AVAILABLE ONLINE AT IJO.IAURASHT.AC.IR

IRANIAN JOURNAL OF OPTIMIZATION

Volume 4, issue 1,winter 2012,292-303
Research Paper

Islamic Azad University
Rasht Branch

Some Results on facets for linear inequality in 0-1 variables

Das Sashi Bhusan[1], Biswal Bagaban[2], J. P Tripathy [3*]

[1]Department of Mathematics, Balasore College of Engg . & Technology Teach. Sergarh , Balasore , Orissa , India.

[2]Ms.student of Mathematics, F. M. Autonomous College, Balasore , Orissa ,India

[3]Department of Mathematics Gurukul Institute of Technology Bhubaneswar, Orissa, India

*Correspondence E-mail: J. P Tripathy, jp_tripathy@yahoo.com

Online version is available on: www.ijo.iaurasht.ac.ir

Abstract

The facet of Knapsack ploytope, i.e. convex hull of 0-1 points satisfying a given linear inequality has been presented in this current paper. Such type of facets plays an important role in set covering set partitioning, matroidal-intersection vertex-packing, generalized assignment and other combinatorial problems. Strong covers for facets of Knapsack ploytope has been developed in the first part of the present paper. Generating family of valid cutting planes that satisfy inequality with 0-1 variables through algorithms are the attraction of this paper.

Keywords: Convex- hull, set-covering, set-partitioning, Matrodial-intersection, vertex-packing, cutting-planes.

1. Introduction

The facets of Knapsack polytope, i.e. of convex –hull of 0-1 points satisfying a given linear inequality play an important role in set covering, set partitioning, matroidal intersection, vertex packing, generalized assignment and other combinatorial problems. It is known that every lower dimensional facet can be augmented to give rise to one or more facets of full dimension. Every strong cover gives rise to at least one different facet. A necessary and sufficient condition for an inequality with 0-1 coefficients to be a facet is reviewed.

2. Definitions and Notation

Consider the inequality.

$$\sum_{j=1}^{n} a_j \times_j \leq a_o \tag{1}$$

Where $a_0 > a_j > 0$, and $x_j = 0, 1$.

Definition 1: Let N = {1,2,……………., n}. Let S be a subset of N. Then S is called cover for (1.2.1), if

(i) $\sum_{j \in S} a_j > a_0$

(ii) $\sum_{i \in T_1} a_j \leq a_0, j \in S \text{ and } T_1 = S - \{j\}$

Let $S^{/} = \{j \in N - S \,|\, a_j \geq a_{j_1}\}$

Where $j_1 = \max_{j \in S} a_j$, $E(S) = SUS^{/}$, $T_2 = N - E(S)$

(iii) If $T_2 \neq \Phi. Then \sum_{i \in T_3} a_i \leq a_0, T_3 = (S - \{j_1\}) \cup \{i_1\}, where\, i_1$

Defined by $a_{i_1} = \max_{j \in T_2} a_j$

Definition 2: A strong cover S of (1) is called a strong q-cover of (1.2.1) if $j_1 = q$.

Definition 1.2.3. Given a strong q-cover S, the set E(S) is called the extension of S, where $S^{/} = \{j \in N - S \,|\, a_j \geq a_{j_1}\}$

Let δ be the set of all strong cover of (1.2.1). Balas and Jeroslow [5] have shown that if δ be the family of minimal covers S for (1.2.1), then (1.2.1) is wquivalent to the set of (canonical) inequalities.

$$\sum_{j \in E(S)} x_j \leq |S| - 1, \text{ for all } S \in \delta \tag{2}$$

In the sense that $x \in R^n$, $x_j = 0$ or 1, $j \in N$, satisfies (1.2.1) if and only if it satisfies (1.2.2). Further it was shown that (1.2.1) is also equivalent to the set

$$\sum_{j\in S} x_j \le |S|-1,\ S \in \delta \qquad (3)$$

It was empirically observed that strong canonically inequalities often defined facets of convex hull of 0 -1, points satisfying (1.2.1), i.e. facet of the Knapsack polytope.

