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# *Analytic-Approximate Solution For An Integro-Differential Equation Arising In Oscillating Magnetic Fields Using Homotopy Analysis Method*

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***Abstract***

In this paper, we give an analytical approximate solution for an integro- differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields is considered. The homotopy analysis method (HAM) is used for solving this equation. Several examples are given to reconfirm the efficiency of these algorithms. The results of applying this procedure to the integro-differential equation with time-periodic coefficients show the high accuracy, simplicity and efficiency of this method.

***Keywords:*** Homotopy analysis method, Integro-differential equations, approximate-analytic solution, homotopy-derivative, homotopy perturbation method.

## 1.*Introduction*

Most scientific problems in engineering are inherently nonlinear. Except a few number of them, majority of nonlinear problems do not have analytical solution. Therefore, these nonlinear equations should be solved using other methods Such as numerical or Perturbation method. In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results each effective parameter should be solved iteratively [4]. In the perturbation method, the small parameter is inserted in the equation. Thus, finding the small parameter and exerting it into the equation is one of the deficiencies of this method [11]. One of the semi-exact methods for solving nonlinear equation which does not need small/large parameters is Homotopy Analysis Method (HAM), first proposed by Liao in 1992

[5,7,9] . Since Liao's book [6] for the homotopy analysis method was published in 2003, more and more researchers have been successfully applying this method to various nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [2], the KdV-type equations [1], finance problems [12], and so on. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h-curve, it is easy to determine the valid regions of h to gain a convergent series solution.

The integro-differential equation [10]

$$\frac{d^2y}{dt^2} = -a(t)y(t) + b(t)\int_0^t \cos(\omega_p s)y(s)ds + g(t), \qquad (1)$$

where $a(t), b(t)$ and $g(t)$ are given periodic functions of time may be easily found in the charged particle dynamics for some field configurations. Taking for instance the three mutually orthogonal magnetic field components $B_x = B_1\sin(\omega_p s)$, $B_y = 0$ and $B_z = B_0$) the nonrelativistic equations of motion for a particle of mass $m$ and charge $q$ in this field configuration are

$$m\frac{d^2x}{dt^2} = q(B_0\frac{dy}{dt}), \qquad (2)$$

$$m\frac{d^2y}{dt^2} = q(B_1\sin(\omega_p t)\frac{dy}{dt} - B_0\frac{dx}{dt}), \qquad (3)$$

$$m\frac{d^2z}{dt^2} = q(-B_1\sin(\omega_p t)\frac{dy}{dt}), \qquad (4)$$

By integration of (2) and (4) and replacement of the time first derivatives of $z$ and $x$ in (3) one has (1) with

$$a(t) = \omega_c^2 + \omega_f^2\sin^2(\omega_p t), \qquad (5)$$

$$b(t) = \omega_f^2\omega_p\sin(\omega_p t), \qquad (6)$$

$$g(t) = \omega_f\sin(\omega_p t)z'(0) + \omega_c^2 y(0) + \omega_c x'(0), \qquad (7)$$

where $\omega_c = q\frac{B_0}{m}$ and $\omega_f = q\frac{B_1}{m}$ Making the additional simplification that $x'(0) = 0$ and $y(0) = 0$, equation (1) is finally written as

$$\frac{d^2y}{dt^2} = \omega_c^2 + \omega_f^2 \sin^2(\omega_p t)y + \omega_f \sin(\omega_p t) z'(0) \qquad (8)$$

$$+\omega_f^2 \omega_p \sin(\omega_p t)\int_0^t \cos(\omega_p s) y(s) ds,$$

In this study, we consider the equation (1) with the following initial conditions

$$y(0) = \alpha, y'(0) = \beta \qquad (9)$$

The aim of this paper is to use homotopy analysis method for solving integro differential equations arising in oscillating magnetic fields.

## 2. *Basic idea of HAM*

To describe the basic ideas of the HAM, we consider the following differential

equation

$$\mathrm{N}[y(t)] = 0, \qquad (10)$$

where N is a nonlinear operator that represent the whole equation, t denote independent variable, and y(t) is an unknown functions, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [5] constructed the so-called zero-order deformation equation

$$(1-q)\mathrm{L}[\phi(t;q) - y_0(t)] = qh\mathrm{H}(t)\,\mathrm{N}[\phi(t;q)], \qquad (11)$$

Where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathrm{H}(t) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $y_0(t)$ is an initial guess of $y(t)$ and $\phi(t;q)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM.

Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t;0) = y_0(t), \quad \phi(t;1) = y(t), \qquad (12)$$

respectively. Thus, as $q$ increases from 0 to 1, the solution $y(t,q)$ varies from the initial guess $y_0(t)$ to the solution $y(t)$. Expanding $y(t,q)$ in Taylor series with respect to $q$, we have

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t) q^m, \tag{13}$$

Where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}\Big|_{q=0}. \tag{14}$$

$D_m$ is called the mth-order homotopy-derivative of $\phi$.

If the auxiliary linear operator, the initial guess, the auxiliary parameter $h$, and the auxiliary function are so properly chosen, the series (13) converges at $q=1$, then we have

$$y(t) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t), \tag{15}$$

Which must be one of solutions of original nonlinear equation, as proved by Liao [6], as h =-1 and H(x, t) = 1, Eq. (11) becomes

$$(1-\mathrm{q})\mathrm{L}\,[\phi(t;q) - y_0(t)] + q\mathrm{H}(t)\,\mathrm{N}[\phi(t;q)] = 0, \tag{16}$$

Which is used mostly in the homotopy perturbation method [3], where as the solution obtained directly, without using Taylor series. According to the definition (14), the governing equation can be deduced from the zero-order deformation equation (11). Define the vector

$$\vec{y}_n = \{y_0(t), y_1(t), \ldots, y_n(t)\}.$$

Differentiating equation (11) $m$ times with respect to the embedding parameter $q$ and then setting $q=0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)R_m(\vec{y}_{m-1}(t)), \tag{17}$$

where

$$R_m(\vec{y}_{m-1}) = D_{m-1}(N[\phi]) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \qquad (18)$$

and

$$\chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases} \qquad (19)$$

It should be emphasized that $y\ (t)$ for $m \ge 1$ is governed by the linear equation (17) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work. If Eq. (10) admits unique solution, then this method will produce the unique solution. If equation (10) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

## *3.Applications*

In this section, to illustrate the description above and to show the efficiency of the mentioned method for solving equation (1), we include some examples with known analytical solutions.

Example 1: Consider equation (1) with

$$\omega_p = 2,$$

$$a(t) = \cos(t), \qquad b(t) = \sin(\frac{t}{2}),$$

$$g(t) = \cos(t) - t\sin(t) + \cos(t)(t\sin(t) + \cos(t)) - \sin(\frac{t}{2})(\frac{2}{9}\sin(3t) - \frac{t}{6}\cos(3t) + \frac{t}{2}\cos(t)),$$

and $\alpha = 1, \beta = 0,$ $y(t) = t\sin(t) + \cos(t)$ is the exact solution of this equation. Then we have

$$y''(t) = -\cos(t)y(t) + \sin(\frac{t}{2})\int_0^t \cos(2s)y(s)ds + g(t), \qquad (20)$$

$$y(0) = 1, \quad y'(0) = 0$$

To solve Eq. (20) by means of the HAM, we choose the initial approximations

$$y_0(t) = 1, \qquad (21)$$

and the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2}, \tag{22}$$

Furthermore, we define the nonlinear operator

$$N[\phi(t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2} + \cos(t)\phi(t;q) - g(t;q)$$
$$+ \sin(\frac{t}{2})\int_0^t \cos(2s)\phi(s;q)ds,$$

We have the mth-order deformation equation

$$y_m(t) = \chi_m y_{m-1}(t) + L^{-1}[hH(t)R_m(\vec{y}_{m-1})], \tag{23}$$

Where

$$R_m(\vec{y}_{m-1}) = \frac{\partial^2 y_{m-1}}{\partial t^2} + \cos(t)y_{m-1} - (1-\chi_m)g(t;q) - \sin(\frac{t}{2})\int_0^t \cos(2s)y_{m-1}ds,$$

