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Uncertainty Analysis Based on Zadeh's Extension Principle

Ghiyam Eslami
Department of Mechanical Engineering. Ahar Branch, Islamic Azad University, Ahar, Iran
Email: gh.eslami@gmail.com

Abstract

In this paper, it is discussed how Zadeh's extension principle (ZEP) can be used for uncertainty analysis of a system. For this end, basic concepts of the fuzzy mathematics including fuzzy sets, fuzzy numbers and ZEP are briefly presented. A comparison made among the results obtained by the sensitivity analysis, ZEP and Monte Carlo (MC) methods. It is shown that ZEP gives the same outputs as the MC method and is in full agreement with the concept of "uncertainty". The sensitivity analysis result is not the same as the uncertainty analysis and, often results in smaller range for the output parameters.

Keywords: Uncertainty, Sensitivity, Fuzzy mathematics, Zadeh's extension principle, Monte Carlo method

1. Introduction

Every system or model includes several input parameters. Often, values of these parameters directly or indirectly are collected from experimental measurements. For example, there is tremendous information on thermos-physical properties of different materials in Handbooks which report the experimental data. An important point should be considered whenever this information is used. That is an inevitable uncertainty associated with this information [1]. The uncertainty pertains to both the conditions in which the experiments have been conducted and limited accuracy of the measuring instruments. None of the two uncertainty sources can be completely removed. A part from the uncertainty inherited in the input data of a system, the system itself is exposed to some uncertainties. Because, often, the real conditions of the understudy system, differ from the conditions represented in Handbooks. Therefore, it is vital to know how accurate is the outputs of a study. Particularly, whenever more accurate data are needed for decision making, the importance of the uncertainty increases [2].

Suppose that a system involves with several interval-valued inputs. What will be the output interval? The answer of this question comes from the uncertainty analysis. In a more general definition, the uncertainty analysis determines the probability of an output variable based on the probability of input parameters [3].

In engineering investigations, often, the uncertainty of a parameter is presented by adding and subtracting a value from the main magnitude of the parameter. For instance, the electrical current is reported as $I = 5 \pm 0.2A$ .This means that, I , would take any value in the interval $[4.8, 5.2]A$ and there is no difference between the values belonging the interval $[4.8, 5.2]A$. However, in practice, based on the experience and opinion of experts, might it be desired not to treat them equally. For this purpose, the input parameters should be considered as fuzzy numbers [4]. By using the concept of fuzzy sets, it is possible to put a difference among the authorized values of a parameter with the aim of “membership degree function”.

Due to its vital importance, there are a huge number of researches in the field of uncertainty [1-5]. Depending on the problem, there are various approaches for uncertainty analysis [6]. Usually, the accuracy of these methods are evaluated by the Monte Carlo sampling (MCS) method in which a random value is chosen for every input parameter corresponding to its authorized interval. Then the system is run to produce the output. This process is done for many times (often more than 200 tines) and the outputs are determined.

In this paper, we use the so-called interval arithmetic based on the Zadeh’s extension principle for the uncertainty analysis of a system. This method can be very useful, specifically, when an explicit expression is available for the output. The results are compared with the MC method showing very good agreement. Also, it is shown that the sensitivity analysis cannot be replaced with the uncertainty analysis, the case that often occurs in the engineering literature [5,7].

2. Basic concepts in fuzzy Mathematics

In this part, some basic and necessary definitions are stated in the field of fuzzy mathematics.

2.1. Fuzzy number definition

Fuzzy numbers are generalizations of classical real numbers. Whenever there is an uncertainty in a numerical variable, for example, an expression like almost 3 or close to 5.5, it is desired to use fuzzy numbers. In fact, fuzzy numbers are fuzzy subsets of real numbers that also have some characteristics.

