Application of $\dfrac{G'}{G}$ – expansion method to the (2+1)- dimensional dispersive long wave equation

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Abstract

In this work $\dfrac{G'}{G}$ – expansion method has been employed to solve (2+1)-dimensional dispersive long wave equation. It is shown that $\dfrac{G'}{G}$ – expansion method, with the help of symbolic computation, provides a very effective and powerful mathematical tool, for solving this equation.

Keywords: $\dfrac{G'}{G}$ – expansion method; (2+1)-dimensional dispersive long wave equation.

1. Introduction

Mathematical modeling of many real phenomena lead to a non-linear ordinary or partial differential equations in various fields of physics and engineering. There are some methods to obtain approximate or exact solutions of these kinds of equations, such as the tanh method [1-2], sine–cosine method [3], homotopy perturbation method [4-5], vibrational iteration method [6-7], Adomian decomposition method [8], Exp-function method [9-11], and many others [12-13]. Most recently, a novel approach called $\dfrac{G'}{G}$ – expansion method [14-15] has been developed to obtain solutions of various nonlinear equations. The solution

procedure of this method, by the help of any mathematical packages, say Matlab or Maple, is of utter simplicity.

In this paper, we will consider the $(2 + 1)$-dimension nonlinear dispersive long wave equation, in the following form, to illustrate the $\frac{G'}{G}$ – expansion method.

$$
\begin{aligned}
u_{ty} + v_{xx} + \frac{1}{2}(u^2)_{xy} &= 0, \\
v_t + u_x + (uv)_x + u_{xxy} &= 0.
\end{aligned}
\tag{1}
$$

If we let x be equal to y, the $(2 + 1)$-dimension nonlinear dispersive long wave equation can be reduced to the $(1 + 1)$-dimension nonlinear dispersive long wave equation that describes the travel of the shallow water wave. The DLWE were first obtained by Boiti et al. [16] as a compatibility condition for a ''weak'' Lax pair. Recently considerable effort has been devoted to the study of this system. For more detail about results about this system, the reader is advised to see the remarkable achievements in Refs. [17–22].

2. The $\frac{G'}{G}$ – expansion method

Consider a nonlinear partial equation, in two independent variables say x and t, in the form

$$
p(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0.
\tag{2}
$$

Where $u = u(x, t)$ is an unknown function, p is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. Using the transformation
$$\xi = x - ct,$$
(3)
where c is constant ,we can rewrite Eq. (2) as the following nonlinear ODE:
$$Q(u, u', u'', u''', \ldots) = 0.$$
(4)
Where the superscripts denote the derivatives with respect to ξ .

Step 2. Suppose that the solution of ODE (4) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$
u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G} \right)^i
\tag{5}
$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{6}$$

α_i, λ and μ are constants to be determined later with $\alpha_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance the highest order derivatives and highest order nonlinear appearing in ODE (4).

Step 3. Substituting Eq. (5) into Eq. (4) and using the second order LODE, Eq. (6) yields an algebraic equation involving powers of $\frac{G'}{G}$. Equating the coefficient of each power of $\frac{G'}{G}$ to zero gives a set of algebraic equations for determining α_i, c, λ, and μ.

Step 4. Assuming that the constants α_i, c, λ, *and* μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (6), depending on the sign of $\Delta = \lambda^2 - 4\mu$, are well known for us, by substituting α_i, c and the general solutions of Eq. (6) into Eq. (5), solutions of the nonlinear evolution Eq. (2) can be obtained.

3. Application to the DLWE

To apply $\frac{G'}{G}$ − expansion method on Eq. (1), let's introduce a complex variable ξ, defined as

$$\xi = x + ky - wt. \tag{7}$$

So, Eq. (1) turns to the following system of ordinary different equation,

$$-kwu'' + v'' + \frac{1}{2} k (u^2)'' = 0,$$
$$-wv' + u' + (uv)' + ku''' = 0. \tag{8}$$

Where k and λ are constants to be determined. By taking twofold integral from the first equation, to derive a simple form of the solution, let's take the integral constant zero, we obtain

$$v = kwu - \frac{1}{2} ku^2. \tag{9}$$

By integrating the second equation we derive

$$-wv + u + uv + ku'' = c_1, \qquad (10)$$

where c_1 is an integration constant that is to be determined later.
Substituting Eq. (9) into Eq. (10), leads to the following

$$(1 - kw^2)u + \frac{kw}{2}u' - \frac{k}{2}u^3 + ku'' = c_1. \qquad (11)$$

Suppose that the solution of ODE Eq. (11) can be expressed by a polynomial in $\frac{G'}{G}$ as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i,$$