Now, the solution of the mth-order deformation Eq. (23) for $m \geq 1$ becomes

$$y_1 = h(-\frac{2208142}{1157625} - 2\cos(t) - t\sin(t) + \frac{1}{4}t\cos(t)\sin(t) - \frac{1}{4}t^2 - \frac{1}{2}\sin^2(t)$$
$$-\frac{77}{1125}\cos(\frac{5}{2}t) + \frac{40}{3087}\cos(\frac{7}{2}t) + \frac{1}{147}t\sin(\frac{7}{2}t) - \frac{1}{75}t\sin(\frac{5}{2}t) - \frac{1}{27}\cos(\frac{3}{2}t) \quad .$$
$$-\frac{1}{9}t\sin(\frac{3}{2}t) + 4\cos(\frac{1}{2}t) + t\sin(\frac{1}{2}t)).$$

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The absolute errors $|y(t) - \phi_6(t)|$ have been calculated for $h = -1$ in Table 1 and Fig. 2. Fig. 1 show the graphs of the HAM solutions and exact solution of the problem. We can see that the solutions obtained by the proposed method are in excellent agreement with the exact solution. Our calculations indicate that the series (13) converges if $-1.4 \leq h \leq -0.4$, we can investigate the influence of $h$ on the convergence of, by plotting the curve of it versus $h$, as shown in Fig. 3.

Table 1. The numerical results of the 6th-order HAM ( $h = -1$ ).

| t | HAM | Exact | Absolute |
|---|---|---|---|
| 0.0 | 1.0000e+000 | 1.0000e+000 | 1.1324e-014 |
| 0.1 | 1.0050e+000 | 1.0050e+000 | 1.1324e-014 |
| 0.2 | 1.0198e+000 | 1.0198e+000 | 1.1324e-014 |
| 0.3 | 1.0440e+000 | 1.0440e+000 | 1.0880e-014 |
| 0.4 | 1.0768e+000 | 1.0768e+000 | 1.1102e-014 |
| 0.5 | 1.1173e+000 | 1.1173e+000 | 1.1324e-014 |
| 0.6 | 1.1641e+000 | 1.1641e+000 | 1.6431e-014 |
| 0.7 | 1.2158e+000 | 1.2158e+000 | 5.4401e-014 |
| 0.8 | 1.2706e+000 | 1.2706e+000 | 2.4936e-013 |
| 0.9 | 1.3266e+000 | 1.3266e+000 | 1.0298e-012 |
| 1.0 | 1.3818e+000 | 1.3818e+000 | 3.5798e-012 |

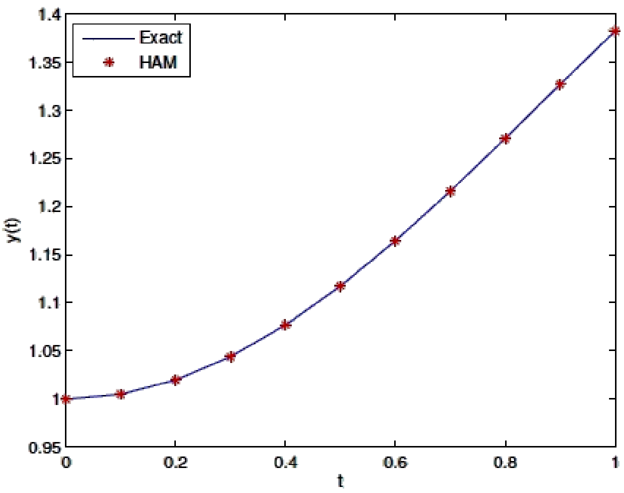


Fig. 1. The comparison of the 6th-order HAM and exact solution for example 1.