A fuzzy set u is called a fuzzy number if it holds the following properties [8]:

1. u is normal, that is $\exists x_0 \in \mathbb{R}, \quad \mu_u(x_0) = 1$,

2. u is fuzzy convex, that is for $0 \le t \le 1$ and $x, y \in \mathbb{R}$:
$\mu_u(tx + (1-t)y) \ge \min\{\mu_u(x), \mu_u(y)\}$,

3. u is upper-semicontinuous, that is $\mu_u(x_0) \ge \lim_{x \to x_0} \sup \mu_u(x)$,

4. u has a compact support. In other phrase: $\overline{\{x \in \mathbb{R} | \mu_u(x) > 0\}}$ is a compact set.

In a simpler way, a fuzzy number u is a upper semicontinuous function (membership function) defined on interval $[a,d]$ where there exist $b, c \in [a,d]$ $(b \le c)$ such that u is non-decreasing on $[a,b]$, non-increasing on $[c,d]$ and equal to 1 on $[b,c]$. The set of all fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$ The most common and applicable fuzzy numbers are triangular and trapezoidal fuzzy numbers due to their simplicity.

2.2. Trapezoidal fuzzy number

A trapezoidal fuzzy number is denoted the by $u = (a,b,c,d)$ and its membership function is:

$$\mu_u(x) = \begin{cases} \frac{x-a}{b-a} & a < x \le b \\ 1 & b < x \le c \\ \frac{d-x}{d-c} & c < x \le d \\ 0 & otherwise \end{cases} \tag{1}$$

A typical trapezoidal fuzzy number is seen in Fig. 1. (a triangular and interval fuzzy number can be produced by the trapezoidal fuzzy number)

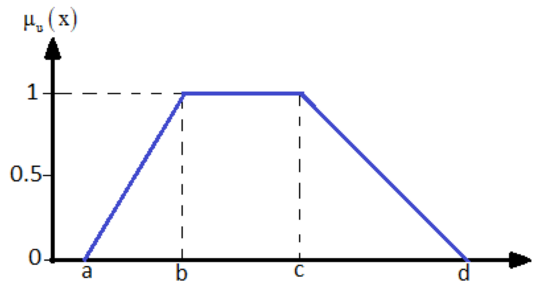

Fig. 1. A typical trapezoidal fuzzy number

2.3. Triangular fuzzy number.

This is the most common fuzzy number and has been used in tremendous investigations [9]. A triangular fuzzy number is denoted by the triple $u=(a,b,c)$ and its membership function is:

$$\mu_u(x)=\begin{cases}\frac{x-a}{b-a} & a<x\le b\\ \frac{c-x}{c-b} & b<x<c\\ 0 & otherwise\end{cases} \tag{2}$$

A typical triangular fuzzy number is seen in Fig. 2.

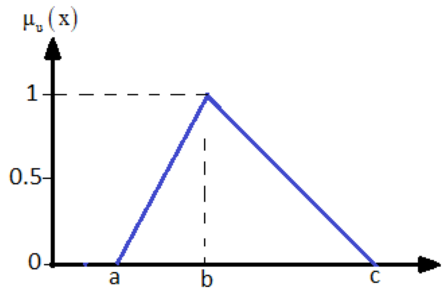

Fig. 2. A typical triangular fuzzy number

2.4. Interval fuzzy number

The interval numbers can be considered as special case of the trapezoidal fuzzy number. Its membership function reads:

$$\mu_u(x)=\begin{cases}1 & a\le x\le b\\ 0 & otherwise\end{cases} \tag{3}$$

A typical interval fuzzy number is seen in Fig. 3.

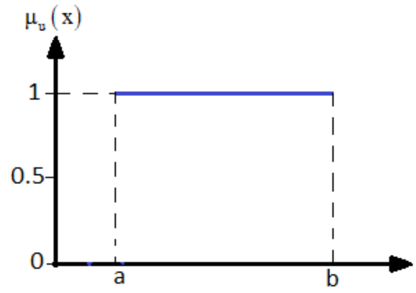

Fig. 3. A typical interval fuzzy number

2.5. Uncertainty of a fuzzy number.

The uncertainty of a fuzzy number u is denoted by Un_u, or Δu which is equal to the length of its support, that is, if $Suppu=[a,c]$, then:

$$Un_u=c-a. \tag{4}$$

It is clear that the uncertainty defined by Eq. (2) is an absolute quantity with a dimension. A dimensionless parameter named percentage uncertainty is defined as:

$$Un_u\%=\frac{c-a}{c+a}\times 100\%. \tag{5}$$

2.6. r-cuts.

