Rotated Unscented Kalman Filter for Two State Nonlinear Systems

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ABSTRACT

In the several past years, Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF) have became basic algorithm for state-variables and parameters estimation of discrete nonlinear systems. The UKF has consistently outperformed for estimation. Sometimes least estimation error doesn't yield with UKF for the most nonlinear systems. In this paper, we use a new approach for a two variable state nonlinear systems which it is called Rotated UKF (R_UKF). R_UKF can be reduced estimation error and reached for least error in state estimation.

KEYWORDS: *Extended kalman filter (EKF), unscented kalman filter (UKF), rotated UKF (R-UKF)*

1. INTRODUCTION

It is derived from the Kalman filter based on successive linearization of the signal process and observation map [1]. The algorithm of EKF is sub-optimal, so can be became divergence. UKF is a free-divergence approach and it can be used instead of EKF for state estimation [2]. The UKF has been used for state-estimation and parameter-estimation at nonlinear systems. The UKF is a derivative free alternative to the EKF. R_UKF is a new suggestion algorithm for estimation two variable state nonlinear systems. This paper is organized as follows. After the introduction in section 1, discussion UKF algorithm is discussion for estimation of parameter nonlinear system in section 2, section 3 present equal R_UKF for nonlinear system with two state variables. Next approach R_UKF and UKF for as sample system and compare simulation result in section 4. Finally the conclusions is given in section5.

2. UNSCENTED KALMAN FILTER

The basic usage for the UKF is estimation of the state of a discrete-time nonlinear dynamic system. The UKF utilizes the Unscented Transformation (UT). For this application, the system nonlinear state equations are expressed in the discretized form:

$$X_{K+1} = F(X_K, U_K) + V_K$$
$$Y_K = H(X_K) + n_K \qquad (1)$$

X_K as a parameter that represents the state-variable of the system, U_K is a known input and Y_K is the output measurement signal. The process noise V_K drives dynamic system and the measurement noise is given by n_K. V_K and n_K in the UKF are assumed Gaussian noise. Sometimes UKF is used for system identification, with nonlinear mapping $Y_K = G(X_K, W)$, where X_K is input, Y_K is output, and the nonlinear map $G(.)$ is parameterized by vector W.

$$W_{K+1} = W_K + r_K$$
$$d_K = G(X_K, W_K) + e_K \qquad (2)$$

The process noise r_K drives the dynamic system, and the measurement noise is given by e_K .The output d_K corresponds to a nonlinear observation on W_K .The EKF have inherent flaws, because EKF uses of linearization approach for calculating the mean and covariance of a random variables [2,3,4]. In the UKF eliminated these flaws by using a deterministic sampling approach to calculate covariance and mean. Essentially, $2L+1$sigma points are chosen based on a square-root decomposition of the previous covariance. These sigma points are spread to the true nonlinearity, without approximation, and then a weighted mean and covariance is calculated. A simple illustration of this approach is shown in Figure 1 for a two-dimensional system. Figure 1-a shows the true mean and covariance propagation using Monte-Carlo sampling; the center plots show the results using a linearization approach (EKF) , the right plots show the performance of the UKF approach for 5 sigma points. The standard UKF involves the recursive application of this sampling approach to the state-space equations. The standard UKF shown in Algorithm 1 for state-estimation, with using the following definitions:

$$W_0^m = \lambda / (L + \lambda)$$
$$W_0^c = \lambda / (L + \lambda) + (1 - \alpha^2 + \beta)$$
$$W_i^c = W_i^m = 1 / \{2(L + \lambda)\} \quad i = 1, 2, \ldots, 2L \qquad (3)$$

W_i Is the weight associated with the i-th sigma point so that $\sum_{i=0}^{2L} W_i = 1$.