P=conv $\{x \in R^n \mid \sum_{j\in R^n} a_j x_j \le a_0, x_j = 0\ or 1,\ j \in N\}$

3. Canonical facets of the Knapsack Polytope

Let d be the dimension of P. The inequality $\sum_{j\in M} x_j \le K^k$ (4)

Where M $\subseteq$ N and k$\geq$ 0, is said to define a facet [(d-1) – dimensional face] of P if and only if the half space defined by $\sum_{j\in M} x_j = k$ (5)

Contains exactly d (affinely) independent points of P.Proposition 1. 3. 1. d =n-n$^{/}$, where n$^{/}$=|N$^{/}$| and N$^{/}$= {j$\in$N|aj> a_0}

Proof .d being the dimension of P, d$\geq$n-n$^{/}$, since P contains n-n$^{/}$ unit vectors e_j, j$\in$ N-N$^{/}$. Also d$\leq$n-n$^{/}$, since x_j=0 for all J$\in$ n$^{/}$ for any x$\in$P.

Hence d=n-n$^{/}$

Proposition 1: The inequality $x_j \le 1$ defines facet of P if and only if

$a_{j*}\, a_j \le a_o$

Where $a_{j*} = \max_{i\in N-\{j\}} a_i$ (1.3.2)

Proof. Let e_i be the ith unit vector, i$\in$N. Let e_j be any arbitrary unit vector j$\in$N. The vectors e_j, e_i +e_j for all i$\in$ N – {j} are also linearly independent which are contained in P for which the inequality (1.3.2) holds. Thus if (1.3.2) holds $x_j \le 1$ defines a facet of P.

Suppose $x_j \le 1$ defines a facet of P. Then $x_j^i = 1$ for n linearly independent vectors x^i of P which are the vertices of P for i=1, 2………n. Let X be the matrix whose rows are the vectors x^i. If (1.3.2) does not hold good, then $x_j^i = 0$, i=1, 2…….n, then the matrix X is a singular matrix. But X is not singular. So (1.3.2) must hold.

Example-1: Let P be the convex hull of 0-1 points satisfying

$10x_1 + 10x_2 + 4x_3 + 7x_4 + 2x_5 + 6x_6 + x_7 + x_8 \le 15.$

Then $x_j \leq 1$, defines a facet of p for j=3, 5, 7, 8 but not for j=1, 2, 4, 6. Following algorithm is being introduced to determine facets of the convex hull P.

Algorithm 1:

Step 1... Find $j^* \in N$ such that

$$a_{j^*} = \max_{i \in N-\{j\}} a_i$$

Step 2: is $a_{j^*} + a_j \leq a_0$?

If yes: then $x_j \leq 1$, $j \in N$ is s facet.

If no: then $x_j \leq 1$, $j \in N$ is not a facet.

Step 3: Redefine N: N =N-{j}

Is N =Φ?

If yes: Stop, all facets have been found out.

If no: Go to step 2.

Theorem 2: If $|M| \geq k+1 \geq 2$ and (1.3.1) defines a facet of P, then for each $i \in N$, P has a vertex $\bar{x}$ satisfying (1.3.1a) and such that $\bar{x}_i = 1$, and for each $i \in M$, P has a vertex x satisfying (1.3.1a) and such that $x_i = 0$

Proof . We shall prove this theorem by the method of contradiction. Since (1.3.1) defines a facet of P,(1.3.1a) is satisfied by n linearly independent vertices x^h, h=1,….,n of P. If X is the nxn matrix whose rows are these vertices x^h, then $x_i^h = 0$, h=1, 2,…, n for some $i \in N$ implies that x is singular. Hence for each $i \in N$, $x_i^h = 1$ for some $h \in \{1, \ldots, n\}$.

Also, if X_M is the sub matrix of X whose columns are indexed by M, then each row of X_M has exactly k entries equal to 1, and therefore, $x_i^h = 1$ for h =1,…., n and some $I \in M$, then the ith column of X_M is the sum of the remaining columns of X_M devided by k-1, hence the columns of X are linearly dependent. Thus for each $i \in M$, $x_i^h = 0$ for some $h \in \{1, \ldots, n\}$.

The following algorithm, for facets is given according above theorem.

Algorithm 2

Step 1: Choose $S \subset N$ a strong cover for Knapsack polytope P.

Step 2: Find out K^k from $|S| = k+1$

Choose j_1 such that $a_{j_1} = \max_{j \in S} a_j$

Set M =S ∪ {j∈ N –S | $a_j \geq a_{j_1}$}

Is M ≥ 2?

If yes: Go to step 3.

If no equation (1.3.1) does not defines a facet,

go to step 4.

Step 3: Set T^1 ={S-{j_1}}

Choose j_2 such that $a_{j_z} = \max_{j \in T^1}\{a_j\}$

Set T^2 = {S-{j_1, j_2}}∪ {1}

Is $\sum_{j \in T^2} a_j \leq a_0$

If yes: Infer equation (1.3.1) defines a facet.