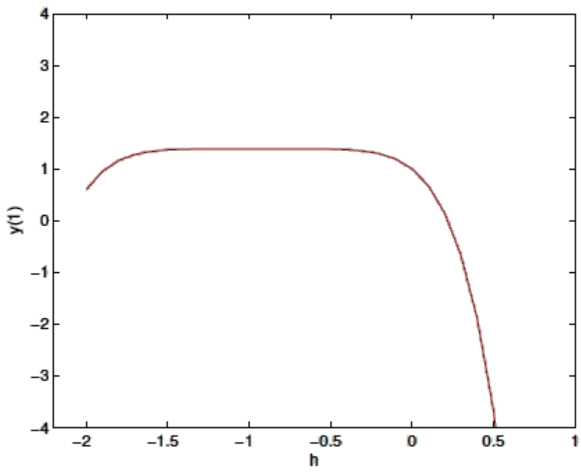


Fig. 2. The h-curve of based on the $6^{th}$-order HAM for example 1,

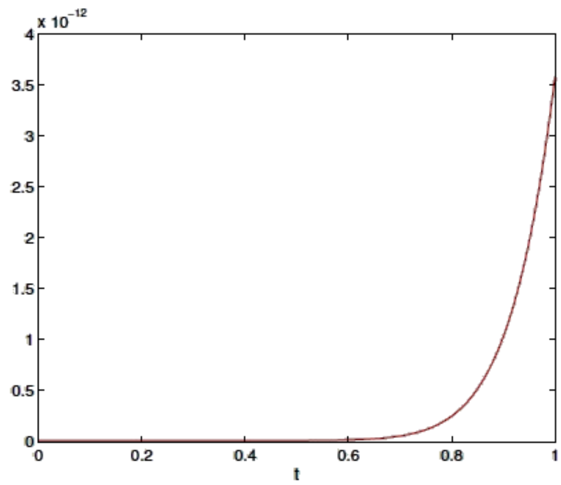


Fig. 3. The absolute error $6^{th}$-order HAM for example 1.

Example 2: Consider equation (1) with

$$\omega_p = 1,$$

$$a(t) = -\sin(t), \quad b(t) = \sin(t),$$

$$g(t) = -t^3 + t^2 - 11t + 4 - (\sin(t) + \cos(t))(-\frac{t^3}{3}\sin(3t)$$
$$-\frac{t^2}{3}\cos(3t) - \frac{13}{27}\cos(3t), -\frac{13}{9}t\sin(3t) + \frac{t^2}{3}\sin(3t)$$
$$+\frac{16}{27}\sin(3t) + \frac{2}{9}t\cos(3t) + \frac{13}{27}.$$

and $\alpha = 1,\ \beta = 0,\ \ y(t) = t\sin(t) + \cos(t)$ is the exact solution of this equation. Then we have

$$y''(t) = -y(t) + g(t) + (\sin(t) + \cos(t))\int_0^t \cos(3s)y(s)ds + g(t), \tag{24}$$

$$y(0) = 2, \quad y'(0) = -5.$$

The exact solution is $y(t) = -t^3 + t^2 - 5t + 2$.

To solve Eq. (24) and by means of the HAM, we choose the initial approximations

$$y_0(t) = -5t + 2, \tag{25}$$

and the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2}, \tag{26}$$

Furthermore, we define the nonlinear operator

$$N[\phi(t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2} + \phi(t;q) - g(t;q) - (\sin(t)$$
$$+\cos(t))\int_0^t \cos(3s)\phi(s;q)ds,$$

We construct the zeroth-order and mth-order deformation equations where

$$R_m(\vec{y}_{m-1}) = \frac{\partial^2 y_{m-1}}{\partial t^2} + y_{m-1}(t,q) - (1-\chi_m)g(t;q) - (\sin(t) + \cos(t))$$
$$\int_0^t \cos(3s)y_{m-1}(s,q)ds,$$

Now, the solution of the mth-order deformation Equation for $m \geq 1$ becomes

$$y_1 = h(\frac{19}{256} + \frac{2}{27}\sin(t) - t^2 + \frac{2}{27}\cos(t) + t^3 - \frac{1}{12}t^4 + \frac{1}{24}t^3\cos(2t)$$
$$-\frac{1}{96}t^3\cos(4t) - \frac{1}{8}\cos(2t) - \frac{161}{6912}\cos(4t) + \frac{1}{64}t^2\sin(4t) - \frac{5}{24}t^2\sin(2t)$$
$$-\frac{1}{256}\sin(4t) + \frac{35}{108}\sin(2t) - \frac{59}{144}t\cos(2t) + \frac{3}{256}t\cos(4t)) + \frac{1}{8}t^2\cos(2t) \quad .$$
$$+\frac{7}{192}t^2\cos(4t) - \frac{107}{2304}t\sin(4t) - \frac{3}{16}t\sin(2t) + \frac{1}{96}t^3\sin(4t) + \frac{1}{24}t^3$$
$$\sin(2t) + \frac{1}{20}t^5 - \frac{79}{256}t.$$