Scaling parameters are:

$$\lambda = \alpha^2 (L + \kappa) - L$$
$$\gamma = \sqrt{(L + \lambda)}$$
$$\kappa = 3 - L \qquad (4)$$

The constant α determines the spread of the sigma points around $\hat{X}$ ($10^{-4} \le \alpha \le 1$). κ is second scaling parameter which is usually set to $\kappa = 3 - L$ and β is used to incorporate prior knowledge of the distribution of X(for Gaussian distributions), $\beta = 2$ is optimal Also note that we use linear algebra operation by adding a column to a matrix. Now it is choosed a set of $2L+1$ weighted samples X_i (sigma points) deterministically so that they completely represent the true mean and covariance of state X.

$$\hat{X}_0 = \hat{X}$$
$$\hat{X}_i = \hat{X} + (\gamma \sqrt{P_{K-1}})_i \qquad i = 1, \ldots, L$$
$$\hat{X}_i = \hat{X} - (\gamma \sqrt{P_{K-1}})_i \qquad i = n+1, \ldots, L$$
$$X_{k-1} = \left[\hat{X}_{k-1} \quad \hat{X}_{k-1} + \left(\gamma \sqrt{P_{k-1}}\right) \quad \hat{X}_{k-1} - \left(\gamma \sqrt{P_{k-1}}\right) \right] \qquad (5)$$

The mean and covariance of Y are approximated by the weighted average mean and covariance of the transformed sigma point.

$$\hat{y} = \sum_{i=0}^{2L} W_i Y_i$$
$$P_y = \sum_{i=0}^{2L} W_i (Y_i - \hat{y})(Y_i - \hat{y})^T \qquad (6)$$

Figure 2 shows the standard UKF algorithm.

3. ROTATED UNSCENTED KALMAN FILTER

However, standard UKF approach has more advantages with comparing EKF [3,4], but it couldn't calculate state variables without error and bios. In this paper we discuss a new algorithm for calculating sigma points, which it chooses sigma points better than UKF. Percentage of improving is correlated to type of nonlinear system [5]. This new algorithm can be used for nonlinear systems with only two state variables.

$$X_{K+1} = F(X_K, U_K) + V_K$$
$$Y_K = H(X_K) + n_K \qquad (7)$$

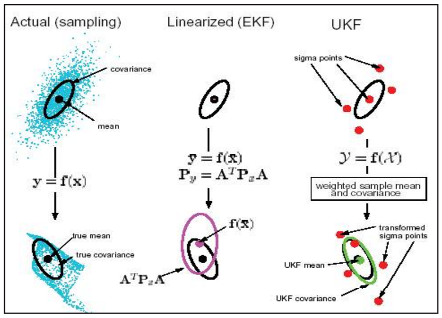

Fig.1. Example of mean and covariance propagation.

Initial with:

$$\hat{X}_0 = E[X_0] \qquad P_0 = E\left[\left(X_0 - \hat{X}_0\right)\left(X_0 - \hat{X}_0\right)^T\right]$$

For $K \in \{1,2,....,\infty\}$

Calculate Sigma points:

1: $X_{k-1} = \left[\hat{X}_{k-1} \quad \hat{X}_{k-1} + \left(\gamma\sqrt{P_{k-1}}\right) \quad \hat{X}_{k-1} - \left(\gamma\sqrt{P_{k-1}}\right)\right]$

Time update:

2: $X^*_{k|k-1} = F\left[X_{k-1}, U_{k-1}\right]$

3: $\hat{X}^-_k = \sum_{i=0}^{2L} W_i^m X^*_{i,k|k-1}$

4: $P^-_k = \sum_{i=0}^{2L} W_i^c \left[X^*_{i,k|k-1} - \hat{X}^-_k\right]\left[X^*_{i,k|k-1} - \hat{X}^-_k\right]^T + R^v$

R^v = process noise covariance

5: $X_{k|k-1} = \left[\hat{X}^-_k \quad \hat{X}^-_k + \left(\gamma\sqrt{P^-_k}\right) \quad \hat{X}^-_k + \left(\gamma\sqrt{P^-_k}\right)\right]$