(12)
where $G = G(\xi)$ satisfies the second order LODE (6). To determine m, we construct some terms of the Eq. (11), which leads to higher order involving m. Using (12) and (6)

$$u^3 = \alpha_m^{\ 3} \left(\frac{G'}{G}\right)^{3m} + ... \qquad (13)$$

$$u'' = m(m+1)\alpha_m \left(\frac{G'}{G}\right)^{m+2} + ... \qquad (14)$$

Considering the homogeneous balance between u'' and u^3 in Eq. (11), based on (13) and (14) we required that $m = 1$, so we can write (12) as the following simple form

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0, \qquad (15)$$

By substituting (15) into Eq. (11) and collecting all terms with the same power of $\frac{G'}{G}$ together, the left-hand side of Eq. (11) is converted into another polynomial in $\frac{G'}{G}$. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for $\alpha_1, \alpha_0, c_1, k, w, \lambda$ and μ as follows:

$$\left(\frac{G'}{G}\right)^0 : \quad -\frac{1}{2}kw\,\alpha_1\,\mu + k\,\alpha_1\,\lambda\,\mu + \alpha_0 - c_1 - \frac{1}{2}k\,\alpha_0^3 - kw^2\,\alpha_0 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : \quad \alpha_1 - kw^2\alpha_1 - \frac{1}{2}kw\,\alpha_1\lambda - \frac{3}{2}k\,\alpha_0^2\alpha_1 + 2k\,\alpha_1\mu + k\,\alpha_1\lambda^2 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : \quad -\frac{1}{2}kw\,\alpha_1 - \frac{3}{2}k\,\alpha_0\alpha_1^2 + 3k\,\alpha_1\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^3 : \quad -\frac{1}{2}k\,\alpha_1^3 + 2k\,\alpha_1 = 0.$$

(16)

Solving this algebraic equations above, yields to

First solution set:

$$\alpha_1 = 2, k = \frac{2}{75\alpha_0^2 - 150\lambda\alpha_0 + 76\lambda^2 - 4\mu}, w = -6\alpha_0 + 6\lambda, c_1 = \frac{2(4\mu\alpha_0 - 4\lambda\mu + \alpha_0^3 - 3\lambda\alpha_0^2 + 2\alpha_0\lambda^2)}{75\alpha_0^2 - 150\lambda\alpha_0 + 76\lambda^2 - 4\mu}.$$

(17)

Second solution set:

$$\alpha_1 = -2, k = \frac{2}{75\alpha_0^2 + 150\lambda\alpha_0 + 76\lambda^2 - 4\mu}, w = 6\alpha_0 + 6\lambda, c_1 = \frac{2(4\mu\alpha_0 + 4\lambda\mu + \alpha_0^3 + 3\lambda\alpha_0^2 + 2\alpha_0\lambda^2)}{75\alpha_0^2 + 150\lambda\alpha_0 + 76\lambda^2 - 4\mu}.$$

(18)

Where λ, μ and α_0 are arbitrary constants.

By substituting (17) and (18) into (15), we drive

$$u(\xi) = \pm 2\left(\frac{G'}{G}\right) + \alpha_0$$

(19)

Where

$$\xi = x + \frac{2}{75\alpha_0^2 \mp 150\lambda\alpha_0 + 76\lambda^2 - 4\mu}\, y - (\mp 6\alpha_0 + 6\lambda)t.$$

(20)

Substituting the general solutions of Eq. (6) into (19) we would have three types of traveling wave solutions of the DLWE as follows:

When $\lambda^2 - 4\mu > 0,$

$$u_{1,2}(\xi) = \pm\sqrt{\lambda^2 - 4\mu}\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + B\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}{A\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi + B\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\xi}\right) \mp \lambda + \alpha_0.$$

(21)

When $\lambda^2 - 4\mu < 0,$

$$u_{3,4}(\xi) = \pm\sqrt{4\mu - \lambda^2}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + B\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}{A\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi + B\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\xi}\right) \mp \lambda + \alpha_0. \quad (22)$$

When $\lambda^2 - 4\mu = 0$,

$$u_{5,6}(\xi) = \frac{\pm 2B}{A + B\xi} \mp \lambda + \alpha_0.$$

(23)

Where A and B are arbitrary constants and

$$\xi = x + \frac{2}{75\alpha_0^2 \mp 150\lambda\alpha_0 + 76\lambda^2 - 4\mu}\,y - (\mp 6\alpha_0 + 6\lambda)t. \quad (24)$$

4. Conclusion

In this article, we have been looking the exact solution of the (2 + 1)-dimensional dispersive long wave equation. We achieved the solution by applying $\frac{G'}{G}$ -expansion method. The free parameters can be determined using any related to initial or boundary conditions. The result shows that $\frac{G'}{G}$ -expansion method is a powerful tool for obtaining exact solution. Applications of $\frac{G'}{G}$ -expansion method for other kinds of nonlinear equations are under study in our research group. The computations associated in this work were performed by using Maple 11.

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