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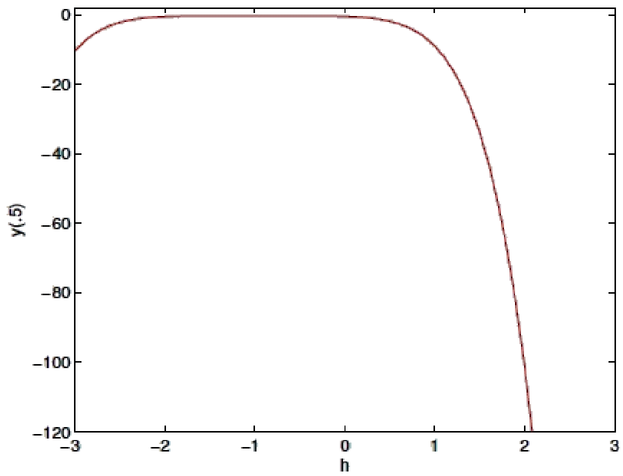


Fig. 4. The h-curve of based on the 6th-order HAM for example 2,

The absolute errors $|y(t) - \phi_6(t)|$ have been calculated for $h = -1$ in Table 2 and Fig. 6. Fig. 5 show the graphs of the HAM solutions and exact solution of the problem. We can see that the solutions obtained by the proposed method are in excellent agreement with the exact solution. Our calculations indicate that the series (13) converges if $-2 \leq h \leq 0$, we can investigate the influence of $h$ on the convergence of, by plotting the curve of it versus $h$, as shown in Fig. 4.

| x | HAM | Exact | Absolute |
|---|---|---|---|
| 0.0 | 2.0000e+000 | 2.0000e+000 | 0 |
| 0.1 | 1.5090e+000 | 1.5090e+000 | 0 |
| 0.2 | 1.0320e+000 | 1.0320e+000 | 2.2204e-016 |
| 0.3 | 5.6300e-001 | 5.6300e-001 | 2.2204e-016 |
| 0.4 | 9.6000e-002 | 9.6000e-002 | 1.6653e-016 |
| 0.5 | -3.7500e-001 | -3.7500e-001 | 2.2204e-016 |
| 0.6 | -8.5600e-001 | -8.5600e-001 | 2.2204e-016 |
| 0.7 | -1.3530e+000 | -1.3530e+000 | 2.2204e-016 |
| 0.8 | -1.8720e+000 | -1.8720e+000 | 2.2204e-016 |
| 0.9 | -2.4190e+000 | -2.4190e+000 | 4.4409e-016 |
| 1.0 | -3.0000e+000 | -3.0000e+000 | 4.4409e-016 |

Table 2. The numerical results of the $6^{th}$-order HAM ( $h=-1$ ).

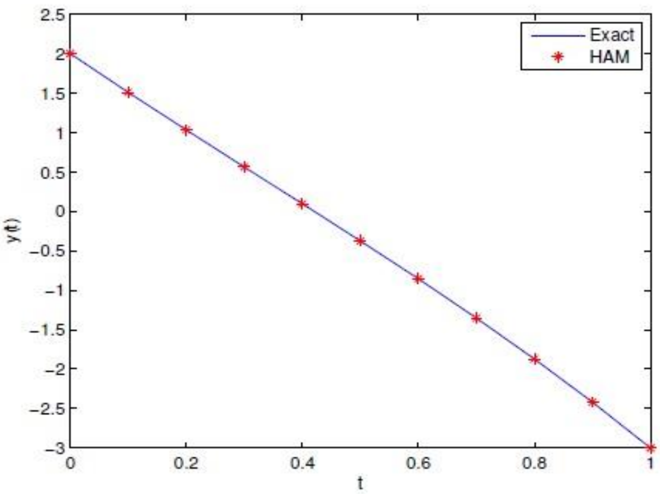


Fig. 5. The comparison of the $6^{th}$-order HAM and exact solution for example 2.