6: $Y_{k|k-1} = H\left[X_{k|k-1}\right]$

7: $\hat{Y}^-_k = \sum_{i=0}^{2L} W_i^m Y_{i,k|k-1}$

Measurement updates equations:

8: $P_{\hat{y}_k \hat{y}_k} = \sum_{i=0}^{2L} W_i^c \left[Y_{i,k|k-1} - \hat{y}^-_k\right]\left[Y_{i,k|k-1} - \hat{y}^-_k\right]^T + R^n$

R^n = measurement noise covariance

9: $P_{x_k y_k} = \sum_{i=0}^{2L} W_i^c \left[X_{i,k|k-1} - \hat{x}^-_k\right]\left[Y_{i,k|k-1} - \hat{y}^-_k\right]^T$

10: $K_k = P_{x_k y_k} P^{-1}_{\hat{y}_k \hat{y}_k}$

11: $\hat{X}_k = \hat{X}^-_k + K_k\left(y_k - \hat{y}^-_k\right)$

12: $P_k = P^-_k - K_k P_{\hat{y}_k \hat{y}_k} K_k^T$

Fig.2. Standard UKF algorithm.

State variables: $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

For two state variables nonlinear systems the UKF algorithm calculates sigma points with equations (8):

$$\hat{X}_0 = E[X_0] \qquad P_0 = E\left[\left(X_0 - \hat{X}_0\right)\left(X_0 - \hat{X}_0\right)^T\right]$$

$$\gamma\sqrt{P_{k-1}} = \begin{bmatrix} a1 & a2 \\ b1 & b2 \end{bmatrix} \tag{8}$$

P_{K-1} is state covariance matrix:

$$\left(\gamma\sqrt{P_{k-1}}\right)_1 = \begin{bmatrix} a1 \\ b1 \end{bmatrix}$$

$$\left(\gamma\sqrt{P_{k-1}}\right)_2 = \begin{bmatrix} a2 \\ b2 \end{bmatrix} \tag{9}$$

Sigma points are follows:

$$X_{k-1} = \left[\hat{X}_{k-1} \quad \hat{X}_{k-1} + \left(\gamma\sqrt{P_{k-1}}\right)_1 \quad \hat{X}_{k-1} + \left(\gamma\sqrt{P_{k-1}}\right)_2 \quad \hat{X}_{k-1} - \left(\gamma\sqrt{P_{k-1}}\right)_1 \quad \hat{X}_{k-1} - \left(\gamma\sqrt{P_{k-1}}\right)_2\right] \tag{10}$$

It is used equation (11) for calculating update sigma points:

$$X_{k|k-1} = \left[\hat{X}^-_k \quad \hat{X}^-_k + \left(\gamma\sqrt{P^-_k}\right)_1 \quad \hat{X}^-_k + \left(\gamma\sqrt{P^-_k}\right)_2 \quad \hat{X}^-_k - \left(\gamma\sqrt{P^-_k}\right)_1 \quad \hat{X}^-_k - \left(\gamma\sqrt{P^-_k}\right)_2\right] \tag{11}$$

In R_UKF , it is utilized following equations for sigma points calculating:

Let:

$$M\theta1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \& \quad M\theta2 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \tag{12}$$

In these equations θ can be represented by: $0 \le \theta \le 180^o$ Or $-90^o \le \theta \le +90^o$

New sigma points are follows:

$$X_{k-1} = \begin{bmatrix} \hat{X}_{k-1} \\ \hat{X}_{k-1} + \left(\gamma\sqrt{P_{k-1}}\right)_1 .M\theta1 \\ \hat{X}_{k-1} + \left(\gamma\sqrt{P_{k-1}}\right)_2 .M\theta2 \\ \hat{X}_{k-1} - \left(\gamma\sqrt{P_{k-1}}\right)_1 .M\theta1 \\ \hat{X}_{k-1} - \left(\gamma\sqrt{P_{k-1}}\right)_2 .M\theta2 \end{bmatrix}^T \tag{13}$$

