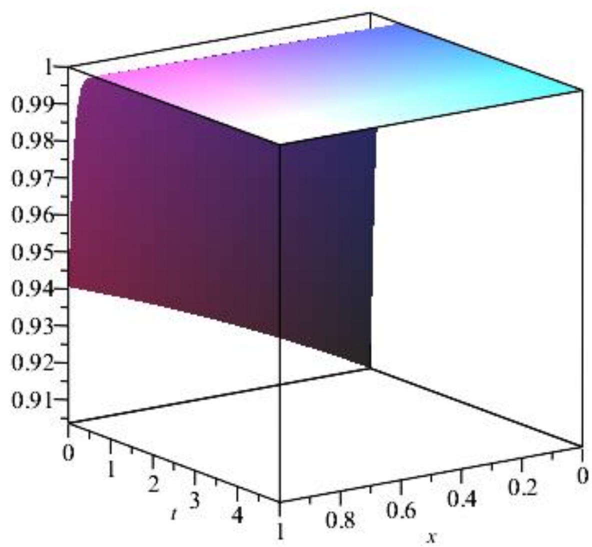

Figure 6. Plot of $u_6(x,t)$ with $D=1, S=6, \lambda=-3, \mu=1, c_1=2.$

With using of $(1/G)$-expansion method and $D=1, S=-1,$ the following solutions are obtained:

$$u(x,t) = \frac{\mu^2}{\left(\mu + \lambda^2 c_1\left(\cosh\left(\frac{5\lambda t + x}{6\lambda}\right) + \sinh\left(\frac{5\lambda t + x}{6\lambda}\right)\right)\right)^2}$$

Where μ, λ and c_1 are parameters. According to the initial condition of problem, two solutions with $\mu=1, \lambda=\frac{1}{6}, c_1=36$ and $\mu=-1, \lambda=\frac{1}{6}, c_1=-36,$ will be obtained.

$$u_{1,1}(x,t) = \frac{1}{\left(1+\left(\cosh\left(\frac{5}{6}t+x\right)+\sinh\left(\frac{5}{6}t+x\right)\right)\right)^2}$$

$$u_{2,1}(x,t) = -\frac{1}{\left(1+\left(\cosh\left(\frac{5}{6}t+x\right)+\sinh\left(\frac{5}{6}t+x\right)\right)\right)^2}$$

Two above two solutions are shown in Figures 7 and 8. These two solutions are different from the approximate solution with the homotopy method and also the exact solution in reference [17]. Therefore we found new solutions to the Fisher KPP initial value problems (13).

All the results of this section are presented using Maple software

CONCLUSIONS

Difference in genotypes is responsible for biodiversity, adaptation, differences in dispersal rates and growth rates of individuals. Mutation, as a powerful mechanism for genotypes alternations, may favor certain genotypes and enhance their survival or even spatial spreading into areas which are occupied by other wild genotypes. Fisher KPP equation is used for studying gene mutation and genes spreading between populations. In this article, the $(1/G)$-expansion method has been successfully used to find traveling wave solutions of the Fisher KPP partial differential equation for gene diffusion in a population.

In the $(1/G)$-expansion method, because the solutions of the second-order differential equation in the form (7) are used, obtained solutions have five parameters that show a wide class of solutions obtained. By using this method, not only exact solutions will obtain but also calculation volumes will be decreased.

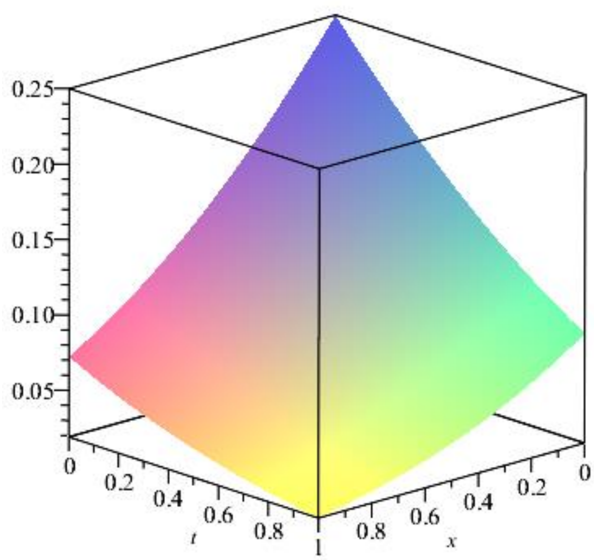

Figure 7. Plot of $u_{1,1}(x,t)$

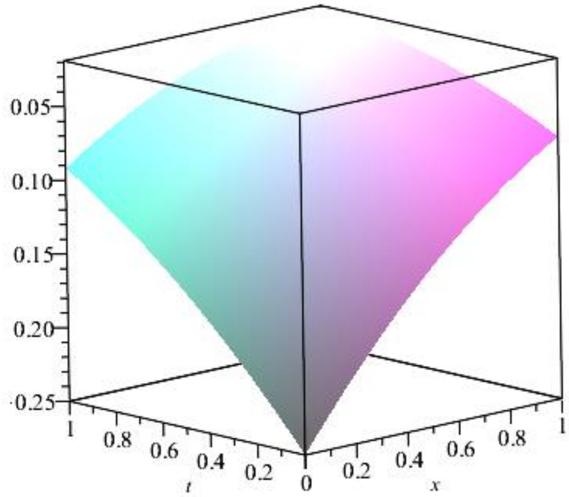

Figure 8. Plot of $u_{2,1}(x,t)$

CONFLICT OF INTERESTS

The authors declare that they have no conflict of interest.

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