If no: Equation (1.3.1) does not define a facet, go to step 4.

Step 4: Redefine S: N-S

Is S =Φ

If yes: Stop, All facets has been found out.

If no: Go to Step 2.

Example 2: Let P be the convex hull of 0-1 points satisfying

$4x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 + x_6 \leq 5$.

Here, N = (1, 2, 3, 4, 5, 6), a_0=5, a_1=4, a_2=4, a_3=3, a_4=2, a_5=2, a_6=1

Let us choose S = (3, 4)

∴|S| =2⇒ k+1=2⇒k =1.

$a_{j_1} = \max_{j \in S} a_j \Rightarrow a_{j_1} = a_3 \Rightarrow j_1 = 3$

T^1=S – {j_1} = {4}

$a_{j_2} = \max_{j \in T^1} a_j \Rightarrow 3 = a_4 \Rightarrow j_2 = 4$

$S^{/}$= {j∈ N- S: $a_j \geq a_j$}

= {j ∈ {1, 2, 5, 6}: aj $\geq a_{j_1}$ } = {1, 2}

M =E(S) =SU$S^{/}$ = {3, 4}∪ {1, 2} ={1,2,3,4}

∴T^2 =(S-{j_1,j_2}∪ {1}=({3,4} – {3,4}) ∪ {1}={1}

$$\sum_{j\in T^2} a_j = a_1 = 4 < a_0 (=5)$$

Thus the condition of Algorithm 1.3.1 is satisfied.

So for |M| ≥ 2,$= \sum_{j\in M} \times_j \leq k$ is a facet

$x_1+x_2+x_3+x_4 \leq 1$ is a facet

Let S = {1, 2, 6}

∴|S| = 3 $\Rightarrow k+1=3 \Rightarrow k=2$

$$a_{j_1} = \max_{j\in S} a_j \Rightarrow a_{j_1} = 1 \Rightarrow j_1 = 1$$

T^1 =S-{j_1} = (2, 6)

$$a_{j_2} = \max_{j\in T^1} a_j \Rightarrow \max_{j\in\{2,\sigma)} a_j = a_2 \Rightarrow j_2 = 2$$

$$S^{/} = \{ j \in N - S : a_j \geq a_{j_1} \} = \Phi$$

M=E(S) = SU$S^{/}$ = {1, 2, 6}∪Φ = {1, 2, 6}

∴T^2=(S-{j_1,j_2}∪{1}=({1,2,6} –{1,2}∪{1} ={1,6}

$$\sum_{j\in T^2} a_j = a_1 + a_\sigma = 4+1 = 5 = a_0 (=5)$$

Thus Algorithm (1.3.2) is satisfied.

Here |M| =3>2

So $\sum_{j\in M} \times_j \leq k$ defines a facet of given inequality.

$x_1+x_2+x_6 \leq 2$ is facet.

Similarly by choosing {1,2,6},{2,3,6}, {2,4,6},{2,5,6}, {3,4,6},{3,5,6} and {4,5,6} as strong cover we can develop the following facets by theorem (2.3.1)

$x_1 + x_2 \qquad\qquad\qquad + x_6 \leq 2$

$x_1 + x_2 + x_3 \qquad\qquad + x_6 \leq 2$

$x_1 + x_2 + \quad + x_4 \qquad + x_6 \leq 2$

$x_1 + x_2 \qquad\qquad + x_5 + x_6 \leq 2$

$x_1 + x_2 + x_3 + x_4 \qquad + x_6 \leq 2$

$x_1 + x_2 + x_3 \qquad + x_5 + x_6 \leq 2$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 2$

Theorem 3: The inequality

$$\sum_{j \in N} \pi_j x_j \leq \pi_0 \qquad (1.3.6)$$

Where $\pi_0 \geq 0$ is an integer, satisfied by all $x \in P$, if n ca be partitioned into (q+1) subsets N_h, h =0, 1, 2 , …. q, $1 \leq q \leq \pi_0$ such that

(α)π_j = h for all $j \in N_h$, h = 0,1,2,….q,

(β) M=$\bigcup_{n=1}^{q} N_h$ is the extension of some minimal cover for S for (1.2.1) such that $S \subseteq N$ and $|S| = \pi_0 + 1$

(γ)=N_0=N-M, N_1=M-Z, where Z $= \cup_{h=z}^{q} N_h$

$N_h = \{i \in N : \sum_{j \in S_{h+1}} a_j \leq a_i \leq \sum_{j \in S_{h+1}} a_j\}$, h =2,…., q

Where S_h is the set of the first h elements of S, h = 2…., q+1. If in addition to (α),(β) and (γ), one also has (η) $\sum_{j \in S - S_{h+1}} a_j + a_i \leq a_0$ for all $i \in N_h$ =0,1….,q

Then (1.3.6) defines a facet of P.