The most important concept related to a fuzzy number is r-cut [10]. This concept is a key tool for computing with fuzzy numbers. In addition to the membership function, the fuzzy number u is denoted by r-cut, which is equal to the members of the universal set P whose degree of membership is at least equal to r and is denoted by u_r. The r-cut of a fuzzy number is always a closed and bounded interval and is given as a subscript

$$u_r = \left[u_r^-, u_r^+\right] = \begin{cases} \overline{\{x \in \mathbb{R} | \mu_u(x) \ge \alpha\}} & 0 < \alpha \le 1 \\ \overline{\{x \in \mathbb{R} | \mu_u(x) > \alpha\}} & \alpha = 0 \end{cases} \quad (6)$$

3. Extension Principle

In the real world, in most cases we deal with equations with uncertain parameters, in such cases we have to find the fuzzy equivalent form of operators that we perform on real numbers in the deterministic state. For example, assume that we want to find the solution of the equation $ax+b=0$ under the conditions that a,b are uncertain. In such cases, the extension principle should be used. In fact, the extension principle is an important mathematical tool that is used to develop theories and operators of classical mathematics in fuzzy environments. The extension principle allows us to calculate the parameters of a function in fuzzy sets.

3.1. One-dimensional extension principle

Assume that $f: X \to Y$ is a function where X and Y are classical sets. Then, f can be generalized to a fuzzy function, where the domain and range of the function are the set of fuzzy subsets of X and Y denoted by $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, respectively, i.e. $F: \mathcal{F}(X) \to \mathcal{F}(Y)$, such that if $u \in \mathcal{F}(X)$, then $v = F(u) \in \mathcal{F}(Y)$ with the membership function

$$\mu_v(y) = \begin{cases} \sup\{\mu_u(x) | x \in X, y = f(x)\} & f^{-1}(y) \ne \varnothing \\ 0 & f^{-1}(y) = \varnothing \end{cases} \quad (7)$$

where $\varnothing$ is the empty set [11].

3.2. Multi-dimensional extension principle

Let $X_1, X_2, \ldots, X_n$ be n sets and $X = X_1 \times X_2 \times \ldots \times X_n$ be their Cartesian multiplication and let $f: X \to Y$.

If $A_1, A_2, \ldots, A_n$ are fuzzy subsets of $X_1, X_2, \ldots, X_n$, then $B = f(A_1, A_2, \ldots, A_n)$ is fuzzy subset of Y with the membership function

$$\mu_B(y) = \sup_{y=f(x_1,\ldots,x_n)} \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\} \quad (8)$$

where $f^{-1}(y) \ne \varnothing$ otherwise, $\mu_B(y) = 0$ [11]. Note that the extension principle, often is complicated and tedious to use in general cases. Therefore, the following theorems commonly called “Neguyen theorems” [12] are introduced as an alternative to the extension principl. Based on these theorems, in the cases dealing with fuzzy numbers, the function acts on fuzzy numbers instead of fuzzy sets.

Theorem 1. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. if we show the extension of this function by F , then $F: \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ it means that it will take every fuzzy number to a fuzzy number, and if $u \in \mathbb{R}_{\mathcal{F}}$ then $v = f(u) \in \mathbb{R}_{\mathcal{F}}$ and for every $r \in [0,1]$, $v_r = f(u_r)$. That is, if $v_r = [\mathrm{v}_r^-, \mathrm{v}_r^+]$ then

$$\begin{aligned} \mathrm{v}_r^- &= \inf\{f(x) | x \in u_r\}, \\ \mathrm{v}_r^+ &= \sup\{f(x) | x \in u_r\} \end{aligned} \quad (9)$$

This theorem is also valid in multivariate cases.

