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IRANIAN JOURNAL OF OPTIMIZATION

Volume 6, issue 2,spring 2014,737-744
Research Paper

Islamic Azad University
Rasht Branch

Unweighted p-center problem on extended stars

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Abstract

An extended star is a tree which has only one vertex with degree larger than two. The p-center problem in a graph G asks to find a subset X of the vertices of G of cardinality p such that the maximum weighted distances from X to all vertices is minimized. In this paper we consider the p-center problem on the unweighted extended stars, and present some properties to find solution.

Keywords: Location theory, center problem, extended star

1. Introduction

Let $G=(V,E)$ be an undirected graph with vertex set V and edge set E. Each vertex v_i has a positive weight w_i and the edges of graph have positive lengths. An important problem in the location theory is the p-center problem. In the p-center problem we want to find a subset $X \subseteq V$ of cardinality p such that the maximum weighted distances from X to all vertices is minimized. If all the weights are equal the problem is called unweighted p-center problem.

The p-Center problem has been known to be NP-hard, [5]. Lan et al. in [6] presented a linear-time algorithm for solving the 1-center problem on weighted cactus graphs. Frederickson in [4] solved this problem for trees in optimal linear-time (without necessarily restricting the location of the facilities to the vertices of the tree) using parametric search. Bespamyatnikh et al. in [1] gave an $O(pn)$ time algorithm for this problem on circular-arc graphs. Kariv and Hakimi in [5] addressed the p-center problem on general graphs. In [8], Tamir showed that the weighted and unweighted p-center problems in networks can be solved in $O(n^p m^p log^2 n)$ time and $O(n^{p-1} m^p log^3 n)$ time, respectively. Burkard and Dollani [2] considered the case that some vertices have negative weights and presented an $O(n^2 logn)$ algorithm for p-center problem on a tree. They also presented a linear time algorithm for 1-center problem with pos/neg weights on paths and star graphs. For further literature on the p-median (and center) problem the reader is referred to the books of Mirchandani and Francis [7] and Drezner and Hamacher [3].

In what follows we state the p-center problem on a graph in Section 2. In Section 3 we show the unweighted p-center problem on a path can be solved in a constant time. Section 4 contains some properties of unweighted p-center problem on extended stars that leads to find a solution.

2 Problem formulation

Let $G=(V,E)$ be a graph, where V is the set of vertices with $|V|= n$, and E is the set of edges. Every edge with the end vertices u and v is presented by e_{uv}. We assume the weights of all vertices and edges are the same and equal to one.

In the p-center problem the maximum of the distances is minimized over all $X \subseteq G$ with $|X|= p$, i.e.

$$minF(X)= \max_{i=1,\dots,n} d(X,v_i), \qquad (1)$$

where $d(X,v_i)= min_{x_j \in X} d(x_j,v_i)$ and $d(x,v)$ is the minimum distance between x and v in G.

In this paper we consider the case that G is a extended star. An extended star is a tree which has only one vertex with degree larger than two. We call this vertex with degree larger than two as central vertex.

3 The p-center on a path

Let P be a path with vertex set $\{v_1, v_2, ..., v_n\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2, ..., n-1$. The following results are straightforward, and so we omit a proof.

- The solution of 1-center problem on P is vertex $v_{\lceil \frac{n}{2} \rceil}$.
- The solution of 2-center problem on P is vertices $v_{\lceil \frac{n}{4} \rceil}$ and $v_{\lceil \frac{3n}{4} \rceil}$.
- In general the solution of p-center problem on P is vertices $v_{\lceil \frac{(2i-1)n}{2p} \rceil}$

for $i = 1, ..., p$.

Using above statements we can find a solution on a path in a constant time, i.e.:

Theorem 3.1 The unweighted p-center problem on a path can be solved in $O(1)$ time.