Theorem (1.3.2) lays the ground work for generating a family T of valid cutting planes. i.e. inequalities satisfied by all 0-1 points satisfying (1.2.1), most of which are facets of P. the procedure is given by the following algorithms as the proof of the theorem is described in Balas [2]

Algorithm 3

Step 1: Select $S \subseteq N$

Is $\sum_{j \in S} a_j > a_0$?

If yes, Take S as a cover

If no: Stop, infer S is not a cover. Go to step 4.

Step 2 set T_1 =S-{j}, j$\in$ S

Is $\sum_{i \in T} a_j \leq a_0$

If yes: Take S as minimal cover.

If no: stop, infer S is not minimal cover.

Go to step 4.

Step3 (Choose j_1 such that $a_{j_1} = \max_{j \in S} a_j$

Set E (S) =S$\cup$ {j$\in$N-S |aj$\geq a_{j_1}$ }

Set T_2 = N-E(S)

Is $T_2 \neq \Phi$

If yes: Go to step 4

If no: Infer S is not a strong cover.

Step 4: Choose i_1, such that, $a_{i_1} = \max_{j \in T^2} a_j$

Set T_3 = (S –{j_1}$\cup$ {i_1}

Is $\sum_{j \in T_3} a_j \leq a_0$

If yes: Take S as a strong cover.

If no: S is not strong cover.

Step5. Redefine S: N-S

S: S-{j})$\cup$ {i | i $\in$ N}

Step 6: Is S$\neq\Phi$?

If yes: Stop, all strong covers are found

If no: Go to step 1.

Algorithm 4:

Step1: Set π_0 =|S| -1, S$\in$T = family of cuts or strong covers

π_j =h,j $\in N_h$, h =0,1…, q

N_0=N-E (S), N_1 =E(S) - $\cup_{h=z}^{q} N_h$

$N_h = \{i\in N \mid \sum_{j\in S_h} a_j \le a_i \le \sum_{j\in S_{h+1}} a_j\}$, h=2,…q

S_h is the set of first h elements of S for h =2,…..q+1

Step2. is $\sum_{j\in S-S_{h+1}} a_j + a_i \le a_0$

For I$\in N_h$, h =1,2,…q.

If yes: $\sum_{j\in N_h} \pi_j x_i \le \pi_0$ is a valid cut

If no: S is not a facet.

Step3. Redefine S =F-S

Is S=Φ

If yes: Stop, all facets are found.

If not: Go to step 1.

Example 3: Let P be the convex hull of 0-1 points satisfying.

$10x_1+8x_2+6x_3+5x_4+3x_5+3x_6+3x_7+2x_8+2x_9+x_{10} \le 11$

Find out the facets along with strong covers of above inequality.

Answer: According to given inequality, we a have a_1 =10, a_2=8, a_3= 6

a_4 =5, a_5 =3, a_6 =3, a_7 =3, a_8 =2, a_9 =2, a_{10} =2 and a_0 =11

So, N = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Table 1.3.1: Lists the family of cutting planes characterized by theorem 1.3.2, along with the strong cover from which they are generated of the 24 members of the family all are facets of P. The table is to be read as follows.

Take line 1: S = {2, 3} is a cover, since a_2+ a_3 =8+6 >a_0=11; a minimal cover since 6 $\le$ 11 and 8 $\le$ 11: a strong cover as a_3+a_4=6+4 $\le$ 10. Extension of S is E(S) =SUS$^/$ = {2, 3} $\cup$\{1\} = {1, 2, 3} and N_0 =N-E (S) = {4, 5, 6, 7, 8, 9, 10}. Since S contains only two elements, so it is impossible to set $S_3 \Rightarrow N_h$ =0 for h>1.

$\therefore N_1=E(S) - \bigcup_{h=z}^{q} N_h + E(S) - 0 = E(S) = \{1, 2, 3\}$

Hence the first cut in the family F is

$x_1 + x_2 + x_3 \leq 1$

Which defines a facet of P, since $S\text{-}S_2 = \Phi$ and $a_1 = 10 \leq a_0 = 11$.