Theorem 2. Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. if we show the extension of this function by F , then $F: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ it means that if $u,v \in \mathbb{R}_{\mathcal{F}}$ then $w = f(u,v) \in \mathbb{R}_{\mathcal{F}}$ and for every $r \in [0,1]$, $w_r = f(u_r, v_r)$ that is, if $v_r = [\mathrm{v}_r^-, \mathrm{v}_r^+]$ then

$$\begin{aligned} w_r^- &= \inf\{f(x,y) | x \in u_r, y \in v_r\} \\ w_r^+ &= \sup\{f(x,y) | x \in u_r, y \in v_r\} \end{aligned} \quad (10)$$

The mentioned two theorems make it easier for us to fuzzify functions, so that instead of directly using the expansion principle, we obtain the expansion of the function by using r-cuts and calculating intervals.

4. Fuzzy binary arithmetic operations

Let $u,v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$. Using Eqs. (9-10) presented in the theorem1 and 2, we have [12]:

1. Sum and scalar product:

$$(u+v)_r = \left[u_r^- + v_r^-, u_r^+ + v_r^+\right]$$

$$(\lambda u)_r = \begin{cases} [\lambda u_-^r, \lambda u_+^r] & \lambda \geq 0 \\ [\lambda u_+^r, \lambda u_-^r] & \lambda < 0 \end{cases} \tag{11}$$

2. Subtraction:

$$(u-v)_r = \left[u_r^- - v_r^+, u_r^+ - v_r^-\right] \tag{12}$$

3. Multiplication:

$$(uv)_r = \left[w_r^-, w_r^+\right]$$

Where,

$$w_r^- = \min\left\{u_r^- v_r^-, u_r^- v_r^+, u_r^+ v_r^-, u_r^+ v_r^+\right\}$$
$$w_r^+ = \max\left\{u_r^- v_r^-, u_r^- v_r^+, u_r^+ v_r^-, u_r^+ v_r^+\right\} \tag{13}$$

4. Division:

$$\left(\frac{u}{v}\right)_r = \left[w_r^-, w_r^+\right] \tag{14}$$

where,

$$w_r^- = \min\left\{\frac{u_r^-}{v_r^-}, \frac{u_r^-}{v_r^+}, \frac{u_r^+}{v_r^-}, \frac{u_r^+}{v_r^+}\right\}$$
$$w_r^+ = \max\left\{\frac{u_r^-}{v_r^-}, \frac{u_r^-}{v_r^+}, \frac{u_r^+}{v_r^-}, \frac{u_r^+}{v_r^+}\right\} \tag{15}$$

provided that $0 \notin v_0$.

5. Uncertainty Analysis

Uncertainty is one of the most important subjects in scientific data. In fact, it determines the accuracy of a given variable or the confidence of a given result. Depending on the case, there are different methods for assessing the uncertainty. The uncertainty of a typical parameter, x, is usually denoted by Δx. Also, percentage uncertainty of x is denoted by $Un_x\%$ which is define d as:

$$Un_x\% = \frac{\Delta x}{x} \times 100\%. \tag{16}$$

Whenever there is a formula for an output parameter, a method called sensitivity analysis is probably the most common approach for obtaining the uncertainty. In the following, a brief description is given for the sensitivity analysis

3.1. sensitivity analysis

The sensitivity analysis is used to determine how the uncertainty of the output parameter depends on the uncertainty of all the involving parameters in the case understudy [13]. From the sensitivity analysis it can be realized which parameter or parameters mostly affect the output value. Particularly, in the experimental studies, it is very important and useful to do a robust sensitivity analysis before manufacturing the test loop to identify the more sensitive variables.

Assume that T is a variable dependent to several parameters as:

$$T = T(x_1, x_2, \ldots x_n)$$

The uncertainty of T depends on the uncertainty of x_i through the corresponding sensitivity coefficient, $\partial T / \partial x_i$,

$$\Delta T = \left[\left(\frac{\partial T}{\partial x_1}.\Delta x_1\right)^2 + \left(\frac{\partial T}{\partial x_2}.\Delta x_2\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_n}.\Delta x_n\right)^2 \right]^{1/2} \tag{17}$$

6. Sensitivity analysis vs Zadeh's Extension Priciple

In this section a comparision is made between the results of the sensitivity method and ZEP in terms of the uncertainty.