Example 3.2 Consider the path depicted in Figure 3.2 which all its weights are equal to one. Table 3.2 contains the solutions of unweighted p-center problem on this path for different values of p. The solutions are computed using the statement 3. For example for $p = 4$ the solution is $X^* = \{x_1, x_2, x_3, x_4\}$ where

$$x_1 = v_{\lceil \frac{14}{8} \rceil} = v_2,$$

$$x_2 = v_{\lceil \frac{3 \times 14}{8} \rceil} = v_6,$$

$$x_3 = v_{\lceil \frac{5 \times 14}{8} \rceil} = v_9,$$

and

$$x_4 = v_{\lceil \frac{7 \times 14}{8} \rceil} = v_{13}.$$

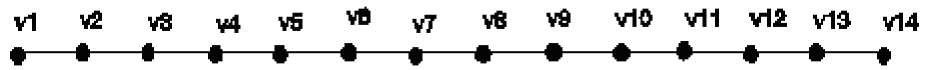

Figure 1: The path for Example 3.2

Table 1: The solutions of unweighted p -center problem on a path.

p	X^*	Value of objective function
1	$\{v_7\}$	7
2	$\{v_4, v_{11}\}$	4
3	$\{v_3, v_7, v_{12}\}$	3
4	$\{v_2, v_6, v_9, v_{13}\}$	2
5	$\{v_2, v_5, v_7, v_{10}, v_{13}\}$	2
6	$\{v_2, v_4, v_6, v_9, v_{11}, v_{13}\}$	2
7	$\{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{13}\}$	1

4 The p-center on extended stars

Now consider the p-center problem on extended stars. We state some properties to decrease computation for finding the solution.

Theorem 4.1 Let S be an extended star and $S' \subseteq S$ be a sub extended star of S contains the p longest branches of S then the solution of the p-center problem on S' for $p > 1$ is also a solution of this problem on S.

Proof. Let $X = \{c_1, ..., c_p\}$ be a solution of p-center problem on S. If the number of branches in S is less than or equal to p then $S = S'$ and the theorem holds. Otherwise let c_t be a center on branch B_i where B_i is not in the p longest branches of S. Also there is a branch B_j which is one of the p longest branches of S and not contains any center $c_r, r = 1,...,p$. Let o be the unique vertex of S with $deg(o) > 2$. If o is assigned to c_t then all vertices on B_j are also assigned to c_t and since $|B_i| < |B_j|$ we can decrease the value of objective function by moving c_t on B_i in the direction of o. Which contradicts that X is a solution of p-center problem. In the other case if o is assigned to $c_k \neq c_t$. Then all vertices on B_j are also assigned to c_k, specially the end vertex v_l of B_j. Since $d(v_l, c_K) > d(v_m, o)$ where v_m is end vertex of B_i, we can set o in X instead of c_t which dos not cause increasing the objective function. By now we showed that there exist a solution $X = \{c_1,...,c_p\}$ such that for $r = 1,...,p$ c_r is in the one of p longest branches of S. Now let S' be the sub extended star of S contains p longest branches. Assignment vertices in S' is the same as S. Suppose o is assigned to c_h then any vertex $v \in S \setminus S'$ is also assigned to c_h so if we delete the vertices in $S \setminus S'$ the solution does not change, just the value of objective function will be increased. This complete the proof. □

Using Theorem 4.1 in the cases $p = 2$ the solutions lies on the longest path or diameter of star so the problem reduces to finding solution on the longest path. Also for the case $p = 1$ the solution lies on the longest path therefore using statements 1 and 2, we can state the following theorem.

Theorem 4.2 Let S be an extended star which its diameter be the path $P = v_1, v_2,...,v_d$ with length d. The solution of the 1-center problem is $v_{\lceil \frac{d+1}{2} \rceil}$ and the solutions of the 2-center problem are $v_{\lceil \frac{d+1}{4} \rceil}$ and $v_{\lceil \frac{3(d+1)}{4} \rceil}$.

Theorem 4.3 There is a solution $X = \{c_1,...,c_p\}$ of the p-center problem on extended star S such that for $i = 1,...,p$ c_i lies on a branch of S which its length grater than or equal to $\frac{d}{2p}$.