Now let us consider a separate set as {4, 5, 6, 7} from below. It is a cover since

$a_4 + a_5 + a_6 + a_7 = 4+ 3+ 3 +2 = 12 > 10$. Further, E(S) ={1, 2, 3, 4, 5, 6, 7} and $N_0 = N\text{-}E(S) = \{8,9\}$, $N_2 = \{1,2\}$, since $a_4 + a_5 \leq a_i \leq a_4 + a_5 + a_6$ for i =1, 2. $N_3 = \Phi$, since

$a_4 + a_5 + a_6 < a_i \leq a_4 + a_5 + a_6 + a_7$ for i=1.

$N_h = 0$, h>2: $N_1 = E(S) - N_2 = \{3, 4, 5, 6, 7\}$

$\pi_j = h$, $j \in N_h$, h=0, 1, 2, $\pi_0 = |S| - 1 = 4 - 1 = 3$

$j \in N_1 = \{3, 4, 5, 6, 7\}$

$\pi_3 = \pi_4 = \pi_5 = \pi_6 = \pi_7 = h = 1$.

$j \in N_2 = \{1, 2\}$

$\pi_2 = \pi_1 = h = 2$.

So the cut associated with S is

$2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 3$.

4. Conclusion

In this paper we have attempted to characterize the convex hull of 0-1 solutions to linear inequalities to a linear inequality. We are motivated by the work of Balas [4] and Wolsy [80]. The purpose of our research was to present algorithms for suitable computer programs for enumerating strong cover planes directly from the definitions

Table 1- The results from example 3

Strong cover{S}	π_1 π_2 π_3 π_4 π_5 π_6 π_7 π_8 π_9 π_{10} π_0
1,5	1 0 0 0 1 0 0 0 0 0 1
1,6	1 0 0 0 0 1 0 0 0 0 1
1,7	1 0 0 0 0 0 1 0 0 0 1

1, 8	1 0 0 0 0 0 0 1 0 0 1
1, 9	1 0 0 0 0 0 0 0 1 0 1
2, 3	1 1 1 0 0 0 0 0 0 0 1
2, 4	1 1 0 1 0 0 0 0 0 0 1
1, 8, 10	1 0 0 0 0 0 0 1 0 1 2
1, 9, 10	1 0 0 0 0 0 0 0 1 1 2
2, 5, 8	1 1 0 0 1 0 0 1 0 0 2
2, 5, 6	1 1 0 0 1 0 0 0 1 0 2
2, 5, 10	1 1 0 0 1 0 0 0 0 1 2
3, 4, 5	1 1 1 1 1 0 0 0 0 0 2
3, 4, 6	1 1 1 1 0 1 0 0 0 0 2
3, 4, 7	1 1 1 1 0 0 1 0 0 0 2
3, 5 6	2 1 1 0 0 1 0 0 0 0 2
3, 5 7	2 1 1 0 1 0 1 0 0 0 2
3, 5, 8	2 1 1 0 1 0 0 1 0 0 2
4, 5, 6, 7	2 2 1 1 1 1 1 0 0 0 3
4, 5, 6, 8	2 2 1 1 1 1 0 1 0 0 3
4, 5, 6, 9	2 2 1 1 1 1 0 0 1 0 3
4, 5, 6, 10	2 2 1 1 1 1 0 0 0 1 3
4, 6, 7, 8	2 2 1 1 0 1 1 1 0 0 3
4, 6, 7, 9	2 2 1 1 0 1 1 0 1 0 3
4, 6, 7, 10	2 2 1 1 0 1 1 0 0 1 3
5, 6, 7, 8, 9	3 2 2 1 1 1 1 1 1 0 4

Reference

[1] Balas E., An Additive Algorithm for solving for linear program with 0-1 variable, operation research, Vol. 13, 517 -545, 1965.

[2] Balas E., Facets of the Knapsack Polytope , Mathematical programming, Vol. 8, 146-164, 1975.

[3] Balas E., and Jeroslow R., Canonical Cuts, on the unit Hypercube, SIAM Journal of Applied Mathematics Vol. 23, 61-69, 1972.

[4] Balas E., Mazzola J.B., Non linear Zero – one programming | I, Linearisation Technique mathematical programming, Vol. 30 , 1-21, 1984a.

[5] Wolfe P., The simplex Algorithm for Quadratic Programming. Econometrical ,Vol . 27, 382-398, 1959.

[6] Walsey LA., Facets for a linear inequality in zero- one variable mathematical programming, Vol. 8, 165 – 178, 1975.

[7] Walsey L.A., A resource decomposition algorithm for general mathematical programming study Vol. 14, 244 -257, 1981.