Without losing generality, assume T is a variable which depends on x_1 and x_2 trough $T(x_1,x_2)=a_1x_1^{m_1}\pm a_2x_2^{m_2}$, where, the coefficients a_1 and a_2 and the powers m_1 and m_2 are crisp values. This means that no uncertainty is considered for them. Also, assume Δx_1 and Δx_2 are uncertainty of x_1 and x_2, respectively. The uncertainty of T can be calculated based on the sensitivity as presented in Eq. (17). Numerical value of T can be reported by $T\pm\Delta T$ which is obviously an interval value as seen in Fig. 3. Therefore, having the fuzzy mathematics concepts in mind, $T(x_1,x_2)$, would be considered as a fuzzy number or fuzzy number-valued function. This gives us a clue that the uncertainty of T can be achieved by the fuzzy arithmetic operations based on ZEP. For this end, x_1 and x_2 should be considered as two interval fuzzy numbers and the relation $T(x_1,x_2)=a_1x_1^{m_1}\pm a_2x_2^{m_2}$ should be done according to one of the fuzzy calculation methods such as ZEP. Some examples are given below for meaningful comparison between the results of the two above-mentioned approaches.

7. Numerical Example

Suppose that it is required to measure the dynamic viscosity of a liquid through a simple experimet as shown in Fig. 4. For this end, a small and smooth ball is slowly dropt in a column of the liquid. The ball diameter and mass are $d=5\pm0.1mm$ and $m=0.54\pm0.04gr$, respectively. After a while, the ball reaches a steady downward motion with the velocity $u_\infty=84\pm2\,mm/s$. Determine the dynamic visosity of the liquid.

Solution: The liquid dynamic viscosity can be determind, straforwardly. Ignoring the wall effect [14], the steady drag force ($3\pi\mu u_\infty d$) wil balance with the gravity force (mg). Therefore:

$$\mu=\frac{mg}{3\pi u_\infty d}=\frac{0.54\times10^{-3}\times9.81}{3\pi\times84\times10^{-3}\times5\times10^{-3}}=1.34Pa.s$$

The uncertainty of m, u_∞ and d are:

$$\begin{cases}\Delta m=0.04gr=4\times10^{-5}kg\\ \Delta u_\infty=2\,mm/s=0.002\,m/s\\ \Delta d=0.1mm=10^{-4}m\end{cases}$$

Using Eq. (17). the uncertainty of μ reads:

$$\begin{aligned}\Delta\mu&=\sqrt{\left(\frac{\partial\mu}{\partial m}.\Delta m\right)^2+\left(\frac{\partial\mu}{\partial u_\infty}.\Delta u_\infty\right)^2+\left(\frac{\partial\mu}{\partial d}.\Delta d\right)^2}\\ &=\frac{g}{3\pi}\sqrt{\left(\frac{\Delta m}{u_\infty d}\right)^2+\left(\frac{m.\Delta u_\infty}{du_\infty^2}\right)^2+\left(\frac{m.\Delta d}{u_\infty d^2}\right)^2}\\ &=\frac{9.81}{3\times3.14}\sqrt{(0.095)^2+(0.031)^2+(0.026)^2}\\ &=0.11Pa.s\end{aligned}$$

Therefore, we have $\mu=1.34\times0.11Pa.s$ or in the form of fuzzy interval number $[1.23,1.45]Pa.s$.