Also, equation (14) in indicates to update sigma points:

$$X_{k-1} = \begin{bmatrix} \hat{X}_{k-1} \\ \hat{X}_{k-1} + \left(\gamma\sqrt{P^-_k}\right)_1 .M\theta1 \\ \hat{X}_{k-1} + \left(\gamma\sqrt{P^-_k}\right)_2 .M\theta2 \\ \hat{X}_{k-1} - \left(\gamma\sqrt{P^-_k}\right)_1 .M\theta1 \\ \hat{X}_{k-1} - \left(\gamma\sqrt{P^-_k}\right)_2 .M\theta2 \end{bmatrix}^T \tag{14}$$

R_UKF approach is shown in figure 2.

Initial with:

$$X=\begin{bmatrix}X_1\\X_2\end{bmatrix}\ \hat{X}_0=E[X_0]\ P_0=E\left[(X_0-\hat{X}_0)(X_0-\hat{X}_0)^T\right]$$

For $K\in\{1,2,....,\infty\}$

$$M\theta 1=\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\ \&\ M\theta 2=\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

Calculate Sigma points:

1: $$X_{k-1}=\begin{bmatrix}\hat{X}_{k-1}\\ \hat{X}_{k-1}+\left(\gamma\sqrt{P_{k-1}}\right)_1.M\theta 1\\ \hat{X}_{k-1}+\left(\gamma\sqrt{P_{k-1}}\right)_2.M\theta 2\\ \hat{X}_{k-1}-\left(\gamma\sqrt{P_{k-1}}\right)_1.M\theta 1\\ \hat{X}_{k-1}-\left(\gamma\sqrt{P_{k-1}}\right)_2.M\theta 2\end{bmatrix}^T$$

Time update:

2: $X^*_{k|k-1}=F[X_{k-1},U_{k-1}]$

3: $\hat{X}^-_k=\sum_{i=0}^{2L}W^m_i X^*_{i,k|k-1}$

4: $P^-_k=\sum_{i=0}^{2L}W^c_i\left[X^*_{i,k|k-1}-\hat{X}^-_k\right]\left[X^*_{i,k|k-1}-\hat{X}^-_k\right]^T+R^v$

R^v = process noise covariance

5: $$X_{k-1}=\begin{bmatrix}\hat{X}_{k-1}\\ \hat{X}_{k-1}+\left(\gamma\sqrt{P^-_k}\right)_1.M\theta 1\\ \hat{X}_{k-1}+\left(\gamma\sqrt{P^-_k}\right)_2.M\theta 2\\ \hat{X}_{k-1}-\left(\gamma\sqrt{P^-_k}\right)_1.M\theta 1\\ \hat{X}_{k-1}-\left(\gamma\sqrt{P^-_k}\right)_2.M\theta 2\end{bmatrix}^T$$

6: $Y_{k|k-1}=H[X_{k|k-1}]$

7: $\hat{Y}^-_k=\sum_{i=0}^{2L}W^m_i Y_{i,k|k-1}$

Measurement updates equations:

8: $P_{\hat{y}_k\hat{y}_k}=\sum_{i=0}^{2L}W^c_i\left[Y_{i,k|k-1}-\hat{y}^-_k\right]\left[Y_{i,k|k-1}-\hat{y}^-_k\right]^T+R^n$

R^n = measurement noise covariance

9: $P_{x_k y_k}=\sum_{i=0}^{2L}W^c_i\left[X_{i,k|k-1}-\hat{x}^-_k\right]\left[Y_{i,k|k-1}-\hat{y}^-_k\right]^T$

10: $K_k=P_{x_k y_k}P^{-1}_{\hat{y}_k\hat{y}_k}$

11: $\hat{X}_k=\hat{X}^-_k+K_k\left(y_k-\hat{y}^-_k\right)$

12: $P_k=P^-_k-K_kP_{\hat{y}_k\hat{y}_k}K^T_k$

Fig.3. Rotate UKF algorithm.