Proof. Let T be an extended star with central vertex O. By Theorem 4.1, there is a solution of the p-center problem on a sub extended star containing the

p longest branches. Let $X = \{c_1, c_2, ..., c_p\}$ be a solution. Suppose L_i is a branch of T with length less than $\frac{d}{2p}$, and let $c_i \in L_i$. Let P be the longest path in T of length $d = diam(T)$. We consider the following cases:

Case 1. $O \notin X$. Let $X_1 = (X \setminus \{c_i\}) \cup \{O\}$. We show that X_1 is a solution. Since $c_i \in L_i$, $|P \cap X| \leq p-1$. Let $Y = X \cap P$, and let $x = maxmind(v, c_i)$, where $c_i \in Y$, and $v \in P - Y$. We observe that $(p-1)x \geq d$, and so $x \geq \frac{d}{2p-2}$. This means that there is a vertex v on P such that the minimum distance from v to Y is at least $\frac{d}{2p-2}$. Since L_i is a branch with length less than $\frac{d}{2p}$, replacing c_i by O does not reduce the maximum distance of a vertex outside X to X_1. This means that X_1 is also a solution.

Case 2. $O \in X$. Let w be a vertex in $P \setminus X$ in a minimum distance from O, and let $X_2 = (X \setminus \{c_i\}) \cup \{w\}$. Similar to case 1 we observe that X_2 is a solution.

We continue the above process until there is no branch L_i with length less than $\frac{d}{2p}$ such that $X \cap L_i \neq \emptyset$. □

Note that by using the Theorems 4.1 and 4.3 we can eliminate some branches and solve the p-center on the remaining sub-tree. The solution will be the same. So the computation will be reduced.

Example 4.4 Consider the extended star depicted in Figure 4.4 which all its weights are equal to one. Table 4.4 shows the branches that we consider to solve unweighted p-center problem for different values of p.

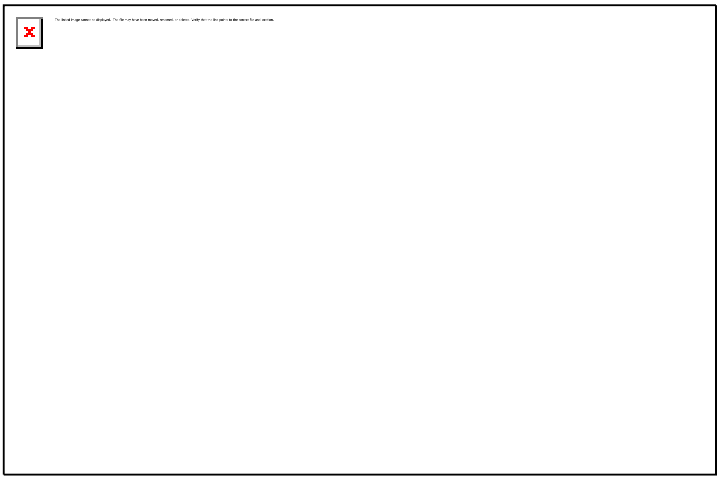

Figure 2: The extended star for Example 4.4

Table 2: The considered branches to solve the unweighted p -center problem on extended star.

p	S'	Value of objective function
1	$\{B_4\}$	4
2	$\{B_4, B_2\}$	3
3	$\{B_4, B_2, B_5\}$	3
4	$\{B_4, B_2, B_5, B_6\}$	3
5	$\{B_4, B_2, B_5, B_6, B_1\}$	2
6	$\{B_4, B_2, B_5, B_6, B_1, B_3\}$	2
7	$\{B_4, B_2, B_5, B_6, B_1, B_3, B_8\}$	2

5 Summary and conclusion

We considered the unweighted p-center problem on paths and extended stars. For the case path we presented an $O(1)$ time algorithm and for extended stars some property are presented to reduce computations.

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