Now, we can obtain a fuzzy interval value for the dynamic viscosity of the fluid, μ, by using ZEP (Eqs. 11-15). For this end, m, u_∞ and d should be rewriten as fuzzy interval values. i.e.

$$\begin{cases}m=[0.5,0.58]gr,\\ u_\infty=[82,86]mm/s,\\ d=[4.9,5.1]mm.\end{cases}$$

Sucequently,

$$u_{\infty}d = [0.082, 0.086]\,m/s \otimes [0.0049, 0.0051]\,m$$
$$= [0.082\times 0.0049, 0.086\times 0.0051]\,m^2/s$$
$$= [0.402, 0.439]\times 10^{-3}\,m^2/s$$

and,

$$\frac{m}{u_{\infty}d} = \frac{[0.5, 0.58]\times 10^{-3}\,kg}{[0.402, 0.439]\times 10^{-3}\,m^2/s} = [1.139, 1.443]\frac{kg.s}{m^2}$$

By subistituding $m/u_{\infty}d = [1.139, 1.443]$ in the formula derived for μ, we have:

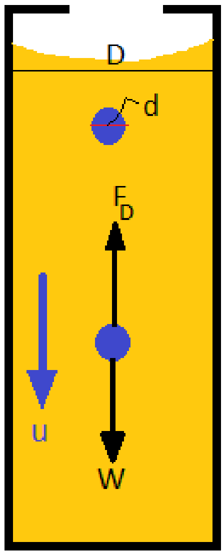

Fig. 4. Schematic of the numerical example

$$\mu = \frac{mg}{3\pi u_{\infty}d} = \frac{9.81 N/kg}{3\times 3.14}\times [1.139, 1.443]\frac{kg.s}{m^2}$$
$$= [1.19, 1.5] Pa.s$$

Let compare the two results: $[1.23, 1.45] Pa.s$ by the sensitivity analysis and $[1.19, 1.5] Pa.s$ by the Zade's extension principle. It is clear that the second interval is almost 41% wider than the first one. The percentage uncertainties based on Eq. (5), are 8.2% and 11.5% for the Sensitivity and ZEP method, respectively.

Naturally, this quary comes to mind that which of them is more correct or more acceptable. For this resion, based on Monte Carlo sampling methods [15], we calculated μ, one thousand times based on the random inputs for m, d and u_{∞} in their corresponding intervals specified in the example and, plotted μ versus μ in Fig. 5. As seen, obviously, some of the resulted points are out of the interval obtained by the sensitivity method (rectangular region in Fig. 5), while, all of them are within the interval calculated by ZEP. Therefore, this

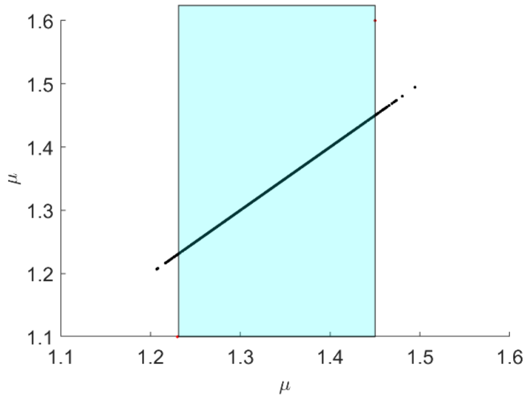

Fig. 5. Plot of Monte Carlo method results for μ.

example shows that the uncertainty obtained by the ZEP is more complete and consequently more acceptable than the sensitivity analysis method. It is worthy to note that as pointed out in several references, the sensitivity analysis and uncertainty analysis are not equivalent [16-17]. A typical sensitivity analysis of a system determines that how much variation would be occurred in a system if an input parameter varies in a certain range. While, it is expected that by the uncertainty analysis of a system, all possible outputs are recognized. However, correct or mistake, in engineering literatures, often, the results of a sensitivity analysis are considered as the uncertainty analysis [5, 7, 18]. The example shows that the arithmetic calculations based on ZEP are more compatible with the concept of the uncertainty.

Conclosions

In this paper, Zadeh's extension principle (ZEP) was used for uncertainty analysis of systems. Results obtained by ZEP, the sensitivity analysis and Monte Carlo methods are compared. It was demonstrated that ZEP is a reliable approach for the uncertainty analysis of systems and gives the same results as the Monte Carlo sampling method. It was emphasized that the sensitivity analysis should not be used as an equivalence for the uncertainty analysis because, it does not include all possible outputs of an uncertain system.

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