For more nonlinear systems can find convenient rotate angle, which errors are minimum [6].We called θ this rotation angle. θ can add to adjusting parameters of standard UKF algorithm [7].

4. SIMULATION

The improvement in error performance of the R_UKF for state estimation is shown in following example. Results of simulation for R_UKF are compared with standard UKF.

$$\dot{X}_0=-X_0-2X_1^2+U$$
$$\dot{X}_1=X_0X_1-X_1^3+U$$
$$\mathrm{Y}=0.8\mathrm{X}_0+\mathrm{X}_1 \quad (15)$$

Initial conditions are:

$$P_0=\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix}\quad X_0=\begin{bmatrix}0\\0\end{bmatrix}$$
$$R_v=\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix}\quad R^n=0.12$$
$$\alpha=1\quad\beta=2\quad\kappa=0 \quad (16)$$

Figure 4 shows relation between Mean Square Error (MSE) of X_0 & X_1 and rotated angle. For this example, least MSE for state X_0 is at rotated angle 14° and least MSE for state X_1 is at rotated 8°. Standard UKF is equal with rotated 0°. In this nonlinear system, the best rotated angle is about 11° for total least MSE.

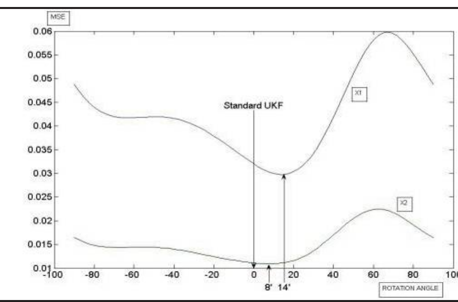

Fig.4. MSE of X_0 & X_1 due rotation angle from -90° to $+90^{\circ}$

Figure 5 & Figure 6 show the superior performance of R_UKF compared with standard UKF for state estimating the white noise (3dB SNR). The performance of R_UKF is superior to the standard UKF for state estimating. The computational of R-UKF is a few complexes, but MSE for states in the R_UKF is approximately 10% less than the standard UKF.

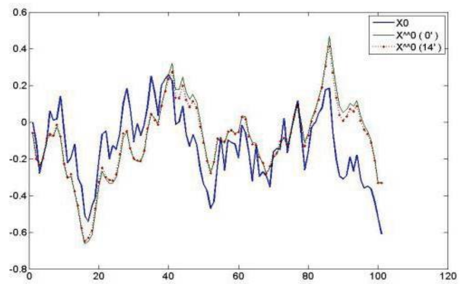

Fig.5. Estimation of X_0 with white noise (3dB SNR) with standard UKF (at 0°) & R_UKF (at 14°).

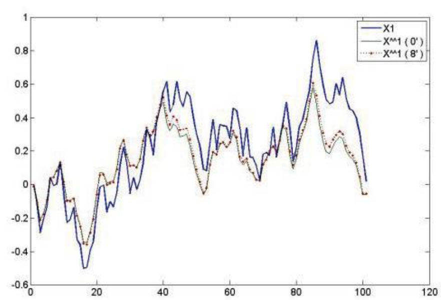

Fig.6. Estimation of X_1 with white noise (3dB SNR) with standard UKF (at 0°) & R_UKF (at 8°).

5. CONCLUSIONS

The R_UKF consistently performs better than or equal than the standard UKF, with the added advantage of R_UKF to standard UKF we can perform best estimation for nonlinear systems. In this paper, it is introduced rotated forms of the UKF. If rotation angle of each nonlinear system adjust finely, the R_UKF has better properties. Nonetheless, this paper discusses for 2 state nonlinear systems, but it may expand to higher number of state nonlinear systems.

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