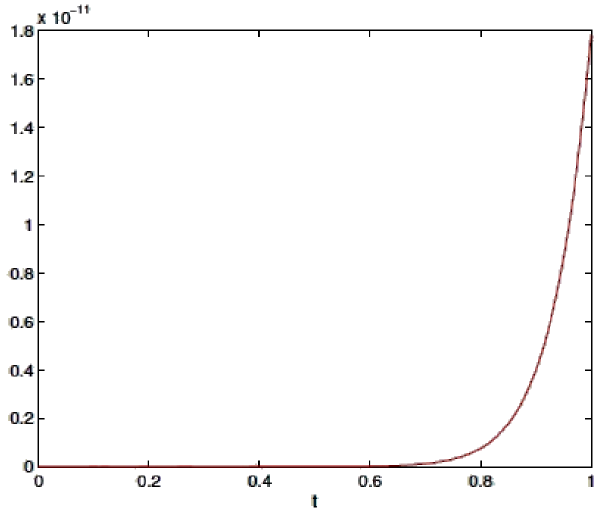


Fig. 6. The absolute error $6^{th}$-order HAM for example 2.

Example 3: Consider equation (1) with

$$\omega_p = 1,$$

$$a(t) = -\sin(t), b(t) = \sin(t)$$

$$g(t) = \frac{1}{9}e^{-\frac{t}{3}} - \sin(t)(e^{-\frac{t}{3}} + t) - \sin(t)(-\frac{3}{10}\cos(t)e^{-\frac{t}{3}} + \frac{9}{10}\sin(t)e^{-\frac{t}{3}} + \cos(t) + t\sin(t) - \frac{7}{10}).$$

and $\alpha = 1, \beta = \frac{2}{3}$, $y(t) = e^{-\frac{t}{3}} + t$ is the exact solution of this equation. Then we have

$$y''(t) = -\sin(t)y(t) + g(t) + \sin(t)\int_0^x \cos(s)y(s)ds, \tag{27}$$

$$y(0) = 1, \quad y'(0) = \frac{2}{3}$$

To solve Eq. (27) by means of the HAM, we choose the initial approximations

$$y_0(t) = 1 + \frac{2}{3}t, \tag{28}$$

and the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2}, \tag{29}$$

Furthermore, we define the nonlinear operator

$$N[\phi(t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2} - \sin(t)\phi(t;q) - g(t;q) - \sin(t)\int_0^t \cos(s)\phi(s;q)ds,$$

We construct the zeroth-order and mth-order deformation equations where

$$R_m(\vec{y}_{m-1}) = \frac{\partial^2 y_{m-1}}{\partial t^2} - \sin(t)y_{m-1} - (1-\chi_m)g(t;q) - \sin(t)\int_0^t \cos(s)y_{m-1}ds,$$

Now, the solution of the mth-order deformation Equation for $m \geq 1$ becomes

$$\begin{aligned} y_1 = h(&-\frac{619139}{205350} + \frac{4361}{4440}t + \frac{31}{30}\sin(t) + \frac{1}{4}t^2 - \frac{2}{3}\cos(t) + \frac{1}{3}\sin(t)\cos(t) \\ &-\frac{1}{3}t\sin(t) + \frac{1}{36}t^3 + \frac{61}{20}e^{-\frac{t}{3}} - \frac{1}{4}\sin(2t) + \frac{1}{8}t\cos(2t) - \frac{1}{6}\cos^2(t) + \frac{2511}{27380} \\ &e^{-\frac{t}{3}}\cos(2t) + \frac{1917}{27380}e^{-\frac{t}{3}}\sin(2t) - \frac{18}{25}e^{-\frac{t}{3}}\sin(t) + \frac{27}{50}e^{-\frac{t}{3}}\cos(t) + \frac{1}{4}\sin(t)), \end{aligned}$$

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The absolute errors $|y(t) - \phi_6(t)|$ have been calculated for $h = -1$ in Table 3 and Fig. 9. Fig. 7 show the graphs of the HAM solutions and exact solution of the problem. We can see that the solutions obtained by the proposed method are in excellent agreement with the exact solution. Our calculations indicate that the series (13) converges if $-1.9 \leq h \leq -0.1$, we can investigate the influence of $h$ on the convergence of, by plotting the curve of it versus $h$, as shown in Fig. 8.

| x | HAM | Exact | Absolute |
|---|---|---|---|
| 0.0 | 1.0000e+000 | 1.0000e+000 | 2.6534e-013 |
| 0.1 | 1.0672e+000 | 1.0672e+000 | 2.6890e-013 |
| 0.2 | 1.1355e+000 | 1.1355e+000 | 2.7245e-013 |
| 0.3 | 1.2048e+000 | 1.2048e+000 | 2.7578e-013 |
| 0.4 | 1.2752e+000 | 1.2752e+000 | 2.7867e-013 |
| 0.5 | 1.3465e+000 | 1.3465e+000 | 2.8155e-013 |
| 0.6 | 1.4187e+000 | 1.4187e+000 | 2.8400e-013 |
| 0.7 | 1.4919e+000 | 1.4919e+000 | 2.8599e-013 |
| 0.8 | 1.5659e+000 | 1.5659e+000 | 2.8755e-013 |
| 0.9 | 1.6408e+000 | 1.6408e+000 | 2.8666e-013 |
| 1.0 | 1.7165e+000 | 1.7165e+000 | 2.7001e-013 |

Table 3. The numerical results of the $6^{th}$-order HAM ($h=-1$).

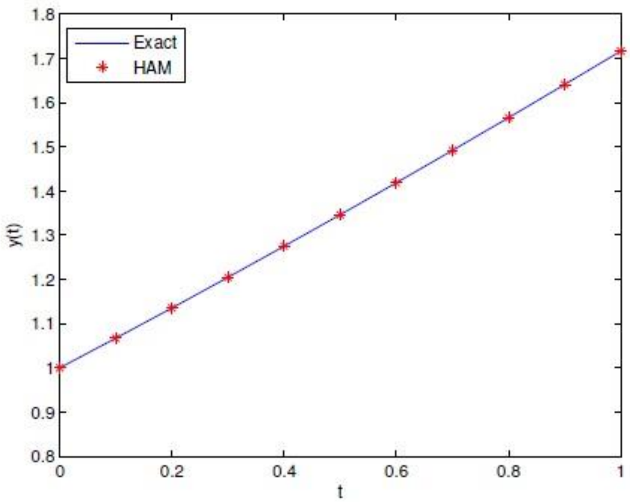


Fig. 7. The comparison of the $6^{th}$-order HAM and exact solution for example3.

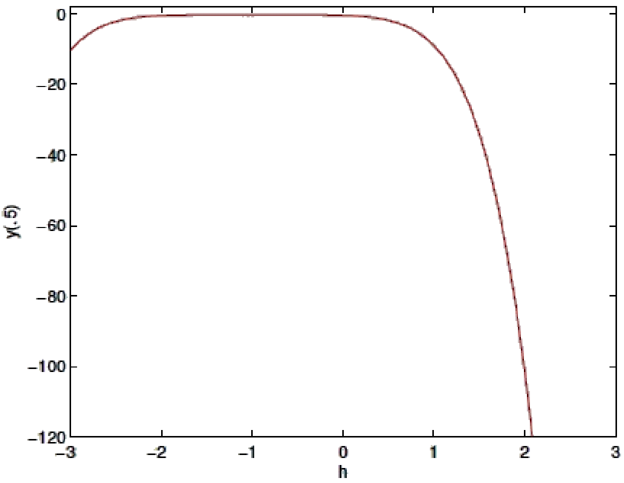


Fig. 8. The h-curve of based on the 6th-order HAM for example 3,

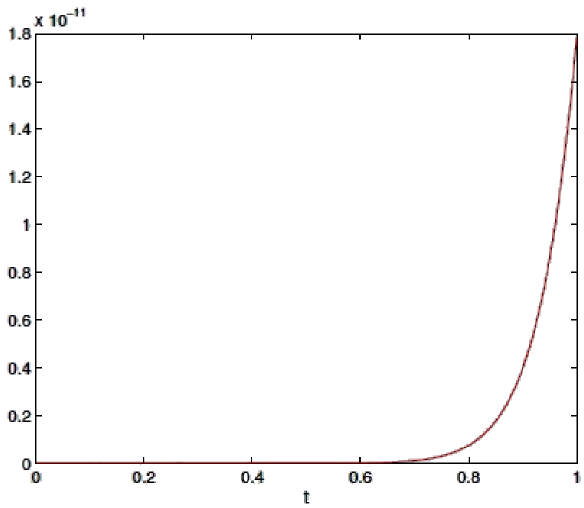


Fig. 9. The absolute error 6th-order HAM for example 3.

## *4.Conclusions*

In this Letter, we have successfully developed homotopy analysis method for solving an integro-differential equation with time-periodic coefficients. This technique was tested on some examples and was seen to produce satisfactory results. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

Matlab has been used for computations in this paper.

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